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On the Projective Special Unitary Groups $PSU_3(q)$ and the Sum of Element Orders

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ABSTRACT: In this paper, we prove that projective special unitary groups $PSU_3(q)$, where $q = 2^n$ and $\frac{q^2-q+1}{\gcd(3,q+1)}$ is a prime number, can be uniquely determined by the even-order components of the group and the set of orders of centralizers of p_m-order elements in G where p_m is the largest element in $\pi(G)$. In the following, we shows that, in a special case, these groups can be recognized by using the sum of the group elements $\psi(G) = \sum_{x\in G} o(x)$ where $o(x)$ denotes the order of $x \in G$.

Keywords: Components of group, even-order components, largest element, sum of element orders.

Contents

1. Introduction

For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$. Let G be a finite group, $\pi_e(G)$ be the set of element orders of G and $\pi(G)$ denote the set of prime divisors of $|G|$. We know that if H is a subgroup with prime index p of finite simple group G, then p is maximal prime p dividing $|G|$. We use p_m to denote the largest element in $\pi(G)$ and $\pi_{p_m}(G)$ to denote the set of orders of centralizers of p_m-order element of G, that is, $\pi_{p_m}(G) = \{|C_G(x)||x \in G, |x| = p_m\}$.

Also, we denote the set of numbers of elements with the same order in G by $nse(G)$ and the sum of element orders of G is denoted by $\psi(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. We assume that $\Gamma(G)$ has $t(G)$ connected components $\pi_1, \pi_2, \ldots, \pi_{t(G)}$. We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. The numbers m_i are called the order components of G. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of G. In the case where the order of group G is even, we always assume that $2 \in \pi_1$ and denote the even-order component of G as $m_1(G)$.

Recently, Zhangjia Han et al. proved that Janko simple groups can be characterized uniquely by the even-order components and $\pi_{p_m}(G)$, see [18]. Also, in [19], Dongyang He et al. proved that some of Alternating group by this method can be characterized. In this article, we prove that the simple projective special unitary groups $PSU_3(q)$, $q = 2^n$ and $\frac{q^2-q+1}{\gcd(3,q+1)}$ is a prime number, is characterizable by even-order components of the group and $\pi_{p_m}(G)$. For this purpose, $\psi(G)$, the sum of element orders of group G, is a suitable criterion.

In [7], its proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(C_n)$ where C_n is the cyclic group of order n. In fact, C_n is characterized by $\psi(C_n)$ and $|C_n|$. Following this publication, many studies have been done on the function $\psi(G)$, for example, see [9,10,11,12,13,14,15,16,17].

We say that the group G is characterized by $\psi(G)$ and $|G|$, whenever there exist the group H, so that if $\psi(G) = \psi(H)$ and $|G| = |H|$, then $G \cong H$. It is clear that the group G cannot be characterized by $nse(G)$ if $\psi(G) = \psi(H)$ and $G \ncong H$. In other words, if group G is not recognizable by the sum of the elements, it can be concluded that it is not recognizable by the number of elements of the same rank. In recent research, some groups have been characterized using this method. Baniasad Azad and Khosravi showed that simple groups $PSL(2,p)$, where $p \in \{11, 13, 17, 19, 23, 29, 37, 61\}$, are determined by their

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orders and the sum of element orders. Also C. Shao in [8] proved that the simple group $PSL(2,2^m)$ with 2^m+1 prime or 2^m-1 prime, is characterizable by $nse(G)$ and its order. But Marefat et al. in [11] proved that the group $PSL(2,64)$ is not characterized by $\psi(G)$. Clearly, this group is not characterized by $nse(G)$.

We prove the following main theorem:

Main Theorem. Let G be a group and $U = PSU_3(q)$ be a groups so that $m_1(G) = m_1(U)$ and $\pi_{p_m}(G) = \pi_{p_m}(U)$. Then $G \cong U$.

2. Notation and Preliminaries

In this section, we give some useful lemmas which will be used in the proof of the main Theorem.

First, we denote $\pi(G)$ the set of prime factors of the order of G, p_m stands for the largest element of $\pi(G)$ and $\pi_{p_m}(G)$ represents the set of orders of centralizers of p_m-order elements in G. In the way we denote $|\pi(G)|$ the number of prime factors of the order of G and also $Aut(G)$ the automorphism group of G. Furthermore, $Out(G)$ be a outer automorphism group of G.

Lemma 2.1 *[5] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Proof: By, Lemma 2.1, the proof is straightforward. □

Lemma 2.2 *An integer $n = p_1^{\alpha_1}...p_k^{\alpha_k}$ is the number of Sylow p-subgroups of a finite solvable group G if and only if $p_i^{\alpha_i} \equiv 1 \pmod p$ for $i = 1, ..., k$.*

Lemma 2.3 *[30] Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$*

Lemma 2.4 *[9] Let $P \in Syl_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 2.5 *[25] Let G_1, G_2 be finite groups, p be a prime number and let a, b be positive integers. Then the following hold:*

1. *$\psi(G_1 \times G_2) = \psi(G_1) \times \psi(G_2)$ if and only if $(|G_1|, |G_2|) = 1$(i.e.ψ is multiplicative);*
2. *$\psi(p^a) = \frac{p^{2a+1}+1}{p+1}$,*
3. *$\psi(p^a) \mid \psi(p^b)$ if and only if $2a+1 \mid 2b+1$,*
4. *$(\psi(p), \psi(p^2)) = (\psi(p), \psi(p^3)) = (\psi(p^2), \psi(p^3)) = 1$.*

Lemma 2.6 *[6] Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.1 *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.7 *[2] Let G be a 2-Frobenius group of even order. Then*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*

2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.8 *[29] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Group	π_1	π_2
A_p, $p \neq 5, 6$, p and p -2 not both prime	$3.4.(p-3)(p-2)(p-1)$	p
A_{p+1}, $p \neq 4, 5$, $p\mid 1$ and $p{+}1$ not both prime	$3.4.(p-2)(p-1)(p+1)$	p
A_{p+2}, $p \neq 3, 4$, p and $p+2$ not both prime	$3.4.(p-1)(p+1)(p+2)$	p
$A_{p-1}(q)$, $(p,q) \neq (3,2), (3,4)$,	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$, $q-1 \mid p+1$,	$q^{p(p+1)/2}(q^{p+1}-1) \prod_{i=2}^{p-1}(q^i-1)$	$\frac{q^p-1}{(q-1)}$
${}^2A_{p-1}(q)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$, $q+1 \mid p+1$, $(p,q) \neq (3,3), (5,2)$	$q^{p(p+1)/2}(q^{p+1}-1) \prod_{i=2}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$,	$2^6.3^4$	5
$B_n(q)$, $n = 2^m \geq 4$, q odd	$q^{n^2}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$	$3^{p^2}(3^p+1) \prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$, $n = 2^m \geq 2$	$q^{n^2}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$, $q = 2, 3$	$q^{p^2}(q^p+1) \prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$, $p \geq 5$, $q = 2, 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$, $q = 2, 3$	$\frac{1}{(2,q-1)} q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
${}^2D_n(q)$, $n = 2^m \geq 4$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$, $n = 2^m+1 \geq 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1) \prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
${}^2D_p(3)$, $p \neq 2^m+1$, $p \geq 5$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$, $n \neq 2^m+1 \neq p$, $m \geq 2$	$\frac{1}{2} 3^{n(n-1)} \prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
$G_2(q)$,$q \equiv \alpha(\mathrm{mod} 3)$, $\alpha = \pm 1$, $q > 2$	$q^6(q^3-\alpha)(q^2-1)(q+\alpha)$	$q^2-\alpha q+1$
${}^3D_4(q)$	$q^{12}(q^6-1)(q^2-1)(q^4+q^2+1)$	q^4-q^2+1
$F_4(q)$,q odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	q^4-q^2+1
$E_6(q)$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5-1)(q^3-1)(q^2-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$,$q > 2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5+1)(q^3+1)(q^2-1)$	$\frac{q^6-q^3+1}{(3,q+1)}$
${}^2F_4(2)'$	$2^{11}.3^3.5^2$	13

Lemma 2.9 *[31] Let q, k, l be natural numbers. Then*

1. $(q^k-1, q^l-1) = q^{(k,l)}-1$.
2. $(q^k+1, q^l+1) =$
   $\{q^{(k,l)}+1$ *if both* $\frac{k}{(k,l)}$ *and* $\frac{l}{(k,l)}$ *are odd,*
   $(2, q+1)$ *otherwise.*$\}$
3. $(q^k-1, q^l+1) =$
   $\{q^{(k,l)}+1$ *if* $\frac{k}{(k,l)}$ *is even and* $\frac{l}{(k,l)}$ *is odd,*
   $(2, q+1)$ *otherwise.*$\}$
   In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k-1, q^k+1) \leq 2$ holds.

Lemma 2.10 *[27] Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

1. *$PSL_n(q)$, $n = 2, 3$, $q \equiv \pm 2 \pmod 5$;*
2. *$G_2(q)$, $q \equiv \pm 2 \pmod 5$;*
3. *$PSU_3(q)$, $q \equiv \pm 2 \pmod 5$;*
4. *${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;*
5. *${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.*

3. Proof of the Main Theorem

In this section, we prove the main theorem. From now on, we denote the simple projective special unitary groups $PSU_3(q)$ by $U_3(q)$. We recall that G is a group with $m_1(G) = m_1(U_3(q))$ and $\pi_{p_m}(G) = \pi_{p_m}(U_3(q))$. First, we know that group $|U_3(q)| = \frac{q^3(q^3+1)(q^2-1)}{(3,q+1)}$.

Proof of the Theorem . First, assume $(3, q+1) = 1$ then we have $m_1(G) = m_1(U_3(q)) = 2(q+1)(q^2-1)$ and $\pi_{p_m}(G) = \pi_{p_m}(U_3(q)) = q^2 - q + 1$. Since $m_1(G) = 2(q+1)(q^2-1)$, so $t(G) \geq 2$ which by Lemma 2.8 implies G is as one of the following case
(1) Frobenius group or 2-Frobenius group
(2)G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$. We prove that G is not Frobenius group. On opposite assume G be a Frobenius group with kernel K and complement H. Then by Lemma 2.6, $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Now, if $2 \in \pi(H)$ the $\pi(H) = \pi_1(G)$ since H is a nilpotent group so $H \cong S_2 \times S_t$, where $t = (q+1)(q^2-1)$ and $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$, for $i = 2, t$. In the way $|K| \mid |Aut(S_2)|$, on the other hand $q^2 - q + 1 \mid |K|$ which is a contradiction. Now, $2 \in \pi(K)$ so Sylow $q^2 - q + 1$-subgroup S_t where $t = q^2 - q + 1$ of G is normal in G and has order t. Now assume p'-subgroup of G act on S_t so must be have a element order $p't$, where this is impossible because $\pi_{p_m}(G) = t$. Therefore we deduce that G is not a Frobenius group. Now, prove that G is not a 2-Frobenius group. On opposite G be a 2-Frobenius group. Then by Lemma 2.7, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$ and $|G/K|$ divides $|Aut(K/H)|$. Since $m_1(G) = m_1(U) = 2(q+1)(q^2-1)$ so $t \in \pi_2(G)$ so K contains an element of order t. Now if t-order element act on the 2-subgroup of$H$ so we must be have the element of order $2t$, which this is impossible, because $\pi_{p_m}(G) = t$. So G is not a 2-Frobenius group. Now, if $(3, q+1) = 3$, then $m_1(G) = m_1(U_3(q)) = \frac{2(q+1)(q^2-1)}{3}$ and $\pi_{p_m}(G) = \pi_{p_m}(U_3(q)) = q^2 - q + 1$. Thus like previous argument, we can show G are not a frobenius and 2-frobenius group. Next, by Lemma 2.8 G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$. On the other hand, we know that $5 \nmid |G|$.Thus K/H is isomorphic to one of the groups in Lemma 2.10. Hence we consider the flowing cases:

(1) If $K/H \not\cong PSL_2(q')$, where $q' \equiv \pm 2(\mod 5)$, $q' = p'^m$. Now, since that $m_1(G) = m_1(PSL_2(q')) = 2$ and also $\pi_{p_m}(G) = \pi_{p_m}(PSL_2(q')) = q' + 1$ we have a contradiction. Since that $m_1(G) = m_1(PSL_2(q')) = 2$ so $2(q+1)(q^2-1) = 2$ it follows that $q^3 + q^2 - q - 2 = 0$, which is impossible. On other hand, $q^2 - q + 1 = q' + 1$ so $q' = q^2 - q$. On the other hand, $q' = 5k \pm 2$ it follows that $5k \pm 2 = q^2 - q - 2$, which is impossible. If $(3, q+1) = 3$, then $\frac{2(q+1)(q^2-1)}{3} = 2$ so $q^3 + q^2 - q - 4 = 0$, which is impossible.

(2) If $K/H \not\cong PSL_3(q')$, where $q' \equiv \pm 2 \pmod 5$. Now since that that $m_1(G) = m_1(PSL_3(q')) = \pi(2(q'^2-1))$ and also $\pi_{p_m}(G) = \pi_{p_m}(PSL_3(q') = \pi(\frac{q'^2+q'+1}{(3,q'-1)}))$. Since that $m_1(G) = m_1(PSL_3(q')) = \pi(2(q^2-1))$ so $2(q+1(q^2-1) = 2(q'^2-1)$ it follows that $q^3 + q^2 - q = (5k \pm 2)^2$. As $2^{3n} + 2^{2n} - 2^n = (5k \pm 2)^2$ which is a contradiction. Now, if $\pi_{p_m}(G) = \pi_{p_m}(PSL_3(q') = \pi(\frac{q'^2+q'+1}{(3,q'-1)}))$ so $q^2 - q + 1 = \frac{q'^2+q'+1}{(3,q'-1)}$. Now, if $(3, q'-1) = 1$ then $q^2 - q + 1 = q'^2 + q' + 1$ it follow that

$q(q-1)=q'(q'+1)$. Hence $q'+1=q$ as $5k+3=2^{2n}-1$ and $5k-2=2^{2n}$, which is a contradiction. For $(3,q'-1)=3$, we have a contradiction. Now if $(3,q+1)=3$ then, $\frac{2(q+1)(q^2-1)}{3}=2(q'^2-1)$. It follows that $q'^2=q^3+q^2-q$, which is impossible.

(3) If $K/H \ncong PSU_3(q')$ then $m_1(G)=m_1(PSU_3(q'))=2\prod_{i=1}^{2}(q'^i-1)$ and and also $\pi_{p_m}(G)=\pi_{p_m}(PSU_3(q')=\pi(\frac{q'^2-q'+1}{(3,q'+1)})))$. First, $m_1(G)=m_1(U_3(q'))=2\prod_{i=1}^{2}(q'^i-1)$ implies $2(q+1)(q^2-1)=2\prod_{i=1}^{2}(q'^i-1)$ it follows that $(q+1)(q^2-1)=(q'-1)(q'^2-1)$. In other words, $(2^n+1)(2^{2n}-1)=(5k\pm 2)-1)(5k\pm 2)^2-1)$ which is contradiction. Now,if $(3,q+1)=3$ then $\frac{2(q+1)(q^2-1)}{3}=2\prod_{i=1}^{2}(q'^i-1)$. So, $(q+1)(q^2-1)=3(q'-1)(q'^2-1)$ it follows that $2^{3n}+2^{2n}-2^n-4=3q'^3-3q'^2+3q'$, which is impossible.

(4) If $K/H \ncong G_2(q')$, where $q'\equiv \pm 2(\mod 5)$, then since that $m_1(G)=m_1(G_2(q'))=\pi(2(q'^2-1)(q'^3+1)$ and also $\pi_{p_m}(G)=\pi_{p_m}(G_2(q')=q'^2+q'+1)$. Since that $m_1(G)=m_1(G_2(q'))=\pi(2(q'^2-1)(q'^3+1)$ so $2(q+1)(q^2-1)=2(q'^2-1)(q'^3+1)$ it follows that $q^3+q^2-q=q'^5-q'^3+q'^2$. As a result, $q(q^2+q-1)=q'^2(q'^3-q'+1)$. Since $(q,q^2+q-1)=1$ so $q^2+q-1=q'^3-q'+1$ it follows that $q(q+1)=q'(q'^2-1)$. Hence, $q+1=q'^2-1$ which is a contradiction. Now, if $(3,q+1)=3$ then $\frac{2(q+1)(q^2-1)}{3}=2(q'^2-1)(q'^3+1)$ so $(q+1)(q^2-1)=3(q'^2-1)(q'^3+1)$, like previous proof, we have a contradiction.

(5) If $K/H \ncong {}^2G_2(q')$, where $q'=3^{2m+1}$, then sine $m_1(G)=m_1({}^2G_2(q')=q'(q'^4-1)$ and also $\pi_{p_m}(G)=\pi_{p_m}({}^2G_2(q'))=q'+\sqrt{3q'}+1$. Since that $2(q+1)(q^2-1)=q'(q'^4-1)$ it follows that $2q^3+2q^2-2q-2=q'^5-q'$. On the other hand, we have $q^2-q+1=q'+\sqrt{3q'}+1$ so $q(q-1)=3^{2m+1}+3^{m+1}$ as $2^n(2^n-1)=3^{m+1}(3^m+1)$. As a result, $2^n=3^{m+1}$ which is a impossible. Now, if $(3,q+1)=3$ then $\frac{2(q+1)(q^2-1)}{3}=q'(q'^4-1)$ thus $2(q+1)(q^2-1)=3q'(q'^4-1)$, which is impossible.

Hence, we deduce that $K/H\cong U_3(q)$. Since that $m_1(G)=m_1(PSU_3(q')=2(q'+1)(q'^2-1)$ and also $\pi_{p_m}(G)=\pi_{p_m}(PSU_3(q'))=q'^2-q'+1$. Since that $m_1(G)=m_1(PSU_3(q')=2(q'+1)(q'^2-1)$ it follows that $2(q+1)(q^2-1)=2(q'+1)(q'^2-1)$ and $q^2-q+1=q'^2-q'+1$. Hence $q=q'$ as $n=n'$. For $(3,q+1)=3$ then $\frac{2(q+1)(q^2-1)}{3}=\frac{2(q'+1)(q'^2-1)}{3}$ so $q=q'$. On the other hand, $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, we deduce that $H=1$, $G=K\cong U_3$.

Example 3.1 *we prove that if G is a group with $|G|=|PSU_3(5)|$ and $\psi(G)=\psi(PSU_3(5))$. Then $G\cong PSU_3(5)$. For this purpose, we have $|G|=126000=2^4.3^2.5^3.7$ and $\psi(G)=845671$. We prove that G is not a solvable group and Lemma 2.3 is satiesfies. Hence, we show if G is a solvable group, then $\psi(G)>845671$. Now, by Lemma 2.2, we have $n_3\in\{1,4,7,16,25,28\}$ and $n_7\in\{1,8\}$. Next, by NC-theorem we have $m_{62}\neq 0$ and $m_{93}\neq 0$. Let Q_{62} and Q_{93} be cyclic subgroups of orders 62 and 93, respectively. if $n_5=1$ so $m_{155}\neq 0$. Thus $\psi(G)\geq\psi(Z_{155})+\phi(62).62+93.\phi(93)+(|G|-155+62+93).2=19551+30.62+60.93+(372000-310).2=19551+(371690).2=770991$, so $\psi(G)>8821051$ as wanted. If G is non-solvable, then by Lemma2.3, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that $K/H\cong PSL(3;5)$ or $PSL(3;31)$ and $|G/K|\mid|Out(K/H)$. First if $K/H\cong PSL(3;5)$. Since that $|G/K|\mid|2$ so $|H|=2^4.3.5^3.31$. Now, we know that $H=Z_{2^4.3.5^3.31}$ then G is a central extension of $Z_{2^4.3.5^3.31}$ by $PSL(3;5)$. Since the Shur multiplier of $PSL(3;5)$ is 1, we get that $G\cong Z_{2^4.3.5^3.31}\rtimes PSL(3;5)$. Hence $\psi(G)>\psi(Z_{2^4.3.5^3.31})\psi(PSL(3,5))>14510693547.8821051$, where this is a contradiction. Now $K/H\cong PSL(3;31)$, so we have $G\cong PSL(3;31)$ and the proof is completed.*

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