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Smarandache Ruled Surfaces with Modified Orthogonal Frame in $\mathbb{R}^3$

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ABSTRACT: This article introduces a new approach for constructing ruled surfaces in Euclidean 3-space, particularly **TN**, **TB** and **NB** Smarandache ruled surfaces. We investigate their Gaussian and mean curvatures to classify them as developable and minimal surfaces. Additionally, we study the geodesic and normal curvatures associated with the base curve of each ruled surface. Moreover, some conditions are derived under which the base curve is a geodesic or an asymptotic line. The striction curve corresponding to each ruled surface is also computed. Finally, we give some examples with their graphical representations.

Keywords: Modified orthogonal frame, Frenet frame, Smarandache surfaces, Gaussian curvature, mean curvature.

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1. Introduction

The study of curves and surfaces holds profound significance in differential geometry, offering valuable insights for solving real-world problems across a range of fields [15], [16], [20]. Researchers have long examined geometric shapes to develop useful tools and methods. Among these shapes, ruled surfaces are a special class. They are generated by a vector traversing along a curve [12], making them relatively easy to parameterise which reduces computational complexities. These properties make ruled surfaces an essential object of study, both in theoretical research and practical applications. The concept of ruled surfaces was first introduced by Monge [22], [23], and since then, numerous advancements have been made in their analysis and applications.

In early works, researchers investigated the singularities of ruled surfaces in $\mathbb{R}^3$ [8], while Ferhat Taş et al. introduced novel approaches for designing ruled surfaces [27]. Further characterisations and applications of ruled surfaces have been discussed in various studies [3], [5], [7], [17], [22], [28]. Recently, S. Ouarab expanded the concept by introducing Smarandache ruled surfaces, where the base curve is a Smarandache curve and the generator is derived from an alternate vector of the Frenet frame [12], [13]. Smarandache curves are special curves whose position vectors are made of the Frenet frame vectors of another curve [13], [23]. These surfaces, along with Smarandache curves, have been extensively studied for their unique geometric properties by many authors, employing different frames in various spaces, such as the Frenet frame [13], [26], the FLC-frame [23], the Darboux frame [12], [25], and the alternative frame [24], etc.

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Further, the investigation of special curves on the ruled surfaces has been explored by various authors. For example, Bertrand curves and cylindrical helices on ruled surfaces were examined by Izumiya and Takeuchi [9]. These curves play an important role in understanding the behavior of ruled surfaces under different conditions. Additionally, Sefure and Emin explored the construction of a family of ruled surfaces that pass through the striction curve [28]. In [14], ruled surfaces are generated from curve lying on regular surface. Ruled surfaces have practical applications in fields like kinematics, the textile industry, Computer Aided Geometric Design (CAGD) and the automobile sector. Ruled surfaces are crucial for efficiently modeling complex shapes, such as industrial parts [19]. In kinematics, ruled surfaces are used to describe the motion in mechanical systems [19]. Also, ruled surfaces are applied to design streamlined vehicle designs, improving both aesthetic appeal and aerodynamic efficiency [19]. These diverse studies underscore the importance of ruled surfaces not only in theoretical differential geometry but also in solving real-world problems across multiple industries.

Motivated by these findings, we investigate Smarandache ruled surfaces generated by a modified orthogonal frame. We aim to enhance the understanding of Smarandache ruled surfaces, highlighting their geometric properties and their relevance in various domains.

2. Preliminaries

In this section, we recall some important results for our later use.

Theorem 2.1 **([4], [11])** *Let $\boldsymbol{\alpha} : I(\subseteq \mathbb{R}) \to \mathbb{R}^3$ be a unit-speed curve with curvature κ and torsion τ. Then,*

$$\begin{aligned} \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s), \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ \mathbf{b}'(s) &= -\tau(s)\mathbf{n}(s), \end{aligned} \tag{2.1}$$

where $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is the Frenet frame of the curve $\boldsymbol{\alpha}$.

Let $\boldsymbol{\alpha}(s)$ be an analytic curve in $\mathbb{R}^3$, parameterised by the arc length s. For the curve $\boldsymbol{\alpha}$, it is assumed that $\kappa(s) \not\equiv 0$. Therefore, an orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ can be given by:

$$\mathbf{T} = \frac{d\boldsymbol{\alpha}}{ds},\ \mathbf{N} = \frac{d\mathbf{T}}{ds},\ \mathbf{B} = \mathbf{T} \wedge \mathbf{N}$$

The relation between the modified orthogonal frame and the Frenet frame is given as:

$$\mathbf{T} = \mathbf{t}, \quad \mathbf{N} = \kappa\mathbf{n}, \quad \mathbf{B} = \kappa\mathbf{b}. \tag{2.2}$$

On differentiating (2.2) with respect to s, we get:

$$\frac{d\mathbf{T}}{ds} = \mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa^2\mathbf{T} + \frac{\kappa'}{\kappa}\mathbf{N} + \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} + \frac{\kappa'}{\kappa}\mathbf{B}.$$

Let $\langle\ ,\ \rangle$ denote the standard inner product in $\mathbb{R}^3$. Then we have:

$$\langle \mathbf{T}, \mathbf{T}\rangle = 1,\ \langle \mathbf{N}, \mathbf{N}\rangle = \kappa^2 = \langle \mathbf{B}, \mathbf{B}\rangle,$$

$$\langle \mathbf{T}, \mathbf{N}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = 0,$$

Therefore the frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is modified orthogonal frame at non-zero points of curvature [2], [10], [21].

Now, we recall the definition of the ruled surface [4], [11]:

Definition 2.1 *Given a one-parameter family of lines $\{\boldsymbol{\alpha}(s), \mathbf{w}(s)\}$, the parameterised surface*

$$\boldsymbol{\zeta}(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{w}(s), \quad s \in I, \quad v \in \mathbb{R}, \tag{2.3}$$

is called the ruled surface generated by the family $\{\boldsymbol{\alpha}(s), \mathbf{w}(s)\}$, where $\boldsymbol{\alpha}(s)$ is the base curve and $\mathbf{w}(s)$ is the generator.

If we denote $\mathbf{U} = \mathbf{U}(s, v)$, the unit normal on the ruled surface $\boldsymbol{\zeta}$ at regular points, then

$$\mathbf{U} = \frac{\boldsymbol{\zeta}_s \wedge \boldsymbol{\zeta}_v}{\|\boldsymbol{\zeta}_s \wedge \boldsymbol{\zeta}_v\|} \tag{2.4}$$

Moreover, the first and second fundamental forms are given by [4], [11]:

$$\begin{aligned} I &= Eds^2 + 2Fdsdv + Gdv^2, \\ II &= Lds^2 + 2Mdsdv + Ndv^2, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} E = \langle \boldsymbol{\zeta}_s, \boldsymbol{\zeta}_s \rangle, \quad F = \langle \boldsymbol{\zeta}_s, \boldsymbol{\zeta}_v \rangle, \quad G = \langle \boldsymbol{\zeta}_v, \boldsymbol{\zeta}_v \rangle, \\ L = \langle \boldsymbol{\zeta}_{ss}, \mathbf{U} \rangle, \quad M = \langle \boldsymbol{\zeta}_{sv}, \mathbf{U} \rangle, \quad N = \langle \boldsymbol{\zeta}_{vv}, \mathbf{U} \rangle. \end{aligned} \tag{2.6}$$

Definition 2.2 ([18]) *Let $\boldsymbol{\zeta}(s, v)$ be a surface with first and second fundamental forms*

$$I = Eds^2 + 2Fdsdv + Gdv^2, \quad \textit{and} \quad II = Lds^2 + 2Mdsdv + Ndv^2,$$

respectively. Then,

$$\kappa_G = \frac{LN - M^2}{EG - F^2}, \qquad \kappa_H \;= \frac{LG - 2MF + NE}{2(EG - F^2)}. \tag{2.7}$$

where κ_G is Gaussian curvature and κ_H is mean curvature at the regular points of $\boldsymbol{\zeta}(s, v)$.

A ruled surface with $\kappa_G = 0$, can be flattened into the plane without any deformation and distortion is called a developable surface [12]. These surfaces include cones, cylinders and tangent developable surfaces.

Also, a ruled surface with zero mean curvature is called a minimal surface [12]. These surfaces are characterised by their ability to minimize area. Plane, catenoid and helicoid are some examples of minimal surfaces.

Definition 2.3 ([18]) *The normal and the geodesic curvature of the surface $\boldsymbol{\zeta}(s, v)$ associated with the curve $\boldsymbol{\alpha}(s)$ on surface is given by:*

$$\kappa_n = \langle \boldsymbol{\alpha}''(s), \mathbf{U} \rangle \quad \textit{and} \quad \kappa_g = \langle \boldsymbol{\alpha}''(s), \mathbf{U} \wedge \boldsymbol{\alpha}'(s) \rangle \tag{2.8}$$

1. The curve $\boldsymbol{\alpha}$ is called an asymptotic line on surface $\boldsymbol{\zeta}$ if $\kappa_n = 0$ [1].

2. The curve $\boldsymbol{\alpha}$ is called a geodesic line on surface $\boldsymbol{\zeta}$ if $\kappa_g = 0$ [1].

Definition 2.4 ([4]) *The striction curve on ruled surface $\boldsymbol{\zeta}(s, v)$ is given by:*

$$\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - \frac{\langle \boldsymbol{\alpha}', \mathbf{w}' \rangle}{\|\mathbf{w}'\|^2} \mathbf{w} \tag{2.9}$$

Definition 2.5 ([6]) *Let $\boldsymbol{\beta}$ be a space curve with modified orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. The Smarandache curve obtained from the vectors $\mathbf{T}$ and $\mathbf{N}$ of the curve $\boldsymbol{\beta}$ can be defined as*

$$\boldsymbol{\alpha}_1 = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}). \tag{2.10}$$

Definition 2.6 ([6]) *Let $\boldsymbol{\beta}$ be a space curve with modified orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. The Smarandache curve obtained from the vectors $\mathbf{T}$ and $\mathbf{B}$ of the curve $\boldsymbol{\beta}$ can be defined as*

$$\boldsymbol{\alpha}_2 = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{B}). \tag{2.11}$$

Definition 2.7 ([6]) *Let $\boldsymbol{\beta}$ be a space curve with modified orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. The Smarandache curve obtained from the vectors $\mathbf{N}$ and $\mathbf{B}$ of the curve $\boldsymbol{\beta}$ can be defined as*

$$\boldsymbol{\alpha}_3 = \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{B}). \tag{2.12}$$

3. Smarandache Ruled Surface According to Modified Orthogonal Frame

In the following section, first we give the definitions of Smarandache ruled surfaces according to modified orthogonal frame in $\mathbb{R}^3$ and we investigate some of their geometric properties.

3.1. TN- Smarandache ruled surface

Definition 3.1 *Let $\boldsymbol{\alpha}(s)$ be an analytic space curve parameterised by arc length s and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be its modified orthogonal frame. The ruled surface generated by* **TN**- *Smarandache curve is given by:*

$$^{1}\boldsymbol{\zeta}(s,v) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N}) + v\mathbf{B} \qquad s \in I, \quad v \in \mathbb{R} \tag{3.1}$$

Theorem 3.1 *The Gaussian and the mean curvatures of ruled surface $^{1}\boldsymbol{\zeta}(s,v)$ are*

$$\kappa_G = -\frac{2}{\varphi_1^2}\left(\frac{\kappa^8\tau^2}{\kappa^6+\kappa^4+(\kappa\kappa')^2+2v^2\kappa^4\tau^2+2\kappa^3\kappa'-2\sqrt{2}v\kappa^4\tau}\right)$$

$$\kappa_H = \frac{2}{\varphi_1}\left(\frac{\kappa^4\lambda_1+\kappa^2\kappa'\lambda_2}{\kappa^6+\kappa^4+(\kappa\kappa')^2+2v^2\kappa^4\tau^2+2\kappa^3\kappa'-2\sqrt{2}v\kappa^4\tau}\right)$$

where

$$\varphi_1 = \left[\left(\frac{1}{\sqrt{2}}+\frac{\kappa'}{\sqrt{2}\kappa}-v\tau\right)^2+\frac{\kappa^6}{2}\right]^{1/2}$$

and

$$\lambda_1 = \left[\kappa^2\left(\frac{\tau^2}{\sqrt{2}}-\frac{\kappa^2}{\sqrt{2}}+v\tau'\right)-v\tau(\sqrt{2}v\tau-2)+\frac{\kappa}{\sqrt{2}}(\kappa'+\kappa'')-\frac{1}{\sqrt{2}}\right],$$

$$\lambda_2 = 2\kappa\left(2v\tau-\sqrt{2}\right)-\frac{3}{\sqrt{2}}\kappa'.$$

Corollary 3.1 *1.* **TN**- *Smarandache ruled surface is developable if the curve $\boldsymbol{\alpha}$ is a planar curve.*

2. **TN**- *Smarandache ruled surface is minimal if*

$$\kappa^4\lambda_1+\kappa^2\kappa'\lambda_2 = 0.$$

Proof: Computing the partial derivatives of (3.1) with respect to s and v to calculate the components of first and the second fundamental form, we get

$$^{1}\boldsymbol{\zeta}_s = \frac{-\kappa^2}{\sqrt{2}}\mathbf{T}+\left(\frac{1}{\sqrt{2}}+\frac{\kappa'}{\sqrt{2}\kappa}-v\tau\right)\mathbf{N}+\left(\frac{\tau}{\sqrt{2}}+\frac{v\kappa'}{\kappa}\right)\mathbf{B} \tag{3.2}$$

$$\begin{aligned} ^{1}\boldsymbol{\zeta}_{ss} = {} & \left(v\kappa^2\tau-\frac{3}{\sqrt{2}}\kappa\kappa'-\frac{\kappa^2}{\sqrt{2}}\right)\mathbf{T} \\ & +\left(\frac{\kappa''}{\sqrt{2}\kappa}+\frac{\kappa'}{\sqrt{2}\kappa}-\frac{\kappa^2}{\sqrt{2}}-\frac{\tau^2}{\sqrt{2}}-\frac{2v\kappa'\tau}{\kappa}-v\tau'\right)\mathbf{N} \\ & +\left(\frac{\tau}{\sqrt{2}}+\frac{\tau'}{\sqrt{2}}+\frac{\sqrt{2}\kappa'\tau}{\kappa}+\frac{v\kappa''}{\kappa}-v\tau^2\right)\mathbf{B}. \end{aligned} \tag{3.3}$$

$$^{1}\boldsymbol{\zeta}_v = \mathbf{B} \tag{3.4}$$

$$^{1}\boldsymbol{\zeta}_{sv} = -\tau\mathbf{N}+\frac{\kappa'}{\kappa}\mathbf{B} \tag{3.5}$$

$$^{1}\boldsymbol{\zeta}_{vv} = 0 \tag{3.6}$$

Unit normal vector on surface can be calculated using (2.4), (3.2) and (3.4)

$$\mathbf{U} = \frac{1}{\varphi_1}\left[\left(\frac{1}{\sqrt{2}} + \frac{\kappa'}{\sqrt{2}\kappa} - v\tau\right)\mathbf{T} + \frac{\kappa^2}{\sqrt{2}}\mathbf{N}\right] \tag{3.7}$$

Using (2.6), (3.2) and (3.4), we have

$$\begin{aligned} E &= \frac{\kappa^4}{2} + \frac{\kappa^2}{2} + \frac{(\kappa')^2}{2} + (v\kappa\tau)^2 + \kappa\kappa' - \sqrt{2}v\kappa^2\tau + \frac{\kappa^2\tau^2}{2} + v^2(\kappa')^2, \\ F &= \frac{\kappa^2\tau}{\sqrt{2}} + v\kappa\kappa', \\ G &= \kappa^2. \end{aligned} \tag{3.8}$$

Using (2.6), (3.3), (3.5) and (3.6), yields

$$\begin{aligned} L = \frac{1}{\varphi_1}\Big(& 2v\kappa^2\tau - 2\sqrt{2}\kappa\kappa' - \frac{\kappa^2}{\sqrt{2}} + 4v\kappa\kappa'\tau - \frac{3}{\sqrt{2}}(\kappa')^2 \\ & - \sqrt{2}v^2\kappa^2\tau^2 + \frac{\kappa^3\kappa''}{\sqrt{2}} + \frac{\kappa^3\kappa'}{\sqrt{2}} - \frac{\kappa^6}{\sqrt{2}} - v\kappa^4\tau' \\ & - \frac{\kappa^4\tau^2}{\sqrt{2}} - 2v\kappa^3\kappa'\tau\Big), \\ M = -&\frac{1}{\varphi_1}(\kappa^4\tau), \\ N = 0&. \end{aligned} \tag{3.9}$$

Using (2.7), (3.8) and (3.9), we get

$$\kappa_G = -\frac{2}{\varphi_1^2}\left(\frac{\kappa^8\tau^2}{\kappa^6 + \kappa^4 + (\kappa\kappa')^2 + 2v^2\kappa^4\tau^2 + 2\kappa^3\kappa' - 2\sqrt{2}v\kappa^4\tau}\right)$$

$$\kappa_H = \frac{2}{\varphi_1}\left(\frac{\kappa^4\lambda_1 + \kappa^2\kappa'\lambda_2}{\kappa^6 + \kappa^4 + (\kappa\kappa')^2 + 2v^2\kappa^4\tau^2 + 2\kappa^3\kappa' - 2\sqrt{2}v\kappa^4\tau}\right)$$

${}^1\boldsymbol{\zeta}(s, v)$ is developable when κ_G vanishes, this implies

$$\kappa^8\tau^2 = 0$$

$$\Rightarrow \qquad \tau = 0$$

${}^1\boldsymbol{\zeta}(s, v)$ is minimal when κ_H vanishes, this implies

$$\kappa^4\lambda_1 + \kappa^2\kappa'\lambda_2 = 0.$$

□

Corollary 3.2 *1. $\boldsymbol{\alpha}$ is an asymptotic line if* $\kappa^2\mu_1 + \kappa\kappa'\mu_2 + \kappa^3\mu_3 - 3(\kappa')^2 = 0$.

where

$$\mu_1 = \sqrt{2}v\tau - 1, \qquad \mu_2 = 3\sqrt{2}v\tau - 4, \qquad \mu_3 = \kappa'' + \kappa' - \kappa^3 - \kappa\tau^3.$$

2. $\boldsymbol{\alpha}$ is a geodesic line if $\kappa^5\epsilon_1 + \kappa^2\epsilon_2 + \kappa^2\tau^2\epsilon_3 + \kappa\kappa'\epsilon_4 + (\kappa')^2\epsilon_5 + \kappa''\tau\epsilon_6 = 0$

where

$$\epsilon_1 = \kappa(\tau + \tau') + 2\kappa'\tau, \qquad \epsilon_2 = \tau(1 - \sqrt{2}v\tau') + \tau',$$

$$\epsilon_3 = \tau - \sqrt{2}v(\kappa^2 + \tau^2 + 1), \qquad \epsilon_4 = \tau^2(\tau - \sqrt{2}v) + \tau(3 - 2\kappa^2) + \tau'(2 - \sqrt{2}v\tau),$$
$$\epsilon_5 = \tau' + \tau\left(4 + \frac{2\kappa'}{\kappa} - 2\sqrt{2}v\tau\right), \qquad \epsilon_6 = \sqrt{2}v\tau - \kappa' - \kappa.$$

Proof: Differentiating (2.10) with respect to s, we get

$$\boldsymbol{\alpha}' = \frac{1}{\sqrt{2}}\left[-\kappa^2\mathbf{T} + \left(1 + \frac{\kappa'}{\kappa}\right)\mathbf{N} + \tau\mathbf{B}\right] \tag{3.10}$$

On differentiating (3.10), we have

$$\boldsymbol{\alpha}'' = \frac{1}{\sqrt{2}}\left[(-\kappa^2 - 3\kappa\kappa')\mathbf{T} + \left(\frac{\kappa''}{\kappa} + \frac{\kappa'}{\kappa} - \kappa^2 - \tau^2\right)\mathbf{N} + \left(\tau' + \tau + \frac{2\kappa'\tau}{\kappa}\right)\mathbf{B}\right] \tag{3.11}$$

Using (2.8), (3.7) and (3.11), the expression for normal curvature κ_n is given by:

$$\kappa_n = \frac{1}{\sqrt{2}\phi_1}\left[\kappa^2(\sqrt{2}v\tau - 1) + \kappa\kappa'(3\sqrt{2}v\tau - 4) + \kappa^3(\kappa'' + \kappa' - \kappa^3 - \kappa\tau^2) - 3(\kappa')^2\right]$$

for $\kappa_n = 0$, $\boldsymbol{\alpha}$ is an asymptotic line, this implies

$$\kappa^2\mu_1 + \kappa\kappa'\mu_2 + \kappa^3\mu_3 - 3(\kappa')^2 = 0$$

Now, we compute $\mathbf{U} \wedge \boldsymbol{\alpha}'$, using (3.7) and (3.10)

$$\begin{aligned}\mathbf{U} \wedge \boldsymbol{\alpha}' = \frac{1}{\sqrt{2}\varphi_1}\Bigg[&\kappa^2\tau\mathbf{T} + \left(\sqrt{2}v\tau^2 - \frac{\kappa'\tau}{\kappa} - \tau\right)\mathbf{N} \\ &+ \left(\frac{2\kappa'}{\kappa} + \kappa^4 - \sqrt{2}v\tau + \left(\frac{\kappa'}{\kappa}\right)^2 - \frac{\sqrt{2}v\kappa'\tau}{\kappa} + 1\right)\mathbf{B}\Bigg]\end{aligned} \tag{3.12}$$

Using (2.8), (3.11) and (3.12), the expression for geodesic curvature κ_g is given by:

$$\kappa_g = \kappa^5\epsilon_1 + \kappa^2\epsilon_2 + \kappa^2\tau^2\epsilon_3 + \kappa\kappa'\epsilon_4 + (\kappa')^2\epsilon_5 + \kappa''\tau\epsilon_6$$

for $\kappa_g = 0$, $\boldsymbol{\alpha}$ is a geodesic line, this implies

$$\kappa^5\epsilon_1 + \kappa^2\epsilon_2 + \kappa^2\tau^2\epsilon_3 + \kappa\kappa'\epsilon_4 + (\kappa')^2\epsilon_5 + \kappa''\tau\epsilon_6 = 0.$$

□

Special case: For a plane $\boldsymbol{\alpha}$ curve with constant curvature, the geodesic curvature κ_g vanishes, this implies $\boldsymbol{\alpha}$ is a geodesic curve.

Theorem 3.2 *Striction curve for* ${}^1\boldsymbol{\zeta}(s, v)$ *is given by:*

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left(\mathbf{T} + \mathbf{N} - \frac{\kappa^2\tau}{(\kappa')^2 - \kappa^2\tau^2}\mathbf{B}\right)$$

Proof: Using (2.3), (2.9) and (3.1), we get

$$\langle\boldsymbol{\alpha}', \mathbf{w}'\rangle = \frac{-\kappa^2\tau}{\sqrt{2}}$$

also,

$$\|\mathbf{w}'\|^2 = (\kappa')^2 - \tau^2\kappa^2$$

this implies,

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left(\mathbf{T} + \mathbf{N} - \frac{\kappa^2\tau}{(\kappa')^2 - \kappa^2\tau^2}\mathbf{B}\right)$$

□

3.2. TB- Smarandache ruled surface

Definition 3.2 *Let $\boldsymbol{\alpha}(s)$ be an analytic space curve parameterised by arc length s and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be its modified orthogonal frame. The ruled surface generated by* **TB**- *Smarandache curve is given by:*

$$^2\boldsymbol{\zeta}(s,v) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{B}) + v\mathbf{N} \qquad s \in I, \quad v \in \mathbb{R} \tag{3.13}$$

Theorem 3.3 *The Gaussian and the mean curvatures of ruled surface $^2\boldsymbol{\zeta}(s,v)$ are*

$$\kappa_G = -\frac{1}{\varphi_2^2}\left[\frac{(\kappa\kappa' + \sqrt{2}v\kappa^2\tau - \sqrt{2}v\kappa^4\tau)^2}{2v^2\kappa^6 + (\kappa\kappa')^2 + 2v^2\kappa^4\tau^2 + 2\sqrt{2}v\kappa^3\kappa'\tau}\right]$$

$$\kappa_H = \frac{1}{\varphi_2}\left(\frac{\kappa\kappa'\rho_1 + v\kappa^4\tau\rho_2 - v\kappa^5\rho_3}{2\sqrt{2}v^2\kappa^6 + \sqrt{2}(\kappa\kappa')^2 + 2\sqrt{2}v^2\kappa^4\tau^2 + 4v\kappa^3\kappa'\tau}\right)$$

where

$$\varphi_2 = \left[\left(\frac{\kappa'}{\sqrt{2}\kappa} + v\tau\right)^2 + v^2\kappa^6\right]^{1/2}$$

and

$$\rho_1 = v\kappa\kappa' + \kappa^2\left(\sqrt{2}v^2\tau + \tau(\sqrt{2}-1) + (1-\sqrt{2})\right),$$

$$\rho_2 = \kappa^2(1-\tau) + \tau + (\sqrt{2}-2),$$

$$\rho_3 = \sqrt{2}v\kappa\tau' + \kappa''.$$

Corollary 3.3 *1.* **TB**- *Smarandache ruled surface is developable if*

$$\kappa\kappa' + \sqrt{2}v\kappa^2\tau - \sqrt{2}v\kappa^4\tau = 0.$$

2. **TB**- *Smarandache ruled surface is minimal if*

$$\kappa\kappa'\rho_1 + v\kappa^4\tau\rho_2 - v\kappa^5\rho_3 = 0.$$

Proof: Computing the partial derivatives of (3.13) with respect to s and v to calculate the components of first and the second fundamental form, we get

$$^2\boldsymbol{\zeta}_s = -v\kappa^2\mathbf{T} + \left(\frac{1}{\sqrt{2}} + \frac{v\kappa'}{\kappa} - \frac{\tau}{\sqrt{2}}\right)\mathbf{N} + \left(\frac{\kappa'}{\sqrt{2}\kappa} + v\tau\right)\mathbf{B} \tag{3.14}$$

$$\begin{aligned} ^2\boldsymbol{\zeta}_{ss} = &\left(\frac{\kappa^2\tau}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}} - 3v\kappa\kappa'\right)\mathbf{T} \\ &+ \left(\frac{v\kappa''}{\kappa} + \frac{\kappa'}{\sqrt{2}\kappa} - \frac{\sqrt{2}v\kappa'\tau}{\kappa} - v\tau^2 - v\kappa^2 - \frac{\tau'}{\sqrt{2}}\right)\mathbf{N} \\ &+ \left(\frac{\tau}{\sqrt{2}} - \frac{\tau^2}{\sqrt{2}} + \frac{2v\kappa'\tau}{\kappa} + \frac{\kappa''}{\sqrt{2}\kappa} + v\tau'\right)\mathbf{B}. \end{aligned} \tag{3.15}$$

$$^2\boldsymbol{\zeta}_v = \mathbf{N} \tag{3.16}$$

$$^2\boldsymbol{\zeta}_{sv} = -\kappa^2\mathbf{T} + \frac{\kappa'}{\kappa}\mathbf{N} + \tau\mathbf{B} \tag{3.17}$$

$$^2\boldsymbol{\zeta}_{vv} = 0 \tag{3.18}$$

Unit normal vector on surface can be calculated using (2.4), (3.14) and (3.16)

$$\mathbf{U} = \frac{1}{\varphi_2}\left[-\left(\frac{\kappa'}{\sqrt{2}\kappa} + v\tau\right)\mathbf{T} - v\kappa^2\mathbf{B}\right] \tag{3.19}$$

Using (2.6), (3.14) and (3.16), we have

$$\begin{aligned} E &= v^2\kappa^4 + \frac{\kappa^2}{2} + \frac{\kappa^2\tau^2}{2} + v^2(\kappa')^2 - \kappa^2\tau + \sqrt{2}v\kappa\kappa' + \frac{(\kappa')^2}{2} + (v\kappa\tau)^2, \\ F &= \frac{\kappa^2}{\sqrt{2}} - \frac{\kappa^2\tau}{\sqrt{2}} + v\kappa\kappa', \\ G &= \kappa^2. \end{aligned} \tag{3.20}$$

Using (2.6), (3.15), (3.17) and (3.18), yields

$$\begin{aligned} L = \frac{1}{\varphi_2}\Big(&\frac{-\kappa\kappa'\tau}{\sqrt{2}} + \frac{\kappa\kappa'}{\sqrt{2}} + \frac{3v(\kappa')^2}{\sqrt{2}} - \frac{v\kappa^2\tau^2}{\sqrt{2}} + \frac{v\kappa^2\tau}{\sqrt{2}} \\ &+ 3v^2\kappa\kappa'\tau - \frac{v\kappa^4\tau}{\sqrt{2}} + \frac{v\kappa^4\tau^2}{\sqrt{2}} - 2v^2\kappa^3\kappa' \\ &- \frac{v\kappa^3\kappa''}{\sqrt{2}} - v^2\kappa^4\tau'\Big), \\ M = \frac{1}{\varphi_2}&\left(\frac{\kappa\kappa'}{\sqrt{2}} + v\kappa^2\tau - v\kappa^4\tau\right), \\ N = 0&. \end{aligned} \tag{3.21}$$

Using (2.7), (3.20) and (3.21), we get

$$\kappa_G = -\frac{1}{\varphi_2^2}\left[\frac{(\kappa\kappa' + \sqrt{2}v\kappa^2\tau - \sqrt{2}v\kappa^4\tau)^2}{2v^2\kappa^6 + (\kappa\kappa')^2 + 2v^2\kappa^4\tau^2 + 2\sqrt{2}v\kappa^3\kappa'\tau}\right]$$

$$\kappa_H = \frac{1}{\varphi_2}\left(\frac{\kappa\kappa'\rho_1 + v\kappa^4\tau\rho_2 - v\kappa^5\rho_3}{2\sqrt{2}v^2\kappa^6 + \sqrt{2}(\kappa\kappa')^2 + 2\sqrt{2}v^2\kappa^4\tau^2 + 4v\kappa^3\kappa'\tau}\right)$$

${}^2\boldsymbol{\zeta}(s, v)$ is developable when κ_G vanishes, this implies

$$\kappa\kappa' + \sqrt{2}v\kappa^2\tau - \sqrt{2}v\kappa^4\tau = 0.$$

${}^2\boldsymbol{\zeta}(s, v)$ is minimal when κ_H vanishes, this implies

$$\kappa\kappa'\rho_1 + v\kappa^4\tau\rho_2 - v\kappa^5\rho_3 = 0.$$

□

Special case: For a plane curve with constant curvature ${}^2\boldsymbol{\zeta}(s, v)$ is always a developable surface. Since

$$\kappa\kappa' + \sqrt{2}v\kappa^2\tau - \sqrt{2}v\kappa^4\tau = 0.$$

at $\tau = 0$, we get,

$$\kappa\kappa' = 0 \qquad \Rightarrow \kappa = \text{constant}$$

Corollary 3.4 *1. $\boldsymbol{\alpha}$ is an asymptotic line if $\kappa\kappa'\psi_1 + \kappa^2\tau^2\psi_2 + \sqrt{2}v\kappa^2\tau\psi_3 - \sqrt{2}v\kappa^3\kappa'' = 0$.*

where

$$\psi_1 = 1 - \tau, \quad \psi_2 = \sqrt{2}v(\kappa^2 - 1), \quad \psi_3 = v(1 - \kappa^2).$$

2. $\boldsymbol{\alpha}$ *is a geodesic line if* $\kappa^4\delta_1 + (\kappa')^2\delta_2 + \kappa\kappa'\delta_3 + \kappa''\delta_4 + \kappa^2\tau^2\delta_5 = 0$

where

$$\delta_1 = \upsilon(2\tau - \tau^2 - 1), \qquad \delta_2 = \frac{\kappa'}{\kappa}\left(\frac{1}{\sqrt{2}} - \sqrt{2}\tau\right) + \upsilon\tau(1 - 2\tau) + \frac{\tau'}{\sqrt{2}},$$

$$\delta_3 = \tau\left(-\upsilon\tau' - \frac{1}{\sqrt{2}}\right) - \frac{\tau^2}{\sqrt{2}}(\tau - 3), \qquad \delta_4 = \frac{\kappa'}{\sqrt{2}}(\tau - 1) + \upsilon\kappa\tau(\tau - 1),$$

$$\delta_5 = \upsilon(2\tau - \tau^2 - 1).$$

Proof: Differentiating (2.11) with respect to s, we get

$$\boldsymbol{\alpha}' = \frac{1}{\sqrt{2}}\left[(1 - \tau)\mathbf{N} + \frac{\kappa'}{\kappa}\mathbf{B}\right] \tag{3.22}$$

On differentiating (3.22), we have

$$\boldsymbol{\alpha}'' = \frac{1}{\sqrt{2}}\left[(\kappa^2\tau - \kappa^2)\mathbf{T} + \left(\frac{\kappa'}{\kappa} - \frac{2\kappa'\tau}{\kappa} - \tau'\right)\mathbf{N} + \left(\tau + \frac{\kappa''}{\kappa} - \tau^2\right)\mathbf{B}\right] \tag{3.23}$$

Using (2.8), (3.19) and (3.23), the expression for normal curvature κ_n is given by:

$$\kappa_n = \frac{1}{\sqrt{2}\phi_2}\left[\frac{\kappa\kappa'}{\sqrt{2}} + \upsilon\kappa^2\tau - \frac{\kappa\kappa'\tau}{\sqrt{2}} - \upsilon\kappa^2\tau^2 - \upsilon\kappa^4\tau - \upsilon\kappa^3\kappa'' + \upsilon\kappa^4\tau^2\right]$$

for $\kappa_n = 0$, $\boldsymbol{\alpha}$ is an asymptotic line, this implies

$$\kappa\kappa'\psi_1 + \kappa^2\tau^2\psi_2 + \sqrt{2}\upsilon\kappa^2\tau\psi_3 - \sqrt{2}\upsilon\kappa^3\kappa'' = 0.$$

Now, we compute $\mathbf{U} \wedge \boldsymbol{\alpha}'$, using (3.19) and (3.22)

$$\begin{aligned}\mathbf{U} \wedge \boldsymbol{\alpha}' = \frac{1}{\sqrt{2}\varphi_2}\Bigg[&(\upsilon\kappa^2 - \upsilon\kappa^2\tau)\mathbf{T} + \left(\frac{1}{\sqrt{2}}\left(\frac{\kappa'}{\kappa}\right)^2 + \frac{\upsilon\kappa'\tau}{\kappa}\right)\mathbf{N} \\ &+ \left(\frac{\kappa'\tau}{\sqrt{2}\kappa} + \upsilon\tau^2 - \frac{\kappa'}{\sqrt{2}\kappa} - \upsilon\tau\right)\mathbf{B}\Bigg]\end{aligned} \tag{3.24}$$

Using (2.8), (3.23) and (3.24), the expression for geodesic curvature κ_g is given by:

$$\kappa_g = \frac{1}{2\phi_2}\left(\kappa^4\delta_1 + (\kappa')^2\delta_2 + \kappa\kappa'\delta_3 + \kappa''\delta_4 + \kappa^2\tau^2\delta_5\right)$$

for $\kappa_g = 0$, $\boldsymbol{\alpha}$ is a geodesic line, this implies

$$\kappa^4\delta_1 + (\kappa')^2\delta_2 + \kappa\kappa'\delta_3 + \kappa''\delta_4 + \kappa^2\tau^2\delta_5 = 0.$$

□

Special case I: For a curve $\boldsymbol{\alpha}$ with unit curvature, and unit torsion i.e., $\kappa = \tau = 1$ the normal curvature κ_n vanishes, this implies $\boldsymbol{\alpha}$ is a asymptotic curve.

Special case II: For a curve $\boldsymbol{\alpha}$ with constant curvature, and unit torsion i.e., $\tau = 1$ the geodesic curvature κ_g vanishes, this implies $\boldsymbol{\alpha}$ is a geodesic curve.

Theorem 3.4 *Striction curve for* ${}^2\boldsymbol{\zeta}(s,v)$ *is given by:*

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left(\mathbf{T} - \frac{\kappa\kappa'}{\kappa^4+(\kappa')^2+\kappa^2\tau^2}\mathbf{N}+\mathbf{B}\right)$$

Proof: Using (2.3), (2.9) and (3.13), we get

$$\langle \boldsymbol{\alpha}', \mathbf{w}'\rangle = \kappa\kappa'$$

also,

$$\|\mathbf{w}'\|^2 = \kappa^4+(\kappa')^2+\kappa^2\tau^2$$

this implies,

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left(\mathbf{T} - \frac{\kappa\kappa'}{\kappa^4+(\kappa')^2+\kappa^2\tau^2}\mathbf{N}+\mathbf{B}\right)$$

□

3.3. NB- Smarandache ruled surface

Definition 3.3 *Let* $\boldsymbol{\alpha}(s)$ *be an analytic space curve parameterised by arc length* s *and* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *be its modified orthogonal frame. The ruled surface generated by* **NB**- *Smarandache curve is given by:*

$$ {}^3\boldsymbol{\zeta}(s,v) = \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B}) + v\mathbf{T} \qquad s\in I, \quad v\in\mathbb{R} \tag{3.25}$$

Theorem 3.5 *The Gaussian and the mean curvatures of ruled surface* ${}^2\boldsymbol{\zeta}(s,v)$ *are*

$$\kappa_G = -\frac{1}{2\varphi_3^2}\left[\frac{(\kappa\kappa'+\kappa^2\tau)^2}{\kappa^2\tau^2+(\kappa')^2+\sqrt{2}v\kappa\kappa'-\sqrt{2}v\kappa^2\tau}\right]$$

$$\kappa_H = \frac{1}{2\varphi_3}\left(\frac{\kappa\kappa'\sigma_1+(\kappa')^2\sigma_2+\kappa\tau\sigma_3+\kappa^2\sigma_4}{\kappa^2\tau^2+(\kappa')^2+\sqrt{2}v\kappa\kappa'-\sqrt{2}v\kappa^2\tau}\right)$$

where

$$\varphi_3 = \frac{1}{\sqrt{2}}\left[(\kappa'+\kappa\tau)^2 + \left(\kappa'-\kappa\tau+\sqrt{2}v\kappa\right)^2\right]^{1/2}$$

and

$$\sigma_1 = \frac{\kappa^2}{2} - \tau' - \sqrt{2}v\tau - \frac{v}{\sqrt{2}}, \qquad \sigma_2 = \frac{v}{\sqrt{2}} - 2\tau,$$

$$\sigma_3 = \kappa'' + \sqrt{2}v\kappa\tau - v^2\kappa, \qquad \sigma_4 = -\frac{v\tau'}{\sqrt{2}} - \tau^3.$$

Corollary 3.5 *1.* **NB**- *Smarandache ruled surface is developable if*

$$\kappa\kappa'+\kappa^2\tau = 0.$$

2. **NB**- *Smarandache ruled surface is minimal if*

$$\kappa\kappa'\sigma_1+(\kappa')^2\sigma_2+\kappa\tau\sigma_3+\kappa^2\sigma_4 = 0.$$

Proof: Computing the partial derivatives of (3.25) with respect to s and v to calculate the components of first and the second fundamental form, we get

$$ {}^3\boldsymbol{\zeta}_s = \frac{-\kappa^2}{\sqrt{2}}\mathbf{T} + \left(\frac{\kappa'}{\sqrt{2}\kappa} - \frac{\tau}{\sqrt{2}} + v\right)\mathbf{N} + \frac{1}{\sqrt{2}}\left(\frac{\kappa'}{\kappa}+\tau\right)\mathbf{B} \tag{3.26}$$

$$
\begin{aligned}
{}^3\boldsymbol{\zeta}_{ss} =& \left(\frac{\kappa^2\tau}{\sqrt{2}} - \frac{3\kappa\kappa'}{\sqrt{2}} - v\kappa^2\right)\mathbf{T} \\
&+ \left(\frac{\kappa''}{\sqrt{2}\kappa} - \frac{\tau'}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}} - \frac{\sqrt{2}\kappa'\tau}{\kappa} + \frac{v\kappa'}{\kappa} - \frac{\tau^2}{\sqrt{2}}\right)\mathbf{N} \\
&+ \left(\frac{\sqrt{2}\kappa'\tau}{\kappa} - \frac{\tau^2}{\sqrt{2}} + v\tau + \frac{\kappa''}{\sqrt{2}\kappa} + \frac{\tau'}{\sqrt{2}}\right)\mathbf{B}.
\end{aligned} \tag{3.27}
$$

$$
{}^3\boldsymbol{\zeta}_v = \mathbf{T} \tag{3.28}
$$

$$
{}^3\boldsymbol{\zeta}_{sv} = \mathbf{N} \tag{3.29}
$$

$$
{}^3\boldsymbol{\zeta}_{vv} = 0 \tag{3.30}
$$

Unit normal vector on surface can be calculated using (2.4), (3.26) and (3.28)

$$
\mathbf{U} = \frac{1}{\varphi_3}\left[\left(\frac{\kappa'}{\sqrt{2}\kappa} + \frac{\tau}{\sqrt{2}}\right)\mathbf{N} - \left(\frac{\kappa'}{\sqrt{2}\kappa} - \frac{\tau}{\sqrt{2}} + v\right)\mathbf{B}\right] \tag{3.31}
$$

Using (2.6), (3.26) and (3.28), we have

$$
\begin{aligned}
E &= \frac{\kappa^4}{2} + (\kappa')^2 + \kappa^2\tau^2 + (v\kappa)^2 + \sqrt{2}v\kappa\kappa' - \sqrt{2}v\kappa^2\tau \\
F &= \frac{-\kappa^2}{\sqrt{2}} \\
G &= 1.
\end{aligned} \tag{3.32}
$$

Using (2.6), (3.27), (3.29) and (3.30), yields

$$
\begin{aligned}
L =& \frac{1}{\varphi_3}\left(\frac{-\kappa^3\kappa'}{2} - \kappa\kappa'\tau' - 2(\kappa')^2\tau + \frac{v(\kappa')^2}{\sqrt{2}} - \frac{\kappa^4\tau}{2} + \kappa\kappa''\tau\right. \\
&- \kappa^2\tau^3 + \sqrt{2}v(\kappa\tau)^2 - \sqrt{2}v\kappa\kappa'\tau - v^2\kappa^2\tau \\
&\left. - \frac{v\kappa\kappa'}{\sqrt{2}} - \frac{v\kappa^2\tau'}{\sqrt{2}}\right), \\
M =& \frac{1}{\sqrt{2}\varphi_3}\left(\kappa\kappa' + \kappa^2\tau\right), \\
N =& 0.
\end{aligned} \tag{3.33}
$$

Using (2.7), (3.32) and (3.33), we get

$$
\begin{aligned}
\kappa_G &= -\frac{1}{2\varphi_3^2}\left[\frac{(\kappa\kappa' + \kappa^2\tau)^2}{\kappa^2\tau^2 + (\kappa')^2 + \sqrt{2}v\kappa\kappa' - \sqrt{2}v\kappa^2\tau}\right] \\
\kappa_H &= \frac{1}{2\varphi_3}\left(\frac{\kappa\kappa'\sigma_1 + (\kappa')^2\sigma_2 + \kappa\tau\sigma_3 + \kappa^2\sigma_4}{\kappa^2\tau^2 + (\kappa')^2 + \sqrt{2}v\kappa\kappa' - \sqrt{2}v\kappa^2\tau}\right)
\end{aligned}
$$

${}^3\boldsymbol{\zeta}(s, v)$ is developable when κ_G vanishes, this implies

$$
\kappa\kappa' + \kappa^2\tau = 0.
$$

${}^3\boldsymbol{\zeta}(s, v)$ is minimal when κ_H vanishes, this implies

$$
\kappa\kappa'\sigma_1 + (\kappa')^2\sigma_2 + \kappa\tau\sigma_3 + \kappa^2\sigma_4 = 0.
$$

□

Special case: For a plane curve with constant curvature ${}^3\boldsymbol{\zeta}(s,v)$ is always a developable surface. Since

$$\kappa\kappa' + \kappa^2\tau = 0$$

at $\tau = 0$, we get,

$$\kappa\kappa' = 0 \qquad \Rightarrow \kappa = \text{constant}$$

Corollary 3.6 *1. $\boldsymbol{\alpha}$ is an asymptotic line if $\kappa\kappa'\eta_1 + \kappa''\eta_2 + \kappa^2\eta_3 = 0$.*

where

$$\eta_1 = 2\sqrt{2}v\tau - 2\tau^2(2+\sqrt{2}) - \kappa^2, \qquad \eta_2 = 2\kappa' + \sqrt{2}v\kappa,$$

$$\eta_3 = \tau'(\sqrt{2}v - 2) - \tau(\kappa^2 + \sqrt{2}v\tau).$$

2. $\boldsymbol{\alpha}$ is a geodesic line if $\kappa'\omega_1 + \kappa^2\tau^2\omega_2 + \kappa\kappa'\omega_3 + \kappa^3\omega_4 = 0$,

where

$$\omega_1 = \sqrt{2}\kappa'\tau - \frac{3\sqrt{2}(\kappa')^2}{\kappa} - 3v\kappa' - \sqrt{2}\kappa^2\kappa'',$$

$$\omega_2 = \sqrt{2}\tau + \sqrt{2}\kappa\kappa' - v\kappa^2 - 1,$$

$$\omega_3 = v\tau - 3\sqrt{2}\tau^2 + 3\tau - \frac{\kappa^4}{\sqrt{2}} - 2v\kappa^2\tau,$$

$$\omega_4 = \frac{\kappa^3\tau}{\sqrt{2}} - v\kappa^3 + \sqrt{2}\kappa\tau\tau' - v\kappa\tau' + v\kappa\kappa''.$$

Proof: Differentiating (2.12) with respect to s, we get

$$\boldsymbol{\alpha}' = \frac{1}{\sqrt{2}}\left[-\kappa^2\mathbf{T} + \left(\frac{\kappa'}{\kappa} - \tau\right)\mathbf{N} + \left(\frac{\kappa'}{\kappa} + \tau\right)\mathbf{B}\right] \tag{3.34}$$

On differentiating (3.34), we have

$$\boldsymbol{\alpha}'' = \frac{1}{\sqrt{2}}\left[(\kappa^2\tau - 3\kappa\kappa')\mathbf{T} + \left(-\kappa^2 - \tau' - \frac{2\kappa'\tau}{\kappa} - \tau^2 + \frac{\kappa''}{\kappa}\right)\mathbf{N} + \left(\frac{2\kappa'\tau}{\kappa} + \tau' - \tau^2 + \frac{\kappa''}{\kappa}\right)\mathbf{B}\right] \tag{3.35}$$

Using (2.8), (3.31) and (3.35), the expression for normal curvature κ_n is given by:

$$\kappa_n = \frac{1}{2\ \phi_3}\left(\kappa\kappa'\eta_1 + \kappa''\eta_2 + \kappa^2\eta_3\right)$$

for $\kappa_n = 0$, $\boldsymbol{\alpha}$ is an asymptotic line, this implies

$$\kappa\kappa'\eta_1 + \kappa''\eta_2 + \kappa^2\eta_3 = 0.$$

Now, we compute $\mathbf{U} \wedge \boldsymbol{\alpha}'$, using (3.31) and (3.34)

$$\mathbf{U} \wedge \boldsymbol{\alpha}' = \frac{1}{\sqrt{2}\varphi_3}\Bigg[\left(\sqrt{2}\left(\frac{\kappa'}{\kappa}\right)^2 + \sqrt{2}\tau^2 + \frac{v\kappa'}{\kappa} - v\tau\right)\mathbf{T} + \left(\frac{\kappa\kappa'}{\sqrt{2}} - \frac{\kappa^2\tau}{\sqrt{2}} + v\kappa^2\right)\mathbf{N} + \left(\frac{\kappa\kappa'}{\sqrt{2}} + \frac{\kappa^2\tau}{\sqrt{2}}\right)\mathbf{B}\Bigg] \tag{3.36}$$

Using (2.8), (3.35) and (3.36), the expression for geodesic curvature κ_g is given by:

$$\kappa_g = \kappa'\omega_1 + \kappa^2\tau^2\omega_2 + \kappa\kappa'\omega_3 + \kappa^3\omega_4$$

for $\kappa_g = 0$, $\boldsymbol{\alpha}$ is a geodesic line, this implies

$$\kappa'\omega_1 + \kappa^2\tau^2\omega_2 + \kappa\kappa'\omega_3 + \kappa^3\omega_4 = 0.$$

□

Theorem 3.6 *Striction curve for* ${}^3\boldsymbol{\zeta}(s, v)$ *is given by:*

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left(\left(\tau - \frac{\kappa'}{\kappa}\right)\mathbf{T} + \mathbf{N} + \mathbf{B}\right)$$

Proof: Using (2.3), (2.9) and (3.25), we get

$$\langle \boldsymbol{\alpha}', \mathbf{w}' \rangle = \kappa\kappa' - \kappa^2\tau$$

also,

$$\|\mathbf{w}'\|^2 = \kappa^2$$

this implies,

$$\overline{\boldsymbol{\alpha}} = \frac{1}{\sqrt{2}}\left[\left(\tau - \frac{\kappa'}{\kappa}\right)\mathbf{T} + \mathbf{N} + \mathbf{B}\right]$$

□

4. Examples

Example 1. Consider a unit speed curve

$$\boldsymbol{\alpha}(s) = \left(\tan^{-1} s, \quad \frac{1}{\sqrt{2}}\log(s^2+1), \quad s - \tan^{-1} s\right)$$

Frenet apparatus for the curve $\boldsymbol{\alpha}(s)$ is given

$$\boldsymbol{t} = \frac{1}{s^2+1}\left(1, \quad \sqrt{2}s, \quad s^2\right)$$

$$\boldsymbol{n} = \frac{1}{s^2+1}\left(-\sqrt{2}s, \quad 1-s^2, \quad \sqrt{2}s\right)$$

$$\boldsymbol{b} = \frac{1}{s^2+1}\left(s^2, \quad -\sqrt{2}s, \quad 1\right)$$

also

$$\kappa = \frac{\sqrt{2}}{s^2+1} \qquad \text{and} \qquad \tau = \frac{\sqrt{2}}{s^2+1}$$

Modified orthogonal frame for the curve $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{T} = \frac{1}{s^2+1}\left(1, \quad \sqrt{2}s, \quad s^2\right)$$

$$\boldsymbol{N} = \frac{1}{(s^2+1)^2}\left(-2s, \quad \sqrt{2}(1-s^2), \quad 2s\right)$$

$$\boldsymbol{B} = \frac{1}{(s^2+1)^2}\left(\sqrt{2}s^2, \quad -2s, \quad \sqrt{2}\right)$$

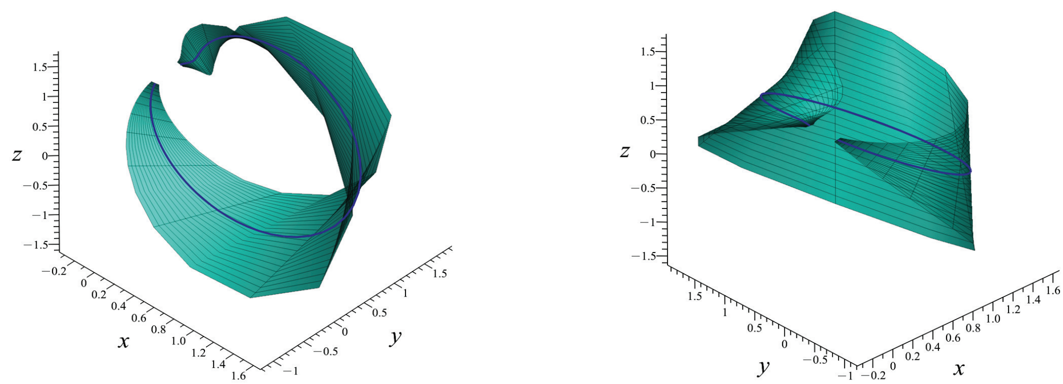
Smarandache ruled surfaces according to modified orthogonal frame are given by

$$^{1}\boldsymbol{\zeta}(s,v)=\frac{1}{\sqrt{2}(s^2+1)^2}\left((s-1)^2+2vs^2,\ \sqrt{2}(s^3-s^2+s+1)-2\sqrt{2}vs,\ s^4+s^2+2s+\sqrt{2}v\right)$$

$$^{2}\boldsymbol{\zeta}(s,v)=\frac{1}{\sqrt{2}(s^2+1)^2}\left(1+s^2(1+\sqrt{2})-2\sqrt{2}vs, \sqrt{2}s^3+s(\sqrt{2}-2)+2v(1-s^2), \sqrt{2}+s^2(s^2+1)+2\sqrt{2}vs\right)$$

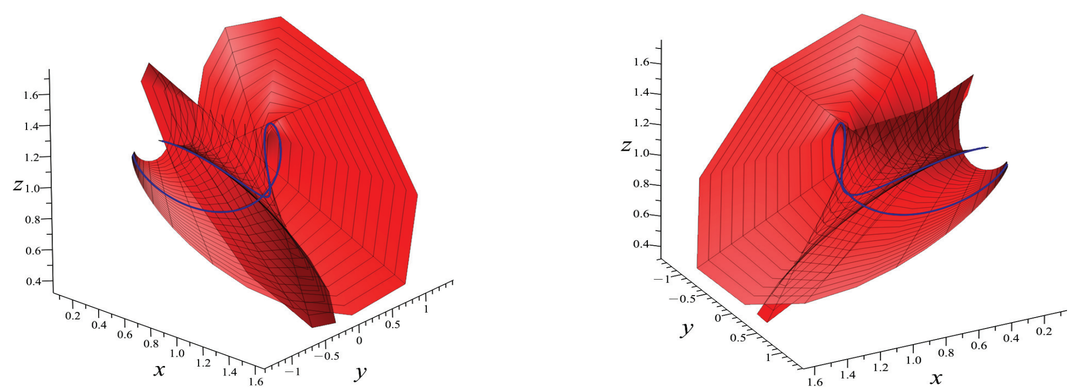
$$^{3}\boldsymbol{\zeta}(s,v)=\frac{1}{\sqrt{2}(s^2+1)^2}\left(\sqrt{2}s^2-2s+\sqrt{2}v(s^2+1),\ \sqrt{2}(1-s^2)-2s+2vs(s^2+1),\ 2s+\sqrt{2}+\sqrt{2}vs^2(s^2+1)\right).$$

For $s\in[-2\pi,2\pi]$ and $v\in[-1,1]$, ruled surfaces are plotted below, with the base curve coloured blue.



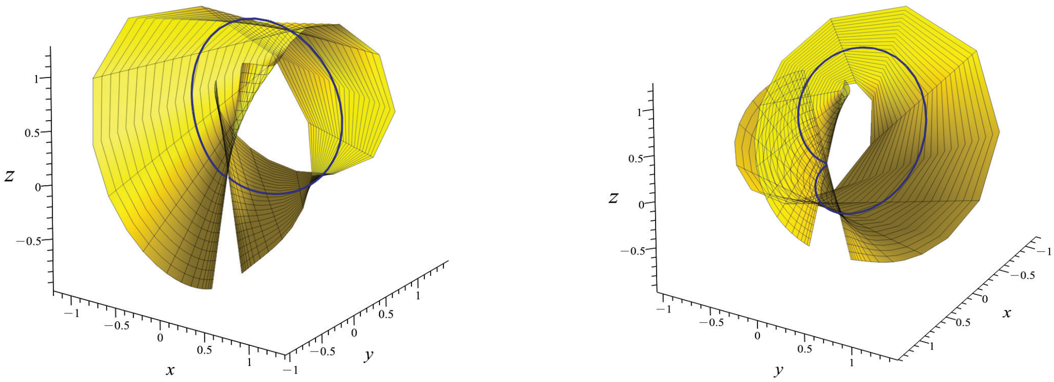
1 (a) 1 (b)

Figure 1: **TN**-Smarandache ruled surfaces with different orientations.



2 (a) 2 (b)

Figure 2: **TB**-Smarandache ruled surfaces with different orientations.



3 (a) 3 (b)

Figure 3: **NB**-Smarandache ruled surfaces with different orientations.

Example 2 Given a unit speed curve

$$\boldsymbol{\beta}(s) = \left(\left(1 + \frac{s^2}{2}\right)^{1/2}, \quad \frac{s}{\sqrt{2}}, \quad \ln\left(\frac{s}{\sqrt{2}} + \left(1 + \frac{s^2}{2}\right)^{1/2} \right) \right)$$

Frenet apparatus for the curve $\boldsymbol{\beta}$ is given by

$$\mathbf{t} = \frac{1}{(s^2+2)^{1/2}} \left(\frac{s}{\sqrt{2}}, \quad \frac{(s^2+2)^{1/2}}{\sqrt{2}}, \quad 1 \right)$$

$$\mathbf{n} = \frac{1}{(s^2+2)^{1/2}} \left(\sqrt{2}, \quad 0, \quad -s \right)$$

$$\mathbf{b} = \frac{1}{(s^2+2)^{1/2}} \left(-\frac{s}{\sqrt{2}}, \quad \frac{(s^2+2)^{1/2}}{\sqrt{2}}, \quad -1 \right)$$

and

$$\kappa = \frac{1}{s^2+2} \quad \text{and} \quad \tau(s) = \frac{1}{s^2+2}$$

Modified orthogonal frame of $\boldsymbol{\beta}$ is given by

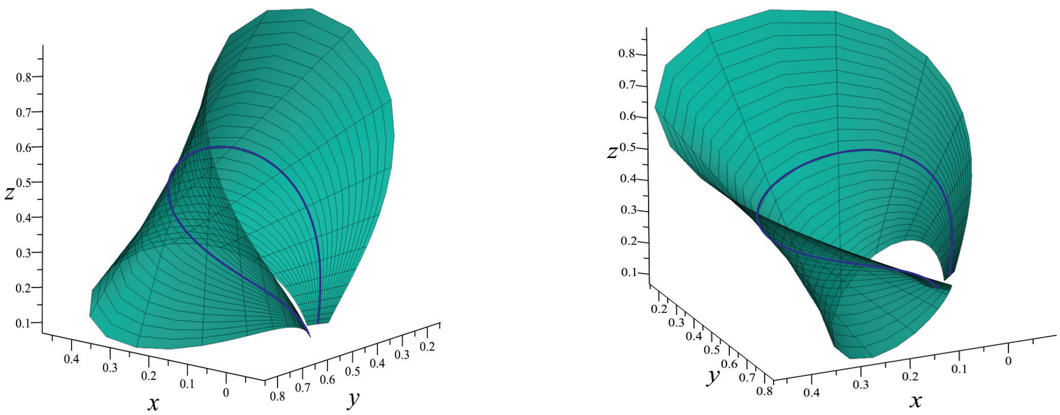
$$\boldsymbol{T} = \frac{1}{(s^2+2)^{1/2}} \left(\frac{s}{\sqrt{2}}, \quad \frac{(s^2+2)^{1/2}}{\sqrt{2}}, \quad 1 \right)$$

$$\boldsymbol{N} = \frac{1}{(s^2+2)^{3/2}} \left(\sqrt{2}, \quad 0, \quad -s \right)$$

$$\boldsymbol{B} = \frac{1}{(s^2+2)^{3/2}} \left(-\frac{s}{\sqrt{2}}, \quad \frac{(s^2+2)^{1/2}}{\sqrt{2}}, \quad -1 \right)$$

Smarandache ruled surfaces according to modified orthogonal frame are given by

$${}^{1}\boldsymbol{\zeta}(s,v) = \frac{1}{2(s^2+2)^{3/2}} \left(s(s^2+2) + 2 - \sqrt{2}vs, \; (s^2+2)^{3/2} + \sqrt{2}v(s^2+2)^{1/2}, \; \sqrt{2}(s^2-s+2) - 2v \right)$$



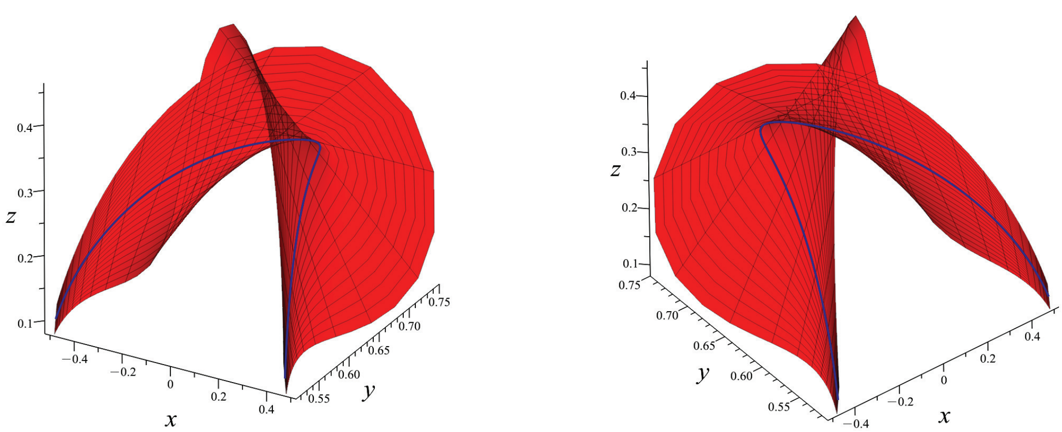
4 (a) 4 (b)

Figure 4: **TN**-Smarandache ruled surfaces with different orientations.

The graphical representation of these ruled surfaces is given for $s \in [-2\pi, 2\pi]$ and $v \in [-1, 1]$.

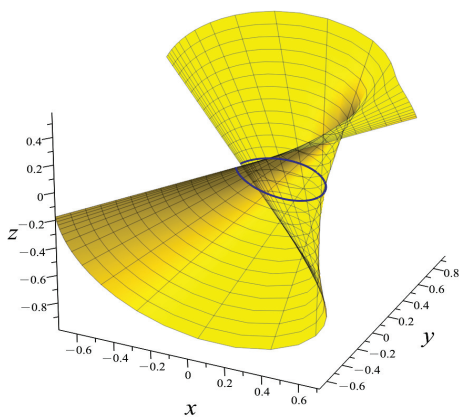
$$^{2}\boldsymbol{\zeta}(s,v) = \frac{1}{2(s^2+2)^{3/2}} \left(s^3 + s + 2v,\ \sqrt{2}(s^2+3)(s^2+2)^{1/2},\ \sqrt{2}(s^2+1-vs) \right)$$

$$^{3}\boldsymbol{\zeta}(s,v) = \frac{1}{2(s^2+2)^{3/2}} \left(2 - s + vs(s^2+2), (s^2+2)^{1/2} + v(s^2+2)^{3/2}, \sqrt{2}[-(s+1) + v(s^2+2)] \right).$$

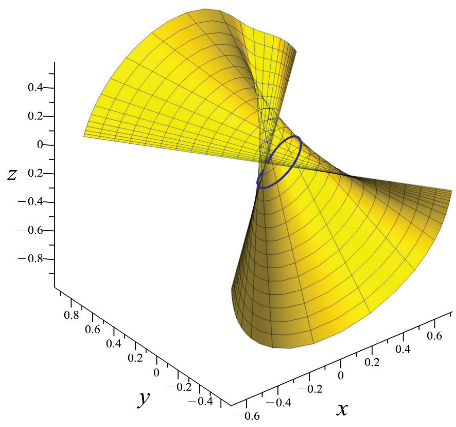


5 (a) 5 (b)

Figure 5: **TB**-Smarandache ruled surfaces with different orientations.



6 (a)



6 (b)

Figure 6: **NB**-Smarandache ruled surfaces with different orientations.

5. Conclusions

In this paper, we introduce new type of ruled surfaces using the modified orthogonal frame i.e. **TN**, **TB** and **NB** Smarandache ruled surfaces. These surfaces are generated by the linear combination of orthogonal frame vectors. We examine I and II fundamental forms of these surfaces, along with their Gaussian and mean curvatures. Also, we derive the conditions under which these surfaces are developable or minimal. Furthermore, we investigate the normal curvature and geodesic curvature corresponding to the base curve for each surface type. We study the conditions for the base curve to be a geodesic or an asymptotic line on each kind of surface. A few examples are included to illustrate the theoretical findings. Further studies could investigate the singularities of these surfaces. This study broadens the opportunities for applications of ruled surfaces in surface modeling, architectural design, and engineering innovations.

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