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A New Multi-Criteria Decision-Making Model for River Pollution Assessment Using Hesitant Fuzzy Soft Multisets: A Case Study of the Haora River, Tripura (Northeast India)

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ABSTRACT: This study introduces a new way to assess river water pollution using a method called the hesitant fuzzy soft multiset (H$\mathfrak{F}$SMS) model. It introduces new concepts such as the LEVEL-H$\mathfrak{F}$SMS, root mean square operator (RMSO), and root mean square matrix (RMSM) to build a reliable multi-criteria decision-making (MCDM) model. To show how this method works, we applied it to evaluate the water pollution levels in the Haora River in Tripura, India. The Haora River is very important for the people living in Tripura. It supplies drinking water to Agartala city and supports fishing, farming, and religious activities. Many people depend on the river for their daily needs. However, pollution from household waste and farming is harming the river. We used 14 important water quality parameters ($\mathcal{WQP}$s), including pH, Electrical conductivity, Total suspended solid, Dissolved oxygen, Biochemical oxygen demand, Chemical oxygen demand, Total hardness, Total alkalinity, Total dissolved solid, Calcium, Magnesium, Chloride, Total Coliform and Faecal coliform, tested at seven different locations during three seasons between April 2022 and March 2023. Our model uses a scoring method called the λ-LEVEL-score to assess the overall water quality. The results showed that this method is effective in dealing with uncertain data and helps in making better decisions about managing river pollution. Overall, the study shows that the H$\mathfrak{F}$SMS-based model is a helpful tool for evaluating water quality in complex real-life situations.

Key Words: Fuzzy logic, soft computing, group decision-making, multi-criteria decision-making, water quality assessment, environmental pollution, hesitant fuzzy set, river water pollution.

Contents

1. Introduction

Rivers are lifelines for human society and ecosystems. They provide water for drinking, agriculture, industry, transportation, and support aquatic life. However, rising pollution from human activities such as untreated domestic wastewater, industrial discharge, and agricultural runoff has severely degraded

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river water quality around the world [43, 58]. In India, this is a growing concern, particularly for rivers in urban and semi-urban regions where monitoring systems are often lacking [13, 18, 20].

The Haora River, one of the most important rivers in Tripura, Northeast India, is one such river under stress. Originating from the confluence of the Raima and Sarma rivers, it plays a vital role in the drinking water supply, fisheries, irrigation, and religious rituals. Despite its importance, increasing pollution—mainly from household sewage and agricultural activities—is posing serious threats to its ecological balance and public health [13, 18].

Evaluating river water quality is a complex task, as it involves analyzing multiple chemical, physical, and biological $\mathcal{WQP}$s over time. Traditional water quality indices ($\mathcal{WQI}$s) such as Brown's $\mathcal{WQI}$ [7], the CCME $\mathcal{WQI}$ [9], and national standards like BIS [8] and ICMR [35] have been widely used. However, these models often fall short in addressing uncertainty, vagueness, and subjectivity in water quality data.

Fuzzy logic [66] is one such tool that has gained attention for handling uncertainty and imprecision. It has been successfully applied in various river systems to better understand pollution levels. For example, Trach et al. [59] used fuzzy logic combined with artificial neural networks to predict water quality with improved precision. Similarly, Pal et al. [53] developed a fuzzy-based method that more accurately categorized water pollution levels, especially in complex environments. Recent advancements in fuzzy sets ($\mathfrak{F}$Ss) [66], hesitant fuzzy sets (H$\mathfrak{F}$Ss) [62], fuzzy soft sets ($\mathfrak{F}$SSs) [42], Weighted H$\mathfrak{F}$SS [64], and hybrid MCDM methods [3, 14, 17] provide better tools to handle the uncertainty in environmental data. These methods allow for flexible modeling of complex relationships and have shown promising results in water quality assessment [1, 31, 55].

Research on fuzzy MCDM has further evolved with the introduction of advanced models like intuitionistic $\mathfrak{F}$SSs and H$\mathfrak{F}$Ss. Das and Granados [15, 16] developed new mathematical frameworks, such as the IFP-hesitant N-soft set, to enhance group MCDM when expert opinions vary or include hesitation. These models allow better handling of conflicting information, which is common in water quality assessments. Several studies have applied fuzzy and soft set techniques to improve water quality prediction and classification. For example, Ahmed et al. [2] used machine learning methods to forecast water quality, while Chauhan & Trivedi [11], Rahi et al. [57], and Patil et al. [56] applied fuzzy-based $\mathcal{WQI}$ models to assess surface and river water. Huo et al. [34] proposed a genetic algorithm-optimized fuzzy model, and Kumaravel & Priya [41] combined AHP with PROMETHEE-II to handle complex MCDM in water resource management.

Moreover, recent research has highlighted the benefits of integrating fuzzy logic with soft computing, artificial intelligence, and geographic information systems for real-time and predictive water quality monitoring [10, 37, 39, 40]. Fuzzy matter-element methods [61] and hybrid neuro-fuzzy models [54] also offer enhanced modeling of nonlinear and uncertain water quality data. Over the years, many researchers have explored fuzzy sets, soft sets, and their extensions to handle uncertainty and improve decision-making models. Das introduced the idea of intuitionistic fuzzy rough relations to better capture uncertainty in mathematical relations [21]. Later, the concept of weighted fuzzy soft multisets was developed to deal with multiple criteria in decision-making problems [22]. Building on this, the FP-intuitionistic multi-fuzzy N-soft set and its hesitant version were proposed to address situations where experts hesitate while giving opinions [23]. A preference intuitionistic fuzzy rough relation was also introduced as a theoretical tool to improve comparisons among alternatives [24]. In addition, a weighted hesitant bipolar-valued fuzzy soft set was designed to handle cases where both positive and negative aspects need to be considered simultaneously [25]. Beyond these, researchers also focused on neutrosophic extensions. For example, a study explored neutrosophic SuperHyper BCI-semigroups and their algebraic significance [26]. Another work presented a weighted hypersoft expert system for group decision-making [27]. Practical applications were also demonstrated, such as using fuzzy soft multi-criteria models for urban river water quality assessment with a focus on human-induced pollution [28]. Granados and Das contributed to neutrosophic modeling by proposing continuous neutrosophic distributions with uncertain parameters [32], and later introduced a weighted neutrosophic soft multiset model for decision-making [33]. Mukherjee and Das made several important contributions. They applied interval-valued intuitionistic fuzzy soft sets in investment decision-making [45, 46], developed relations on intuitionistic fuzzy soft multisets [47], and studied their topological structures [49]. They also applied fuzzy soft multisets to decision-making problems [48] and later extended the approach using Einstein operations [50]. Their recent book provided a comprehensive treatment of fuzzy soft multisets and their applications [51]. Finally, more advanced frameworks have been developed. A generalized interval-valued neutrosophic rough soft set model was introduced for water quality assessment, offering an effective way to deal with uncertainty in environmental decision-making [52].

The need to identify the sources of water pollution has also led to regional and global studies. Diamantini et al. [29] studied three large European river basins and found that human activities, especially land use and agriculture, were the main drivers of water quality changes. In a global context, Dilipkumar and Shanmugam [30] proposed a fuzzy logic-based model using satellite data to map and monitor polluted zones, called "water quality cells," from space. In India, Shah and Joshi [60] applied a traditional $\mathcal{WQI}$ model to the Sabarmati River and highlighted the impact of urban development on water pollution. More recently, Das [13] proposed an integrated model for the Mahanadi River Basin using fuzzy logic, machine learning (random forest), and MCDM tools like AHP-TOPSIS and MOORA to evaluate contamination and prioritize corrective actions. Another notable contribution is from Al Jawei et al. [4] and WHO [65] developed a fuzzy-based water quality model for the Al-Gharraf River in Iraq. Their work demonstrated how fuzzy models can be tailored to different river systems and used across countries to make more informed water management decisions.

Despite these advances, most models still do not account for hesitation in expert opinions or the variability of $\mathcal{WQP}$ weights. To address this gap, this study proposes an H$\mathfrak{F}$SMS that combines fuzzy logic, H$\mathfrak{F}$Ss, and a weighted function. The H$\mathfrak{F}$SMS offers a more reliable and flexible tool for water quality evaluation, especially under uncertain and imprecise conditions [6, 19, 20].

This research focuses on the Haora River and develops a λ-LEVEL-score incorporating the important $\mathcal{WQP}$s like dissolved oxygen, biochemical oxygen demand, total coliform, and electrical conductivity [1,3]. Water quality is evaluated across six strategically selected sites on the river using data collected between April 2022 and March 2023. Data collection and analysis were conducted using standardized methods, as endorsed by the APHA [5] and WHO [65]. By comparing it with existing methods [7, 9, 55], this study evaluates the effectiveness of the proposed H$\mathfrak{F}$SMS framework in evaluating water quality and its impacts on fisheries and the aquatic ecosystem. In addition, the results are also compared with the existing guidelines for water quality management as per ICMR [35] and BIS [8] to make them reliable and applicable. This research addresses the urgent need for sustainable water resource management in polluted areas by proposing a new approach to water quality assessment. This study intends to enhance current methods and highlight the need for informed MCDM for ecological restoration and resource optimization. The research findings aim to augment the resilience of aquatic ecosystems and livelihoods that depend on the Haora River by providing insights to mitigate the negative consequences of declining water quality.

This paper is organized as follows: Section 2 explains the basic concepts like $\mathfrak{F}$Ss, soft sets, and their advanced forms, including hesitant fuzzy and multiset models. Section 3 introduces key ideas like the λ-LEVEL-H$\mathfrak{F}$SMS, RMSO, and RMSM. These are used to build a flexible MCDM model. Section 4 applies this model to assess water pollution in the Haora River. Section 5 compares our method with existing water quality assessment approaches to show its advantages. Section 6 provides a comprehensive conclusion, summarizing key findings and suggestions for future research.

2. Preliminaries

In this present section, we establish the groundwork for comprehending H$\mathfrak{F}$SMS theory. We outline basic concepts and discoveries that serve as the foundation for the creation of our suggested framework, including soft sets, $\mathfrak{F}$Ss, H$\mathfrak{F}$Ss, $\mathfrak{F}$SSs, H$\mathfrak{F}$SSs, multisets, and H$\mathfrak{F}$SMSs.

Assume that $\mathbb{E}(\mathbb{S})$ gives the power set of $\mathbb{S}$, and that $\mathbb{S}$ represents the initial universe. Assume that $\mathbb{E}$ represents a nonempty collection of parameters.

Definition 2.1 *[44]: A soft set (SS) defined over a universe $\mathbb{S}$ is represented by a pair $(\psi, \mathbb{E})$, where ψ is a mapping such that $\psi : \mathbb{E} \to \mathbb{E}(\mathbb{S})$. More precisely, in this case, ψ associates each object of the parameter set $\mathbb{E}$ with a subset of $\mathbb{S}$, which provides a rich framework to deal with the uncertainty or imprecision of*

	List of Abbreviations
$\mathfrak{F}$S	Fuzzy set
SS	Soft set
$\mathfrak{F}$SS	Fuzzy soft set
H$\mathfrak{F}$S	Hesitant fuzzy set
H$\mathfrak{F}$SS	Hesitant fuzzy soft set
H$\mathfrak{F}$SMS	Hesitant fuzzy soft multiset
MCDM	Multi-criteria decision-making
RMSM	Root mean square matrix
RMSO	Root mean square operator
$\mathcal{WQI}$s	Water quality indices
$\mathcal{WQP}$s	Water quality parameters

each parameter. By providing this mapping, the resulting modeling relationships between parameters and the universe become more nuanced and flexible, while also being more complex in the sense that vagueness can be considered in different decision-making or evaluation contexts.

Example 2.1 *Soft Set in Water Pollution:*

A soft set is a way to connect certain $\mathcal{WQP}$s (like water quality indicators) with the specific locations (such as river sites) where they appear. It helps us deal with uncertainty in environmental data by showing which places are affected by which pollution factors.

Let us consider the Haora River, where we want to study water pollution in different parts. Let the set of locations be: $\mathbb{S} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, where each element ζ_i $(i = 1, 2, 3, 4)$ represents a sampling station along the river.

Now, we consider some water quality parameters ($\mathcal{WQP}s$):

$$\mathbb{E} = \{\mathfrak{e}_1 = \textit{high BOD},\ \mathfrak{e}_2 = \textit{high Total Coliform},\ \mathfrak{e}_3 = \textit{low DO},\ \mathfrak{e}_4 = \textit{high TDS}\}.$$

These are the indicators we use to judge the pollution level.

We now define a soft set $(\psi, \mathbb{E})$ to show which stations are affected by which $\mathcal{WQP}$:

- $\psi(\mathfrak{e}_1) = \{\zeta_1, \zeta_4\}$ $\rightarrow$ *Stations ζ_1 and ζ_4 show high BOD levels.*
- $\psi(\mathfrak{e}_2) = \{\zeta_2, \zeta_4\}$ $\rightarrow$ *Stations ζ_2 and ζ_4 have high Total Coliform.*
- $\psi(\mathfrak{e}_3) = \{\zeta_4\}$ $\rightarrow$ *Only station ζ_4 has low DO.*
- $\psi(\mathfrak{e}_4) = \{\zeta_2, \zeta_3\}$ $\rightarrow$ *Stations ζ_2 and ζ_3 have high TDS.*

Hence, the soft set $(\psi, \mathbb{E})$ can be expressed as:

$$(\psi, \mathbb{E}) = \{(\mathfrak{e}_1, \{\zeta_1, \zeta_4\}), (\mathfrak{e}_2, \{\zeta_2, \zeta_4\}), (\mathfrak{e}_3, \{\zeta_4\}), (\mathfrak{e}_4, \{\zeta_2, \zeta_3\})\}.$$

This soft set provides a clear picture of which stations are affected by which pollution problems, thus supporting better decision-making even when the data is uncertain or imprecise.

Definition 2.2 *[66]: A fuzzy set ($\mathfrak{F}S$) Y on $\mathbb{S}$ takes the form $Y = \{(o, \mu_Y(o)) : o \in \mathbb{S}\}$,where a membership function or fuzzy membership function is used to describe the mapping, and $\mu_Y(o)$ signifies the degree of membership or membership value for the object $o \in \mathbb{S}$.*

In this study, let $\mathfrak{F}S(\mathbb{S})$ represent the overall collection of all fuzzy sets on $\mathbb{S}$.

Definition 2.3 *[42]: A pair $(\psi, \mathbb{E})$ is called a fuzzy soft set ($\mathfrak{F}SS$) over $\mathbb{S}$, where $\psi : \mathbb{E} \rightarrow \mathfrak{F}S(\mathbb{S})$ is a mapping. That is, a $\mathfrak{F}SS$ is defined as a pair consisting of a mapping ψ and a subset $\mathbb{E}$ of parameters. Every object of the parameter set $\mathbb{E}$ is assigned to a fuzzy set over $\mathbb{S}$ by the mapping ψ.*

Example 2.2 *$\mathfrak{F}S$ & $\mathfrak{F}SS$ for Water Pollution:*

A $\mathfrak{F}S$ allows us to represent how much a certain item (like a river station) belongs to a group (like "highly polluted") using a membership value between 0 and 1. A value of 0.9 means high pollution, while 0.2 means low pollution.

A $\mathfrak{F}SS$ connects $\mathcal{WQP}$s, (like BOD, coliform, etc.) with fuzzy information from different locations. It helps us model pollution data when it is not exact or clear, but rather somewhat vague or in between.

Let us say we are analyzing water quality at four sampling stations $\mathbb{S} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, and we are evaluating them based on four $\mathcal{WQP}$s:

$$\mathbb{E} = \{\mathfrak{e}_1 = \textit{high BOD}, \mathfrak{e}_2 = \textit{high Total Coliform}, \mathfrak{e}_3 = \textit{low DO}, \mathfrak{e}_4 = \textit{high TDS}\}$$

Now, instead of saying whether a station is simply "polluted" or "not polluted" (0 or 1), we give fuzzy scores (values between 0 and 1) showing how much each station is affected.

So the $\mathfrak{F}SS$ $(\psi, \mathbb{E})$ would look like this:

- $\psi(\mathfrak{e}_1) = \{(\zeta_1, 0.7), (\zeta_2, 0.6), (\zeta_3, 0.3), (\zeta_4, 0.4)\}$ $\rightarrow$ *Indicates levels of high BOD.*
- $\psi(\mathfrak{e}_2) = \{(\zeta_1, 0.4), (\zeta_2, 0.6), (\zeta_3, 0.5), (\zeta_4, 0.4)\}$ $\rightarrow$ *Indicates total coliform presence.*
- $\psi(\mathfrak{e}_3) = \{(\zeta_1, 0.5), (\zeta_2, 0.4), (\zeta_3, 0.3), (\zeta_4, 0.7)\}$ $\rightarrow$ *Indicates low DO severity.*
- $\psi(\mathfrak{e}_4) = \{(\zeta_1, 0.7), (\zeta_2, 0.8), (\zeta_3, 0.3), (\zeta_4, 0.2)\}$ $\rightarrow$ *Indicates high TDS levels.*

So, the complete $\mathfrak{F}SS$ is:

$$(\psi, \mathbb{E}) = \Big\{(\mathfrak{e}_1, \{(\zeta_1, 0.7), (\zeta_2, 0.6), (\zeta_3, 0.3), (\zeta_4, 0.4)\}), (\mathfrak{e}_2, \{(\zeta_1, 0.4), (\zeta_2, 0.6), (\zeta_3, 0.5), (\zeta_4, 0.4)\}),$$
$$(\mathfrak{e}_3, \{(\zeta_1, 0.5), (\zeta_2, 0.4), (\zeta_3, 0.3), (\zeta_4, 0.7)\}), (\mathfrak{e}_4, \{(\zeta_1, 0.7), (\zeta_2, 0.8), (\zeta_3, 0.3), (\zeta_4, 0.2)\})\Big\}.$$

This helps water quality experts understand and compare pollution levels more clearly, even when the data is not black-and-white.

Definition 2.4 *[62]: A hesitant fuzzy set (H$\mathfrak{F}$S) defined on a set $\mathbb{S}$ is denoted by $Z = \{\langle o, h_Z(o)\rangle : o \in V\}$, and it is defined by the terms $h_Z(o)$ when applied to $\mathbb{S}$, where $h_Z(o)$ is a collection of various values in $[0,1]$, reflecting the possible membership degrees for each member $o \in V$, and $h_Z(o)$ is called a hesitant fuzzy element (H$\mathfrak{F}$E). These membership degrees, ranging between 0 and 1, signify the degree of hesitation or uncertainty regarding the membership status of each element.*

Assume that, $H\mathfrak{F}S(\mathbb{S})$ means the collection of all $H\mathfrak{F}S$s on $\mathbb{S}$, providing a comprehensive framework for managing uncertainty and ambiguity within the set.

Definition 2.5 *[63]: A pair $(\psi, \mathbb{E})$ is termed a hesitant fuzzy soft set (H$\mathfrak{F}$SS) over $\mathbb{S}$, where $\psi : \mathbb{E} \rightarrow H\mathfrak{F}S(\mathbb{S})$ is a mapping. That is, a H$\mathfrak{F}$SS is defined as a pair consisting of a mapping ψ and a subset $\mathbb{E}$ of parameters. The mapping ψ assigns each element of the parameter set $\mathbb{E}$ to a hesitant fuzzy set (H$\mathfrak{F}$S) over the universe $\mathbb{S}$.*

H$\mathfrak{F}$Ss allow for the representation of uncertainty or hesitation in decision-making processes, providing a more flexible framework for handling imprecise information compared to traditional $\mathfrak{F}S$s

Example 2.3 *H$\mathfrak{F}$SS in Water Pollution:*

A H$\mathfrak{F}$SS helps us model situations where experts are not certain and may give multiple possible values instead of just one. For example, instead of saying the pollution level at a station is 0.6, an expert might say it is between 0.5 and 0.7 because they are unsure. This kind of hesitation is common in environmental studies.

In a H$\mathfrak{F}$SS, each $\mathcal{WQP}$, (like high BOD or low DO) is connected to multiple fuzzy values at each sampling site, reflecting the hesitation or variation in expert opinions.

Let us again take four river sampling stations:

$$\mathbb{S} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$$

And four $\mathcal{WQP}$s:

$$\mathbb{E} = \{\mathfrak{e}_1 = \textit{high BOD},\ \mathfrak{e}_2 = \textit{high Total Coliform},\ \mathfrak{e}_3 = \textit{low DO},\ \mathfrak{e}_4 = \textit{high TDS}\}$$

Now, experts hesitate while rating the pollution levels, so they give a set of values (not just one) for each station and parameter.

So, the H$\mathfrak{F}$SS $(\psi, \mathbb{E})$ looks like this:

- $\psi(\mathfrak{e}_1) = \{(\zeta_1, \{0.7, 0.5, 0.4\}), (\zeta_2, \{0.7, 0.6, 0.5\}), (\zeta_3, \{0.5, 0.4, 0.3\}), (\zeta_4, \{0.4, 0.3, 0.2\})\}$
- $\psi(\mathfrak{e}_2) = \{(\zeta_1, \{0.6, 0.5, 0.4\}), (\zeta_2, \{0.6, 0.4, 0.3\}), (\zeta_3, \{0.6, 0.4, 0.3\}), (\zeta_4, \{0.5, 0.4, 0.2\})\}$
- $\psi(\mathfrak{e}_3) = \{(\zeta_1, \{0.7, 0.4, 0.3\}), (\zeta_2, \{0.7, 0.5, 0.3\}), (\zeta_3, \{0.5, 0.3, 0.2\}), (\zeta_4, \{0.6, 0.3, 0.2\})\}$
- $\psi(\mathfrak{e}_4) = \{(\zeta_1, \{0.7, 0.5, 0.4\}), (\zeta_2, \{0.8, 0.6, 0.4\}), (\zeta_3, \{0.7, 0.4, 0.3\}), (\zeta_4, \{0.4, 0.2, 0.1\})\}$

Each group of values shows how uncertain or hesitant the experts are when rating pollution. For instance, at ζ_2 under high BOD ($\mathfrak{e}_1$), the values $\{0.7, 0.6, 0.5\}$ mean the pollution is likely between 0.5 *and* 0.7*.*

This approach gives a more realistic picture of pollution because it respects expert hesitation and variation in judgment, something common in environmental studies.

Definition 2.6 *[36]: A count function m_Ω defined as $m_\Omega : \mathbb{S} \to \mathbb{N}$, where $\mathbb{N}$ represents the collection of non-negative integers, and is used to represent a multiset Ω taken from the set $\mathbb{S}$.*

The count of occurrences of the member o in the multiset Ω is denoted by $m_\Omega(o)$. The multiset Ω is derived from $\mathbb{S} = \{o_1, o_2, \ldots, o_n\}$, and it is written as $\Omega = \{m_1/o_1, m_2/o_2, \ldots, m_n/o_n\}$,where m_i is the count of occurrences of the member o_i, $i = 1, 2, 3, \ldots, n$ in the multiset Ω.

Definition 2.7 *[38]: Assume q is a positive integer. An H$\mathfrak{F}$SMS Z of dimension q on a multiset Ω is a set $Z = \{\langle m/o, h_Z(o)\rangle : o \in \Omega\}$, where $h_Z(o) = \{h_i : i = 1, 2, \ldots, q\}$ is a collection of multiple values in the range $[0, 1]$, reflecting the possible multi-membership degrees for each member $o \in \Omega$ and called a H$\mathfrak{F}$E . Here q is said to be the dimension of the H$\mathfrak{F}$SMS Z.*

Assume that, the collection of all H$\mathfrak{F}$SMSs of dimension q over Ω is denoted by $\mathrm{H\mathfrak{F}SMS}(\Omega)^q$ *and* $\mathrm{H\mathfrak{F}E}(\Omega)^q$ *represents the gathering of all the H$\mathfrak{F}$Es on Ω with a dimension of q.*

Definition 2.8 *[38]: A combination $(\psi, \mathbb{E})$ is defined as a hesitant fuzzy soft multiset (H$\mathfrak{F}$SMS) of dimension q across Ω, where ψ is a function derived from $\psi : \mathbb{E} \to H\mathfrak{F}SMS(\Omega)^q$, such that for all $\mathfrak{e} \in \mathbb{E}$,$\psi(\mathfrak{e}) = \left\{\langle m/o,\ h_{\psi(\mathfrak{e})}(o)\rangle : o \in \Omega\right\}$, where $h_{\psi(\mathfrak{e})}(o) = \{h_i : i = 1, 2, \ldots, q\}$ is a collection of multiple values in the range $[0, 1]$, reflecting the possible multi-membership degrees for each member $o \in \Omega$ are called hesitant fuzzy elements (H$\mathfrak{F}$E) in $(\psi, \mathbb{E})$. Here q is said to be the dimension of the H$\mathfrak{F}$SMS $(\psi, \mathbb{E})$.*

Example 2.4 *H$\mathfrak{F}$SMS in Water Pollution:*

An H$\mathfrak{F}$SMS is a model used when:

- *We have multiple samples from a place (like repeated water tests).*
- *Experts are not certain and give several possible pollution levels (between 0 and 1) instead of one fixed value.*
- *Each $\mathcal{WQP}$ (like BOD or pH) is matched to these hesitant values for each sample.*

The "dimension" (q) refers to how many values are listed for each item (e.g., 3 values = dimension 3). This helps handle both repetition (like multiple samples from the same site) and uncertainty in pollution levels.

Let's say we take several water samples from four stations in the Haora River, and name them based on how many times we sampled:

- $3/\zeta_1,\ 3/\zeta_2\ =$ *stations* ζ_1 *and* ζ_2 *sampled 3 times.*
- $2/\zeta_3,\ 2/\zeta_4\ =$ *stations* ζ_3 *and* ζ_4 *sampled 2 times.*

We are checking the river for four $\mathcal{WQP}$*s* $\mathfrak{e}_1$ = *high BOD,* $\mathfrak{e}_2$ = *high Total Coliform,* $\mathfrak{e}_3$ = *low DO,* $\mathfrak{e}_4$ = *high TDS*

Experts give three possible values for how polluted each sample is, reflecting hesitation. So, the H$\mathfrak{F}$*SMS* $(\psi, \mathbb{E})$ *looks like:*

- $\mathfrak{e}_1$: $(3/\zeta_1, \{0.2, 0.3, 0.4\}),\ (3/\zeta_2, \{0.4, 0.5, 0.7\}),\ (2/\zeta_3, \{0.5, 0.6, 0.7\}),\ (2/\zeta_4, \{0.3, 0.5, 0.7\})$
- $\mathfrak{e}_2$: $(3/\zeta_1, \{0.5, 0.7, 0.9\}),\ (3/\zeta_2, \{0.2, 0.5, 0.7\}),\ (2/\zeta_3, \{0.3, 0.4, 0.5\}),\ (2/\zeta_4, \{0.4, 0.5, 0.7\})$
- $\mathfrak{e}_3$: $(3/\zeta_1, \{0.4, 0.5, 0.7\}),\ (3/\zeta_2, \{0.5, 0.6, 0.8\}),\ (2/\zeta_3, \{0.2, 0.5, 0.8\}),\ (2/\zeta_4, \{0.3, 0.4, 0.6\})$
- $\mathfrak{e}_4$: $(3/\zeta_1, \{0.2, 0.5, 0.7\}),\ (3/\zeta_2, \{0.3, 0.4, 0.5\}),\ (2/\zeta_3, \{0.4, 0.5, 0.7\}),\ (2/\zeta_4, \{0.6, 0.7, 0.9\})$

Table 1 presents the H$\mathfrak{F}$*SMS* $(\psi, \mathbb{E})$ *in tabular form with three dimensions.*

Table 1: H$\mathfrak{F}$SMS $(\psi, \mathbb{E})$

Sample	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$
$3/\zeta_1$	$\{0.2, 0.3, 0.4\}$	$\{0.5, 0.7, 0.9\}$	$\{0.4, 0.5, 0.7\}$	$\{0.2, 0.5, 0.7\}$
$3/\zeta_2$	$\{0.4, 0.5, 0.7\}$	$\{0.2, 0.5, 0.7\}$	$\{0.5, 0.6, 0.8\}$	$\{0.3, 0.4, 0.5\}$
$2/\zeta_3$	$\{0.5, 0.6, 0.7\}$	$\{0.3, 0.4, 0.5\}$	$\{0.2, 0.5, 0.8\}$	$\{0.4, 0.5, 0.7\}$
$2/\zeta_4$	$\{0.3, 0.5, 0.7\}$	$\{0.4, 0.5, 0.7\}$	$\{0.3, 0.4, 0.6\}$	$\{0.6, 0.7, 0.9\}$

This model helps us include both repeated sampling and expert hesitation in our pollution assessments, making the data more realistic and flexible for decision-making.

3. λ-LEVEL H$\mathfrak{F}$SMS and its Applications in Water Pollution Evaluation

Using a H$\mathfrak{F}$SMS, we can improve the analysis by setting threshold values for each $\mathcal{WQP}$. These thresholds, called λ-values, help us focus only on the pollution levels that are equal to or higher than a certain level of concern.

Definition 3.1 *Let us fix* $(\psi_\alpha, \mathbb{E})$ *a H*$\mathfrak{F}$*SMS of dimension* q *over* Ω *and* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$*, a vector of thresholds with* $\lambda_k \in [0, 1]$ *associated with every parameter* $\mathfrak{e}_k \in \mathbb{E}$*, for* $k = 1, 2, \ldots, m$*. Then the* λ*-LEVEL-H*$\mathfrak{F}$*SMS is denoted by* $L(\lambda, \psi_\alpha) = \psi_\alpha^\lambda$*, and defined by, for* $\mathfrak{e}_k \in \mathbb{E}$*,* $\psi_\alpha^\lambda(\mathfrak{e}_k) = \{\langle o, h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(o)\rangle : o \in \Omega\}$*, where* $h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(o) = \{h_i \in h_{\psi_\alpha(\mathfrak{e}_k)}(o) : h_i \geq \alpha_k, i = 1, 2, \ldots, q\}$.

This new version is called the λ*-LEVEL-H*$\mathfrak{F}$ *and it filters out the lower values, keeping only the serious pollution readings.*

- *Each* λ*-value matches one* $\mathcal{WQP}$.
- *If a pollution score is less than the* λ*-threshold, we ignore it.*
- *If it's greater than or equal, we keep it in the result.*

Example 3.1 *Let us revisit the previous H*$\mathfrak{F}$*SMS for four river sampling stations:* $\Omega = \{3/\zeta_1, 3/\zeta_2, 2/\zeta_3, 2/\zeta_4\}$*,* $\mathcal{WQP}$*s Thresholds* (λ):

- $\mathfrak{e}_1$ *(BOD)* $\rightarrow \lambda = 0.4$
- $\mathfrak{e}_2$ *(Coliform)* $\rightarrow \lambda = 0.5$

- $\mathfrak{e}_3$ *(DO)* $\rightarrow \lambda = 0.6$
- $\mathfrak{e}_4$ *(TDS)* $\rightarrow \lambda = 0.5$

We apply these thresholds to the values in the H𝔉SMS (Table 1), and remove any value less than the λ-threshold for that $\mathcal{WQP}$.

Table 2: λ-LEVEL H𝔉SMS $-L(\lambda, \psi_\alpha) = \psi_\alpha^\lambda$ (Filtered Pollution Values)

Sampling Point	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$
$3/\zeta_1$	$\{0.4\}$	$\{0.5, 0.7, 0.9\}$	$\{0.7\}$	$\{0.5, 0.7\}$
$3/\zeta_2$	$\{0.4, 0.5, 0.7\}$	$\{0.5, 0.7\}$	$\{0.6, 0.8\}$	$\{0.5\}$
$2/\zeta_3$	$\{0.5, 0.6, 0.7\}$	$\{0.5\}$	$\{0.8\}$	$\{0.5, 0.7\}$
$2/\zeta_4$	$\{0.5, 0.7\}$	$\{0.5, 0.7\}$	$\{0.6\}$	$\{0.6, 0.7, 0.9\}$

Table 2 presents the λ-LEVEL-H𝔉SMS that only includes the pollution levels that are concerning, based on expert-defined thresholds (λ). It filters out the weaker signals and focuses on serious pollution indicators, making it easier to prioritize action for river cleanup and policy planning.

Definition 3.2 *Assume that the collection of all H𝔉SMSs of dimension q over Ω is denoted by H𝔉SMS$(\Omega)^q$ and H𝔉E$(\Omega)^q$ represents the gathering of all the H𝔉Es on Ω with a dimension of q. Then the RMSO on H𝔉E$(\Omega)^q$ is represented by Δ and described by Δ : H𝔉E$(\Omega)^q \rightarrow [0, 1]$ as $\forall h(w) \in$ H𝔉E$(\Omega)^q$,*

$$\Delta(h(w)) = \sqrt{\frac{1}{|h(w)|} \sum_{h_k \in h(w)} (h_k)^2},$$

where $\Delta(h(w))$ is called the root mean square value of the H𝔉E $h(w)$ for each $w \in \Omega$.

Definition 3.3 *The RMSM of λ-LEVEL-H𝔉SMS $L(\lambda, \psi_\alpha)$ is denoted by $L(\lambda, \Delta\psi_\alpha) = \Delta\psi_\alpha^\lambda$ and defined as, for all $\mathfrak{e}_k \in \mathbb{E}$,*

$$\Delta\psi_\alpha^\lambda(\mathfrak{e}_k) = \{\langle w, \Delta(h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(w))\rangle : w \in \Omega\},$$

where $\Delta(h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(w))$ is the root mean square value of the H𝔉E $h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(w)$ in the λ-LEVEL-H𝔉SMS $L(\lambda, \psi_\alpha)$.

3.1. RMSO and RMSM in Water Pollution

After filtering serious pollution values using the λ-H𝔉SMS, we can go one step further by summarizing these values using a mathematical method called the Root Mean Square (RMS).

3.1.1. RMSO.

- Suppose we have multiple pollution values for a sampling station and a $\mathcal{WQP}$ (like BOD at ζ_1).
- To summarize these values into one number, we calculate the RMS, which gives a kind of "average" that gives more weight to higher values.
- The formula is:

$$\Delta(h(w)) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} h_i^2},$$

  where $h(w)$ is the set of pollution values for a specific station and $\mathcal{WQP}$s.

3.1.2. RMSM.

- We calculate the RMS for every entry in the λ-LEVEL-H$\mathfrak{F}$SMS table (every cell).
- The result is a new table where each fuzzy pollution score set is replaced by its RMS value.
- This table is called the RMSM, which gives a single score per station per $\mathcal{WQP}$, making it easier to compare and rank.

Example 3.2 *Let us apply this to the λ-LEVEL-H$\mathfrak{F}$SMS shown in Table 2. Then, we obtain the RMSM $L(\lambda, \Delta\psi_\alpha) = \Delta\psi_\alpha^\lambda$ of $L(\lambda, \psi_\alpha)$ as shown in Table 3.*

Table 3: The RMSM $L(\lambda, \Delta\psi_\alpha)$ for $L(\lambda, \psi_\alpha)$

Sampling Point	$\mathfrak{e}_1(0.4)$	$\mathfrak{e}_2(0.5)$	$\mathfrak{e}_3(0.6)$	$\mathfrak{e}_4(0.7)$
$3/\zeta_1$	0.4000	0.5167	0.7000	0.4967
$3/\zeta_2$	0.5477	0.4967	0.5774	0.5000
$2/\zeta_3$	0.6055	0.5000	0.8000	0.4967
$2/\zeta_4$	0.4967	0.4967	0.6000	0.7439

Remark 3.1
- *These* RMS *values give a clear and fair summary of pollution levels at each site.*
- *Higher* RMS *means worse pollution under that $\mathcal{WQP}$.*
- *For example, $2/\zeta_4$ has the highest* RMS *(0.7439) under TDS, showing it is heavily polluted with dissolved solids.*

This method helps decision-makers compare pollution levels more easily and target the most critical parameters for cleanup or intervention.

We now explain how our method works to solve decision-making problems using the H$\mathfrak{F}$SMS model. This approach is specially designed to handle uncertainty and hesitation that come from real-world data, like $\mathcal{WQP}$s. By using tools like $\mathfrak{F}$SSs, hesitant fuzzy multisets, and weighted H$\mathfrak{F}$SMSs, we can make more accurate and flexible decisions.

3.2. Algorithm

Step 1. Enter a non-empty set $\mathbb{S} = \{o_1, o_2, o_3, \ldots, o_n\}$, a multiset as

$\Omega = \{m_1/o_1, m_2/o_2, \ldots, m_n/o_n\}$, a $\mathcal{WQP}$ set $\mathbb{E} = \{\mathfrak{e}_k : k = 1, 2, \ldots, m\}$, and a group of decision-makers $\{DM_1, DM_2, \ldots, DM_q\}$.

Step 2. Enter the H$\mathfrak{F}$SMS $\{(\psi_k, \mathbb{E}) : k = 1, 2, \ldots, q\}$ as provided by each decision maker $DM_k, k = 1, 2, \ldots, q$.

Step 3. Obtain the resultant H$\mathfrak{F}$SMS $(\psi, \mathbb{E})$ from the $\mathfrak{F}SS$ $\{(\psi_k, \mathbb{E}) : k = 1, 2, \ldots, q\}$.

Step 4. Input the weight $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, corresponding to the $\mathcal{WQP}$ set $\mathbb{E} = \{\mathfrak{e}_k : k = 1, 2, \ldots, m\}$, where $\alpha : \mathbb{E} \to [0, 1]$, such that $\alpha_k = \alpha(\mathfrak{e}_k), k = 1, 2, \ldots, m$.

Step 5. Enter a threshold $\lambda : \mathbb{E} \to [0, 1]$ where $\lambda_k = \lambda(\mathfrak{e}_k)$ associated with each attribute $\mathfrak{e}_k \in \mathbb{E}$.

Step 6. Obtain the λ-LEVEL-H$\mathfrak{F}$SMS $L(\lambda, \psi_\alpha)$ in its tabular form.

Step 7. Calculate the RMSM $L(\lambda, \Delta\psi_\alpha) = \Delta\psi_\alpha^\lambda$ of the λ-LEVEL-H$\mathfrak{F}$SMS $L(\lambda, \psi_\alpha)$.

Step 8. Compute the λ-LEVEL-score $S_\lambda(o) = \frac{\sum_{k=1}^{m} \left[\alpha(\mathfrak{e}_k) \times \Delta(h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(o))\right]}{\sum_{k=1}^{m} \alpha(\mathfrak{e}_k)}, \quad \forall o \in \Omega.$

Step 9. Interpret the λ-LEVEL score: A score $S_\lambda(o_t)$ approaching 0 indicates excellent water quality, and a score approaching 1 indicates poor water quality.

Steps of the Proposed MCDM

1. Start with a list of sampling sites (e.g., river stations), a multiset of observations, and a group of experts.
2. Collect $\mathfrak{F}$SSs from each expert for different $\mathcal{WQP}$s.
3. Combine these into a H$\mathfrak{F}$SMS.
4. Assign weights to each $\mathcal{WQP}$ (like BOD, DO, etc.).
5. Set threshold values (λ) for each $\mathcal{WQP}$ to filter serious pollution levels.
6. Create a λ-LEVEL- H$\mathfrak{F}$SMS, which filters out less significant values.
7. Calculate the RMSM to summarize pollution levels.
8. Compute the λ-LEVEL-score for each station, combining all weighted scores.
9. Interpret the result: A score near 0 means clean water; a score near 1 means high pollution.

We apply this step-by-step model to assess water pollution in the Haora River, showing how this H$\mathfrak{F}$SMS-based method works effectively in real situations.

4. Assessment of water pollution of Haora River using H$\mathfrak{F}$SMS based model

This section uses a model based on the H$\mathfrak{F}$SMS to evaluate the levels of water contamination in Tripura's Haora River. We hope to learn more about the amount and causes of contamination in the river basin by using this model. The Haora River in Tripura, which is located in India, is graded for pollution using our method for evaluating water quality indicators. The Haora River is important because it provides the majority of the water used for drinking for Agartala, the capital of Tripura, which is located in the lower valley of the river. The river, which spans 47.21 km, rises 247 meters above sea level on the western side of the Baramura range close to Champaknagar. As shown in Figures 1 and 2, it flows west across the plains before entering the Titas River in Bangladesh at Banganj. It crosses the border close to Ganganagar.

An important metric in evaluating the Haora River's water quality with a H$\mathfrak{F}$SMS-based method is the λ-LEVEL-score. Excellent water quality is indicated by a λ-LEVEL score that is close to zero, whereas poor quality is indicated by a value that is closer to one. To generate a complete single parametric value, the suggested MCDM aggregates multiple $\mathcal{WQP}$s and assigns relative weights. As indicated in Figures 1 and 2 and Tables 4, seven sampling locations along the river were chosen based on the distribution of point sources of trash discharge, ranging from Champaknagar (23.8020N, 91.4880E) to Rajnagar (Indo-Bangla border) (23.8270N, 91.2570E). A Garmin eTrex Vista Cx GPS unit was used to capture the geographic location. To ensure complete mixing and an accurate representation of the river's quality, between April 2022 and March 2023, triplicate water samples were collected from every location. In order to properly blend the released trash with the river water, the distance between locations was carefully considered.

The samples were taken in 1.5-liter polypropylene bottles that had been cleaned with sample water and pre-washed with 10% HCl. A total of 14 significant $\mathcal{WQP}$s were chosen for this investigation, including pH ($\mathfrak{e}_1$), Electrical conductivity ($\mathfrak{e}_2$), Total suspended solid ($\mathfrak{e}_3$), Dissolved oxygen ($\mathfrak{e}_4$), Biochemical oxygen demand ($\mathfrak{e}_5$), Chemical oxygen demand ($\mathfrak{e}_6$), Total hardness ($\mathfrak{e}_7$), Total alkalinity ($\mathfrak{e}_8$), Total dissolved solid ($\mathfrak{e}_9$), Calcium ($\mathfrak{e}_{10}$), Magnesium ($\mathfrak{e}_{11}$), Chloride ($\mathfrak{e}_{12}$), Total Coliform ($\mathfrak{e}_{13}$) and Faecal coliform ($\mathfrak{e}_{14}$), were carefully measured at every location. The standard procedures outlined in APHA [5] were used for the sampling and analysis.

Tables 5,6, and 7 present the outcomes as clear data highlights of $\mathcal{WQP}$s for each of the seven places that were chosen for the pre-monsoon, monsoon and post-monsoon seasons, respectively. The standards established by BIS [8], CPCB [12], ICMR [35], and WHO [65], which are listed in Table 8, were then contrasted with these values.

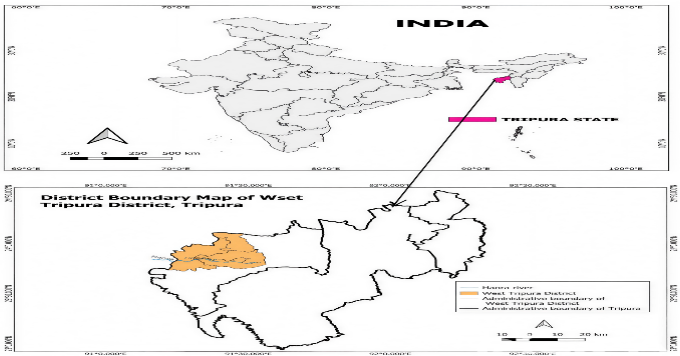

Figure 1: Study Area Sampling Stations

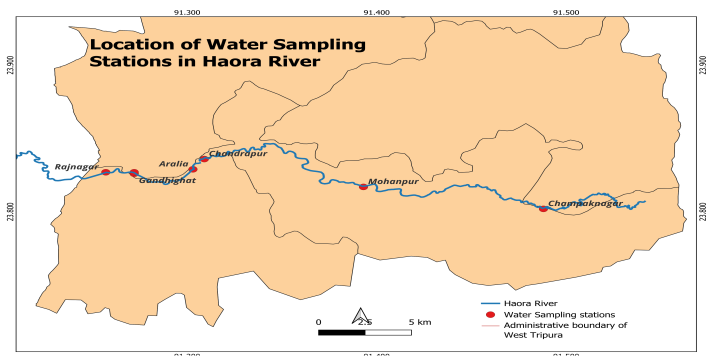

Figure 2: Sampling Stations

Table 4: Sampling stations ($\mathbb{S}$)

$\mathbb{S}$	Sampling stations	Location	Coordinates
ζ_1	Champaknagar, Tripura, India	Bathing Ghat	23.802° N, 91.488° E
ζ_2	Mohanpur, Tripura, India	Bathing Ghat	23.817° N, 91.393° E
ζ_3	Chandrapur, Tripura, India	Bathing Ghat	23.836° N, 91.309° E
ζ_4	Aralia, Tripura, India	Intake Point	23.829° N, 91.303° E
ζ_5	Gandhighat, Tripura, India	Intake Point	23.826° N, 91.272° E
ζ_6	Dashamighat, Tripura, India	Bathing Ghat & Immersion Ghat	23.827° N, 91.272° E
ζ_7	Rajnagar (Indo-Bangla border area)	Downstream to Bangladesh	23.827° N, 91.257° E

Table 5: Seasonal overview of $\mathcal{WQP}$s in Haora River: crisp data for pre-monsoon (2022-23)

$\mathbb{S}$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{c}_4$	$\mathfrak{c}_5$	$\mathfrak{c}_6$	$\mathfrak{c}_7$	$\mathfrak{c}_8$	$\mathfrak{c}_9$	$\mathfrak{c}_{10}$	$\mathfrak{c}_{11}$	$\mathfrak{c}_{12}$	$\mathfrak{c}_{13}$	$\mathfrak{c}_{14}$
ζ_1	6.58	165.7	98.68	6.79	3.96	5.53	90.71	156.66	166.6	19.85	4.19	9.46	236.66	176.66
ζ_2	6.62	202.13	82.65	5.98	5	7.43	70.05	104.16	156.13	17.38	4.86	10.08	293.33	231.66
ζ_3	6.57	200.18	55.31	6.48	6.75	7.66	76	94.16	147.5	11.57	5.55	12.91	305	233.33
ζ_4	6.45	171.31	88.68	5.84	7.04	8.28	81	84.5	170.1	12.53	6.34	11.63	343.33	258.33
ζ_5	6.42	181.36	92.48	4.9	7.73	8.93	78.52	86.66	166.58	12.56	6.73	11.93	411.66	293.33
ζ_6	6.39	168.85	74.05	4.7	10.95	10.8	72	90	157.11	12.57	8.48	14.38	431.66	295
ζ_7	6.05	172.31	94.95	5.23	11.97	11.23	82.02	162.09	151.2	15.26	10.85	15.87	491.66	340

Table 6: Monsoon season water quality assessment: crisp data analysis for Haora River (2022-23)

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
ζ_1	6.65	200.66	91.22	6.28	3.5	5.39	82.78	98.56	170.62	15.86	3.85	9.54	253.33	183.33
ζ_2	6.63	189.75	73.81	6.2	6.74	6.76	62.19	97.14	178.12	12	5.36	10.5	321.66	250
ζ_3	6.56	168.61	80.91	6.22	4.8	3	64.9	92.85	147.95	11.52	5.59	9.51	338.33	260
ζ_4	6.38	178.73	95.04	5.96	5	4	67.54	83.85	145.27	12.88	6.3	14.09	408.33	285
ζ_5	6.22	167.36	89.37	5.05	8.04	8.87	65.8	84.28	134.4	11.67	6.52	13.89	501.66	346.66
ζ_6	6.21	158.58	73.92	4.95	8.25	10.35	56.32	86.42	148.97	12.71	8.3	12.74	540	363.33
ζ_7	6.15	151.9	92.86	5.34	8.78	11.27	62.87	78.09	139.7	11.25	9.81	11.91	676.66	516.66

Table 7: Post-monsoon water quality evaluation: crisp data summaries for Haora River (2022-23)

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
ζ_1	6.56	187.42	79.77	6.18	4.37	7.76	81.75	161.62	169.05	17.54	3.61	10.16	223.33	158.33
ζ_2	6.56	201.09	69.95	5.53	6.35	7.36	71.82	114.37	176.12	18.52	4.28	9.93	273.33	190
ζ_3	6.58	180.91	71.38	6.28	7.51	8.41	61.23	95.75	152.73	11.54	4.13	10.55	293.33	198.33
ζ_4	6.32	169.82	96.71	5.94	7.81	8.93	51.5	86.12	153.08	13.41	5.02	14.51	326.66	235
ζ_5	6.28	170.09	88.26	4.7	7.9	9.4	72	84.37	155.75	11.09	7.53	9.24	385	263.33
ζ_6	6.34	175.78	75.57	5.31	8.8	9.6	59.47	88.87	153.23	10.88	9.56	13.22	420	281.66
ζ_7	6.32	167.98	92.12	4.75	9.75	11	55.87	85	126.53	11.61	10.7	11.95	468.33	318.33

Table 8: Reference ideal and standard values for water quality factors: guidelines for Haora River assessment

$\mathcal{WQP}$s	$\mathcal{WQP}$ Name	Standard Value	Ideal Value	Recommending Agencies
$\mathfrak{e}_1$	pH	6.5–8.5	7	ICMR [35] / BIS [8]
$\mathfrak{e}_2$	Electrical Conductivity (mho/cm)	300	0	ICMR [35]
$\mathfrak{e}_3$	Total Suspended Solids (mg/l)	500	0	WHO [65]
$\mathfrak{e}_4$	Dissolved Oxygen (mg/l)	5.00	14.6	ICMR [35] / BIS [8] / WHO [65]
$\mathfrak{e}_5$	Biological Oxygen Demand (mg/l)	5.00	0	ICMR [35]
$\mathfrak{e}_6$	Chemical Oxygen Demand (mg/l)	10	0	WHO [65]
$\mathfrak{e}_7$	Total Hardness (mg/l)	300	0	ICMR [35] / BIS [8]
$\mathfrak{e}_8$	Total Alkalinity (mg/l)	200	0	BIS [8]
$\mathfrak{e}_9$	Total Dissolved Solids (mg/l)	500	0	ICMR [35] / BIS [8]
$\mathfrak{e}_{10}$	Calcium (mg/l)	75	0	ICMR [35] / BIS [8]
$\mathfrak{e}_{11}$	Magnesium (mg/l)	30	0	ICMR [35] / BIS [8]
$\mathfrak{e}_{12}$	Chlorides (mg/l)	250	0	ICMR [35]
$\mathfrak{e}_{13}$	Total Coliform (MPN/100ml)	50	0	CPCB [12]
$\mathfrak{e}_{14}$	Faecal Coliform (MPN/100ml)	20	0	CPCB [12]

Let $\mathbb{S} = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6, \varsigma_7\}$ represent the group of Haora River sample stations, as shown in Table 4. We define the set $\mathbb{E}$ as the collection of $\mathcal{WQP}$s, which are as follows:

$$\begin{aligned}\mathbb{E} = \{&\mathfrak{e}_1 = \text{pH},\ \mathfrak{e}_2 = \text{Electrical conductivity},\ \mathfrak{e}_3 = \text{Total suspended solid},\ \mathfrak{e}_4 = \text{Dissolve oxygen},\\ &\mathfrak{e}_5 = \text{Biochemical oxygen demand}, \mathfrak{e}_6 = \text{Chemical oxygen demand},\ \mathfrak{e}_7 = \text{Total hardness},\\ &\mathfrak{e}_8 = \text{Total alkalinity},\ \mathfrak{e}_9 = \text{Total dissolved solid},\ \mathfrak{e}_{10} = \text{Calcium},\\ &\mathfrak{e}_{11} = \text{Magnesium},\ \mathfrak{e}_{12} = \text{Chloride},\ \mathfrak{e}_{13} = \text{Total Coliform},\ \mathfrak{e}_{14} = \text{Faecal coliform}\}\end{aligned}$$

For the $\mathcal{WQP}$s as given in Table 8, we refer to standard and ideal values for which we define the fuzzy membership functions of each $\mathcal{WQP}$, as indicated in Table 9.

Table 9: Fuzzy membership functions for important measurements of water quality used to evaluate the Haora River.

$\mathcal{WQP}$s	**Fuzzy membership functions**	$\mathcal{WQP}$s	**Fuzzy membership functions**
pH ($\mathfrak{e}_1$)	$\mu_{\mathfrak{e}_1}(x) = \begin{cases} \frac{7.0-x}{7.0-6.5}, & 6.5 \le x \le 7.0, \\ \frac{x-7.0}{8.5-7.0}, & 7.0 \le x \le 8.5, \\ 1, & \text{otherwise} \end{cases}$	Total alkalinity ($\mathfrak{e}_8$)	$\mu_{\mathfrak{e}_8}(x) = \begin{cases} \frac{x-0}{200-0}, & 0 \le x \le 200, \\ 1, & x \ge 200 \end{cases}$
Electrical conductivity ($\mathfrak{e}_2$)	$\mu_{\mathfrak{e}_2}(x) = \begin{cases} \frac{x-0}{300-0}, & 0 \le x \le 300, \\ 1, & x \ge 300 \end{cases}$	Total dissolved solid ($\mathfrak{e}_9$)	$\mu_{\mathfrak{e}_9}(x) = \begin{cases} \frac{x-0}{500-0}, & 0 \le x \le 500, \\ 1, & x \ge 500 \end{cases}$
Total suspended solid ($\mathfrak{e}_3$)	$\mu_{\mathfrak{e}_3}(x) = \begin{cases} \frac{x-0}{500-0}, & 0 \le x \le 500, \\ 1, & x \ge 500 \end{cases}$	Calcium ($\mathfrak{e}_{10}$)	$\mu_{\mathfrak{e}_{10}}(x) = \begin{cases} \frac{x-0}{75-0}, & 0 \le x \le 75, \\ 1, & x \ge 75 \end{cases}$
Dissolved oxygen ($\mathfrak{e}_4$)	$\mu_{\mathfrak{e}_4}(x) = \begin{cases} \frac{14.6-x}{14.6-5}, & 5 \le x \le 14.6, \\ 0, & x \le 5, \\ 1, & x \ge 14.6 \end{cases}$	Magnesium ($\mathfrak{e}_{11}$)	$\mu_{\mathfrak{e}_{11}}(x) = \begin{cases} \frac{x-0}{30-0}, & 0 \le x \le 30, \\ 1, & x \ge 30 \end{cases}$
Biochemical oxygen demand ($\mathfrak{e}_5$)	$\mu_{\mathfrak{e}_5}(x) = \begin{cases} \frac{x-0}{5-0}, & 0 \le x \le 5, \\ 1, & x \ge 5 \end{cases}$	Chloride ($\mathfrak{e}_{12}$)	$\mu_{\mathfrak{e}_{12}}(x) = \begin{cases} \frac{x-0}{250-0}, & 0 \le x \le 250, \\ 1, & x \ge 250 \end{cases}$
Chemical oxygen demand ($\mathfrak{e}_6$)	$\mu_{\mathfrak{e}_6}(x) = \begin{cases} \frac{x-0}{10-0}, & 0 \le x \le 10, \\ 1, & x \ge 10 \end{cases}$	Total Coliform ($\mathfrak{e}_{13}$)	$\mu_{\mathfrak{e}_{13}}(x) = \begin{cases} \frac{x-0}{50-0}, & 0 \le x \le 50, \\ 1, & x \ge 50 \end{cases}$
Total hardness ($\mathfrak{e}_7$)	$\mu_{\mathfrak{e}_7}(x) = \begin{cases} \frac{x-0}{300-0}, & 0 \le x \le 300, \\ 1, & x \ge 300 \end{cases}$	Faecal Coliform ($\mathfrak{e}_{14}$)	$\mu_{\mathfrak{e}_{14}}(x) = \begin{cases} \frac{x-0}{20-0}, & 0 \le x \le 20, \\ 1, & x \ge 20 \end{cases}$

Then, fuzzy membership functions for different water-related measures are defined, followed by the $\mathfrak{F}SS$ transformation of crisp data summaries of these $\mathcal{WQP}$s as $(\psi, \mathbb{E})$, $(\phi, \mathbb{E})$, and $(\sigma, \mathbb{E})$ respectively in Tables 10,11,and12

If the i-th $\mathcal{WQP}$ $\mathfrak{e}_i \in \mathbb{E}$ is one of the recommended standard values, then the relation weight for this $\mathcal{WQP}$ $\mathfrak{e}_i$ in the set $\mathbb{E}$ is calculated by the formula $\alpha_i = \frac{1}{\zeta_i}$, where ζ_i is the corresponding standard value.

For pH ($\mathfrak{e}_1$), for instance, ζ_1 is set to 8.5 as the standard value. Using this formula, the relative weights given to the $\mathcal{WQP}$s in the set $\mathbb{E}$ are computed as follows:

$$\begin{aligned}\alpha = \{&\alpha_1 = \alpha(\mathfrak{e}_1) = 0.118,\ \alpha_2 = \alpha(\mathfrak{e}_2) = 0.003,\ \alpha_3 = \alpha(\mathfrak{e}_3) = 0.002,\ \alpha_4 = \alpha(\mathfrak{e}_4) = 0.2,\ \alpha_5 = \alpha(\mathfrak{e}_5) = 0.2,\\ &\alpha_6 = \alpha(\mathfrak{e}_6) = 0.1,\ \alpha_7 = \alpha(\mathfrak{e}_7) = 0.003,\ \alpha_8 = \alpha(\mathfrak{e}_8) = 0.005,\ \alpha_9 = \alpha(\mathfrak{e}_9) = 0.002,\ \alpha_{10} = \alpha(\mathfrak{e}_{10}) = 0.013,\\ &\alpha_{11} = \alpha(\mathfrak{e}_{11}) = 0.033,\ \alpha_{12} = \alpha(\mathfrak{e}_{12}) = 0.004,\ \alpha_{13} = \alpha(\mathfrak{e}_{13}) = 0.02,\ \alpha_{14} = \alpha(\mathfrak{e}_{14}) = 0.05\}.\end{aligned}$$

Combining the three $\mathfrak{F}$SSs $(\psi, \mathbb{E})$, $(\phi, \mathbb{E})$, and $(\sigma, \mathbb{E})$ from Table 10,11, and12, respectively, along with the unit weight α, yields the resultant hesitant fuzzy soft multiset (H$\mathfrak{F}$SMS) ψ_α as presented in Table 13.

If we consider $\lambda = \{0.4, 0.3, 0.05, 0.7, 0.4, 0.4, 0.1, 0.4, 0.2, 0.1, 0.03, 0.01, 0.7, 0.7\}$ as a fixed threshold, then we obtain the λ-LEVEL- H$\mathfrak{F}$SMS $L(\lambda, \psi_\alpha)$ and its RMSM $L(\lambda, \Delta\psi_\alpha) = \Delta\psi_\alpha^\lambda$, with λ-LEVEL-score $S_\lambda(\zeta_i)$ computed as in Table 14. The last column in Table 14 shows the calculation of the λ-LEVEL-score:

$$S_\lambda(\zeta_i) = \sum_{k=1}^{14} \Big[\alpha(\mathfrak{e}_k) \times \Delta\big(h_{\psi_\alpha^\lambda(\mathfrak{e}_k)}(\zeta_i)\big)\Big], \quad \mathfrak{e}_k \in \mathbb{E}, \forall \zeta_i \in \mathbb{S}.$$

Table 10: The $\mathfrak{F}SS(\psi, \mathbb{E})$

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
ζ_1	0.84	0.552	0.197	0.814	0.792	0.553	0.302	0.783	0.333	0.265	0.140	0.038	1	1
ζ_2	0.76	0.674	0.165	0.898	1	0.743	0.234	0.521	0.312	0.232	0.162	0.040	1	1
ζ_3	0.86	0.667	0.111	0.846	1	0.766	0.253	0.471	0.295	0.154	0.185	0.052	1	1
ζ_4	1	0.571	0.177	0.913	1	0.828	0.270	0.423	0.340	0.167	0.211	0.047	1	1
ζ_5	1	0.605	0.185	1	1	0.893	0.262	0.433	0.333	0.167	0.224	0.048	1	1
ζ_6	1	0.563	0.148	1	1	1	0.240	0.450	0.314	0.168	0.283	0.058	1	1
ζ_7	1	0.574	0.190	0.976	1	1	0.273	0.810	0.302	0.203	0.362	0.063	1	1

Table 11: The $\mathfrak{F}SS(\phi, \mathbb{E})$

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
ζ_1	0.70	0.669	0.182	0.867	0.70	0.539	0.276	0.493	0.341	0.211	0.128	0.038	1	1
ζ_2	0.74	0.633	0.148	0.875	1	0.676	0.207	0.486	0.356	0.160	0.179	0.042	1	1
ζ_3	0.88	0.562	0.162	0.873	0.96	0.300	0.216	0.464	0.296	0.154	0.186	0.038	1	1
ζ_4	1	0.596	0.190	0.900	1	0.400	0.225	0.419	0.291	0.172	0.210	0.056	1	1
ζ_5	1	0.558	0.179	0.995	1	0.887	0.219	0.421	0.269	0.156	0.217	0.056	1	1
ζ_6	1	0.529	0.148	1	1	1	0.188	0.432	0.298	0.169	0.277	0.051	1	1
ζ_7	1	0.506	0.186	0.965	1	1	0.210	0.390	0.279	0.150	0.327	0.048	1	1

Table 12: The $\mathfrak{F}SS(\sigma, \mathbb{E})$

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
ζ_1	0.88	0.625	0.160	0.877	0.874	0.776	0.273	0.808	0.338	0.234	0.120	0.041	1	1
ζ_2	0.88	0.670	0.140	0.945	1	0.736	0.239	0.572	0.352	0.247	0.143	0.040	1	1
ζ_3	0.84	0.603	0.143	0.867	1	0.841	0.204	0.479	0.305	0.154	0.138	0.042	1	1
ζ_4	1	0.566	0.193	0.902	1	0.893	0.172	0.431	0.306	0.179	0.167	0.058	1	1
ζ_5	1	0.567	0.177	1	1	0.940	0.240	0.422	0.311	0.148	0.251	0.037	1	1
ζ_6	1	0.586	0.151	0.968	1	0.960	0.198	0.444	0.306	0.145	0.319	0.053	1	1
ζ_7	1	0.560	0.184	1	1	1	0.186	0.425	0.253	0.155	0.357	0.048	1	1

Table 13: The $H\mathfrak{F}SMS\ \psi_\alpha$

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$
	0.118	0.003	0.002	0.200	0.200	0.100	0.003	0.005	0.002	0.013	0.033	0.004	0.020	0.050
ζ_1	0.84	0.552	0.197	0.814	0.792	0.553	0.302	0.783	0.333	0.265	0.140	0.038	1	1
	0.70	0.669	0.182	0.867	0.70	0.539	0.276	0.493	0.341	0.211	0.128	0.038	1	1
	0.88	0.625	0.160	0.877	0.874	0.776	0.273	0.808	0.338	0.234	0.120	0.041	1	1
ζ_2	0.76	0.674	0.165	0.898	1	0.743	0.234	0.521	0.312	0.232	0.162	0.040	1	1
	0.74	0.633	0.148	0.875	1	0.676	0.207	0.486	0.356	0.160	0.179	0.042	1	1
	0.88	0.670	0.140	0.945	1	0.736	0.239	0.572	0.352	0.247	0.143	0.040	1	1
ζ_3	0.86	0.667	0.111	0.846	1	0.766	0.253	0.471	0.295	0.154	0.185	0.052	1	1
	0.88	0.562	0.162	0.873	0.96	0.300	0.216	0.464	0.296	0.154	0.186	0.038	1	1
	0.84	0.603	0.143	0.867	1	0.841	0.204	0.479	0.305	0.154	0.138	0.042	1	1
ζ_4	1	0.571	0.177	0.913	1	0.828	0.270	0.423	0.340	0.167	0.211	0.047	1	1
	1	0.596	0.190	0.900	1	0.400	0.225	0.419	0.291	0.172	0.210	0.056	1	1
	1	0.566	0.193	0.902	1	0.893	0.172	0.431	0.306	0.179	0.167	0.058	1	1
ζ_5	1	0.605	0.185	1	1	0.893	0.262	0.433	0.333	0.167	0.224	0.048	1	1
	1	0.558	0.179	0.995	1	0.887	0.219	0.421	0.269	0.156	0.217	0.056	1	1
	1	0.567	0.177	1	1	0.940	0.240	0.422	0.311	0.148	0.251	0.037	1	1
ζ_6	1	0.563	0.148	1	1	1	0.240	0.450	0.314	0.168	0.283	0.058	1	1
	1	0.529	0.148	1	1	1	0.188	0.432	0.298	0.169	0.277	0.051	1	1
	1	0.586	0.151	0.968	1	0.960	0.198	0.444	0.306	0.145	0.319	0.053	1	1
ζ_7	1	0.574	0.190	0.976	1	1	0.273	0.810	0.302	0.203	0.362	0.063	1	1
	1	0.506	0.186	0.965	1	1	0.210	0.390	0.279	0.150	0.327	0.048	1	1
	1	0.560	0.184	1	1	1	0.186	0.425	0.253	0.155	0.357	0.048	1	1

Table 14: The RMSM $L(\lambda, \Delta\psi_\alpha)$ for $L(\lambda, \psi_\alpha)$, with λ-LEVEL-score $S_\lambda(\zeta_i)$

$\mathbb{S}$	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{13}$	$\mathfrak{e}_{14}$	$S_\lambda(\zeta_i)$
Weights	0.118	0.003	0.002	0.200	0.200	0.100	0.003	0.005	0.002	0.013	0.033	0.004	0.020	0.050	
ζ_1	0.810	0.617	0.180	0.853	0.792	0.632	0.284	0.709	0.337	0.238	0.130	0.039	1	1	0.7604
ζ_2	0.796	0.659	0.151	0.906	1	0.719	0.227	0.528	0.341	0.216	0.162	0.041	1	1	0.8388
ζ_3	0.860	0.612	0.140	0.862	0.987	0.679	0.225	0.471	0.299	0.154	0.171	0.044	1	1	0.8270
ζ_4	1	0.578	0.187	0.905	1	0.740	0.226	0.424	0.313	0.173	0.197	0.054	1	1	0.8732
ζ_5	1	0.577	0.180	0.998	1	0.910	0.241	0.425	0.305	0.157	0.231	0.048	1	1	0.9217
ζ_6	1	0.560	0.149	0.989	1	0.987	0.210	0.442	0.306	0.161	0.294	0.054	1	1	0.9322
ζ_7	1	0.547	0.187	0.980	1	1	0.226	0.574	0.279	0.171	0.349	0.053	1	1	0.9350

Based on the data shown in Table 14, it can be seen that the λ-LEVEL obtained for water quality at different stations along the Haora River, such as $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$ and ζ_7, was in the poor range and fell from 0.7604 to 0.9350. These scores represent a reduction of water quality, reflecting the introduction of wastewater elevated in total coliform ($\mathfrak{e}_{13}$) and faecal coliform ($\mathfrak{e}_{14}$) levels, which are pollutants that severely degrade the river's ecosystem. The lowest λ-LEVEL score, 0.7604 at ζ_1 (24.802° N, 91.488° E) sampling station, indicates slightly better water quality than the other stations. Even this station's water quality, however, is still suboptimal. The highest λ-LEVEL score of 0.9350 is recorded from ζ_7, near the Indo–Bangla border (23.827° N, 91.257° E), which suggests that the poorest water quality at this site is influenced by both domestic and industrial wastewater pollution of the river.

The analysis underlines the massive pollution that spans across the river, most especially in the parts close to human settlements, resulting in the river water being laced with harmful pathogens. Quantification of the λ-LEVEL scores indicates that the river is suffering nearly insurmountable water quality degradation at all sampling locations, with additional pollutants driven by urbanization and agricultural runoff worsening the problem. This extensive pollution makes the water of the river unfit for drinking, bathing, and general domestic purposes, thereby posing a grave public health risk. Immediate and comprehensive action is necessary to curb pollution sources, particularly the discharge of untreated wastewater, and to remediate the pollution of the Haora River. The findings suggest the urgent need for integrated water management approaches, such as the construction of advanced wastewater treatment facilities, strict regulation of industrial discharges, and public education programs to protect the river's ecosystem and its supporting communities. If the Haora River is to be sustained in the long term, enhanced monitoring, regular water quality assessments, and collaboration between local authorities and stakeholders are essential.

5. Comparative analysis of the proposed model with existing approaches

We compare our proposed model with three influential approaches for water quality assessment in this comparative analysis. Brown et al. [7] developed a $\mathcal{WQI}$ as a composite index to evaluate water quality and initiated the discussion on its practicality for management. The CCME $\mathcal{WQI}$ [9] discusses standardized guidelines for ecosystem health of aquatic systems, and the dynamics of Sabarmati River water quality were modeled using fuzzy logic by Patel and Chitnis [55] with respect to the level of industrialization as well as climate change.

5.1. Comparative assessment of the approached model and the CCME $\mathcal{WQI}$ method

This section compares fuzzy $\mathcal{WQI}$ values derived from the CCME $\mathcal{WQI}$ model and the fuzzy $\mathcal{WQI}$ values based on the proposed model. Using CCME's $\mathcal{WQI}$ model, the clear data summaries of $\mathcal{WQP}$s were analysed resulting in $\mathcal{WQI}$ values for stations $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$, and ζ_7 that are tabulated in Table 15. These stations had values of 49.52, 48.11, 47.79, 44.80, 45.80, 45.51, and 42.70, respectively. Table 15 shows that marginal and poor values of CCME $\mathcal{WQI}$ are as follows: $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$, and ζ_6 have marginal values, while ζ_7 is poor. Our proposed model is similar to the CCME method in that both recognize the same best option, and the water quality assessment values by the two are quite similar. However, differences in the ranking order of the stations are evident (see Table 18). The cause of this divergence is that the CCME $\mathcal{WQI}$ method does not weigh the values of different $\mathcal{WQP}$s as an important factor in developing a good quality assessment. In contrast, with the CCME $\mathcal{WQI}$ model, our proposed model takes standardized and idealized $\mathcal{WQP}$s as inputs, providing a more thorough and accurate measure of water quality.

Table 15: CCME $\mathcal{WQI}$ of Haora River

Station ($\mathbb{S}$)	CCME $\mathcal{WQI}$	Quality
ζ_1	49.52	Marginal
ζ_2	48.96	Marginal
ζ_3	48.61	Marginal
ζ_4	47.64	Marginal
ζ_5	46.15	Marginal
ζ_6	46.14	Marginal
ζ_7	43.56	Poor

5.2. Comparative evaluation of the proposed model and the Patel-Chitnis model

This section also compares the fuzzy $\mathcal{WQI}$ values using the Patel–Chitnis model with those derived by our proposed model. The fuzzy data outlines of $\mathcal{WQP}$s, offered in Tables 10,11, and ref tab12, have been processed in the Patel–Chitnis model to generate values of fuzzy $\mathcal{WQI}$ for various seasons. In Table 16, these values for many stations along the Haora River are summarized. The average fuzzy $\mathcal{WQI}$ values for the sampling stations at $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$, and ζ_7 are provided in the last column of Table 16 with the range of 0.756768 and 0.933717. These station values do indicate poor water quality, suggesting very high pollution in the Haora River. The Patel–Chitnis model, as well as our proposed model, generate similar optimal options and rankings, but differences in the score values stem from the diverse methods used. In our model, we use λ-Level scores while Patel–Chitnis' model uses fuzzy $\mathcal{WQI}$ values. One of the main novelties of our method is the use of the RMSO (instead of the average value used in the Patel–Chitnis method), which offers better stability and feasibility. In addition, the flexibility and adaptability of the λ-Level score function in our MCDM is greater than that provided by the Patel–Chitnis model.

Table 16: Summaries of fuzzy-$\mathcal{WQI}$ for various seasons

$\mathbb{S}$	Pre-monsoon	Monsoon	Post-monsoon	Average fuzzy-$\mathcal{WQI}$
ζ_1	0.7448	0.707562	0.817942	0.756768
ζ_2	0.8336	0.814463	0.863647	0.837237
ζ_3	0.8376	0.774805	0.847627	0.820011
ζ_4	0.8865	0.825976	0.890125	0.867534
ζ_5	0.9189	0.915681	0.925559	0.920047
ζ_6	0.9355	0.934715	0.922726	0.930980
ζ_7	0.9361	0.927042	0.938008	0.933717

5.3. Comparative analysis of the proposed H$\mathfrak{F}$SMS approach and the Brown $\mathcal{WQI}$ method

Table 17 shows that using the Brown Weighted Quality Index Model [7], the average Weighted Arithmetic $\mathcal{WQI}$ values of sampling stations such as $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$ and ζ_7 were found to be. These values vary from 119.5 to 247.87 at the various stations on the Haora River and imply very poor water quality, such that the water is unfit for consumption. The same best option is identified by our suggested model and the Brown $\mathcal{WQI}$ method, and the results are similar. However, differences in rank order (Table 18) can be due to different methodological approaches. The Brown model [7], in turn, uses Weighted Arithmetic $\mathcal{WQI}$ values, whereas our proposed model produces λ-Level scores. Unlike in the Weighted Arithmetic $\mathcal{WQI}$ model, which takes no consideration of the integration of ideal and standard values of $\mathcal{WQP}$s, our model is much more comprehensive and nuanced by incorporating these critical factors.

Table 17: Seasonal summaries of weighted arithmetic $\mathcal{WQI}$ for Haora River: Comparative analysis across pre-monsoon, monsoon, and post-monsoon seasons

$\mathbb{S}$	Pre-monsoon	Monsoon	Post-monsoon	Average	Ranking	Probable usage
ζ_1	118.82	121.13	118.54	119.50	Unsuitable for drinking purposes	Proper treatment is required before any kind of usage
ζ_2	150.64	165.96	143.43	153.34		
ζ_3	159.42	154.06	152.63	155.37		
ζ_4	172.72	167.44	167.21	169.12		
ζ_5	194.84	216.25	184.13	198.41		
ζ_6	216.49	227.33	196.35	213.39		
ζ_7	236.25	287.91	219.45	247.87		

The analysis further shows that our work made major improvements over existing methods. It would, however, be overly optimistic to suggest that the problems of the current literature were resolved. A major contribution to our model is the introduction of the RMSO concept, as opposed to classical scoring functions. This approach, compared to traditional ones, is more stable and practical. Our model is flexible, using the λ-LEVEL score rather than a fixed score value, and this is indeed a key strength of our model. The approach model is well-suited to a variety of real-world applications due to its adaptability. Our work has advanced the field in bridging gaps by addressing the majority of challenges inherent in previous methods. In Table 18, we present a detailed summary of these findings with details on how our model beats state-of-the-art methods in addressing important problems. Indeed, the contributions of this work are significant, yet fundamental questions remain over which further study and refinement are desired. Having a balanced view allows one to keep in mind both the results achieved and still to be done, keeping the continuous improvement effort alive in the domain.

Table 18: Comparative water quality assessment across seasons: Pollution rating for the Haora River

Models	Best optimal choice	Ranking according to good quality	Score Values	Weightage values
Proposed model	ζ_1	$\zeta_1 > \zeta_3 > \zeta_2 > \zeta_4 > \zeta_5 > \zeta_6 > \zeta_7$	$\zeta_1 = 0.7604$, $\zeta_3 = 0.8270$, $\zeta_2 = 0.8388$, $\zeta_4 = 0.8732$ $\zeta_5 = 0.9217$, $\zeta_6 = 0.9322$, $\zeta_7 = 0.9350$	Considered
CCME $\mathcal{WQI}$ model (CCME, 2001)	ζ_1	$\zeta_1 > \zeta_2 > \zeta_3 > \zeta_5 > \zeta_6 > \zeta_4 > \zeta_7$	$\zeta_1 = 49.52$, $\zeta_3 = 47.79$, $\zeta_2 = 48.11$, $\zeta_4 = 44.80$ $\zeta_5 = 45.80$, $\zeta_6 = 45.51$, $\zeta_7 = 42.70$	Not considered
Patel-Chitnis model (Patel and Chitnis, 2022)	ζ_1	$\zeta_1 > \zeta_3 > \zeta_2 > \zeta_4 > \zeta_5 > \zeta_6 > \zeta_7$	$\zeta_1 = 0.7568$, $\zeta_3 = 0.8200$, $\zeta_2 = 0.8372$, $\zeta_4 = 0.8675$ $\zeta_5 = 0.9200$, $\zeta_6 = 0.9310$, $\zeta_7 = 0.9337$	Considered
Brown $\mathcal{WQI}$ model (Brown et al., 1970)	ζ_1	$\zeta_1 > \zeta_2 > \zeta_3 > \zeta_4 > \zeta_5 > \zeta_6 > \zeta_7$	$\zeta_1 = 119.5$, $\zeta_3 = 155.37$, $\zeta_2 = 153.34$, $\zeta_4 = 169.12$ $\zeta_5 = 198.41$, $\zeta_6 = 213.39$, $\zeta_7 = 247.87$	Not considered

6. Conclusions

This study introduces a new method using the H$\mathfrak{F}$SMS-based model to evaluate water pollution in the Haora River, Tripura. The model combines fuzzy logic, hesitant fuzzy information, and a weight function, which helps in handling uncertain or vague water quality data more effectively. The researchers collected and analyzed water samples from seven locations during three different seasons using 14 $\mathcal{WQP}$s. The method calculated a λ-LEVEL-score for each location. Results showed that all locations had poor water quality, with the worst pollution found near the Indo-Bangla border (ζ_7) due to domestic and industrial waste. This approach helped to identify how polluted each site is and which $\mathcal{WQP}$s are most responsible. It also demonstrated the usefulness of advanced fuzzy decision-making models in real environmental situations.

The study strongly suggests that urgent steps—such as building wastewater treatment plants, enforcing strict pollution control, and raising public awareness—are needed to save the river. The proposed model can also be applied to other rivers or environmental assessments where uncertainty is involved.

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Declarations

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