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Approximate Solutions of Second Kind of Nonlinear Volterra Integral Equations

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ABSTRACT: In this article we find out the analytical solutions of some nonlinear problems of second kind of Volterra integral equations (VIE) by Optimal Homotopy asymptotic method (OHAM) and Adomian decomposition method (ADM). From the comparative assessment of both solutions, it is concluded that OHAM is operative, simple and unambiguous as like ADM. To show the effectiveness of these approaches, we showed the quick convergence of OHAM and ADM, also listed a few models to verify the exactness and correctness of these methods. The exactness, precision, and convergence of both methods are evident in graphical analysis. These techniques have mechanized steps which can be easily attained by means of Mathematica.

Key Words: Nonlinear, Volterra Integral equation, OHAM, explicit, convergence, ADM.

Contents

1. Introduction

The majorities of problems are nonlinear, particularly in the fields of engineering and applied sciences. Volterra integral equations have many applications in applied fields, including fluid mechanics, biomechanics, demography and the study of viscoelastic materials. Italian mathematician and physicist Vito Volterra invented them in 1908 in his research of mathematical physics [1].The scientists introduced some approaches to deals such type of problems and also for calculating analytical solution of these problems. These method include [2], Variational Iterative method [3], Differential Transform Method [4], Group Analysis Method [5], and Glowinski in [6] used wavelets to approximate some Volterra integral equations. In this research article we deliberate some nonlinear VIE. Volterra integral equation have the general form which is given below;

$$\xi(h) = f(h) + \lambda \int_0^h K(h,t)G(\xi(t))dt \tag{1.1}$$

The function $G(\xi(h))$ in (1.1) is nonlinear in $\xi(h)$ like $\xi^2(h)$, $\xi^3(h)$, $e^{\xi(h)}$, $in(\xi(h))$ and several others. The symbol λ (lambda), is a parameter in above equation, while $K(h,t)$ is called integration kernel [7].

The limit of integration of the Volterra integral equation is functions of "h " but not a constant number similar in the Fredholm integral equations. In eq. (1.1) the kernel $K(h,t)$ will be assuming a separable kernel.

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2. The Existence of Solutions for Nonlinear Volterra integral Equations

Suppose, we have a general nonlinear Volterra integral equations of the second kind, [7].

$$\xi(h) = f(h) + \int_0^h G(h, t, \xi(t))dt \tag{2.1}$$

Following are some explicit conditions for nonlinear Volterra integral equations under which solution exist;
i. The solution is exist in $a \leq h \leq \beta$, if $f(h)$ is bounded and integrable.
ii. The then solution is exist, if $f(h)$ satisfy the Lipchitz condition in the interval (α, β) it means that $|f(h) - f(y)| < k|h - y|$.
iii.The solution is exist, when function $G(h, t, \xi(t))$ can be integrated and constrained $|G(h, t, \xi(t)) < k$, in $\alpha \leq h, t \leq \beta$.
iv. The solution must be occur, but if Lipchitz condition $|G(h, t, z) - G(h, t, z') < M|z - z'|$, satisfied by function $G(h, t, \xi(t))$.
If all of the above conditions are exist, then solutions of the nonlinear VIE will also exists.

3. The Optimal Homotopy Asymptotic Method

Scientists and engineers have recently known OHAM's applications in linear and nonlinear problems [8] and [9], since this technique always distort difficult problems into very simple problems that are easy to solve. This technique provides an instant method for estimating the sequence and high potential convergence in the scientific and engineering fields to solve nonlinear problems.The method of OHAM was first of all proposed by Marinca and Herişanu [10].

3.1. OHAM analysis of nonlinear Volterra integral equations.
Many researchers have extensively studied several arithmetic methods for integral equations, such as [11] and [12]. Let us consider the general nonlinear problem [13].

$$\pounds g(h) + f(h) + \mu(q(h)) = 0 \tag{3.1}$$

$\pounds$, is called linear operator which is a function, $f(h)$ is also function which is given, μ is said to be nonlinear operator, and $q(h)$ is unidentified function.
Conferring to rules of OHAM [13], we will make a Homotopy:
$\Omega \times [0, 1] \longrightarrow R$ for (3.1) which satisfy,

$$(1 - \rho)[\pounds q(h, \rho) + f(h)] = H(\rho)[\pounds q(h, \rho) + f(h) + \mu(q(h, \rho))] \tag{3.2}$$

in eq (3.2) $H(\rho)$ represents a non-zero auxiliary function for $\rho \neq 0$ and $H(0) = 0$ obviously, When, ρ=0 then it holds that

$$q(h, 0) = q_0(h) \tag{3.3}$$

But when $\rho = 1$ then it holds that

$$q(h, 1) = q(h) \tag{3.4}$$

Consider the auxiliary function $H(\rho)$ can be expressed as,

$$H(\rho) = \sum_{j=1}^{m} C_j(\rho)^j \tag{3.5}$$

In above eqn, $C_j, j = 1, 2, 3...$ are constant.
By introducing $\rho = 0$ in eq. (3.2), it holds that

$$\pounds(q_0(h)) + f(h) = 0 \tag{3.6}$$

Through the sequence of Taylor, OHAM solution can be considered as follows;

$$q(h, \rho, c_j) = q_0(h) + \sum_{k=1}^{m} q_k(h, c_j)\rho^m, \tag{3.7}$$

Also, $j = 1, 2, 3, ...$
When ρ=1, then eq. (3.7) converts

$$q(h, \rho, c_j) = q_0(h) + \sum_{k=1}^{m} q_k(h, c_j) \tag{3.8}$$

$j = 1, 2, 3, ...$
When we Substitute eq. (3.8) into eq. (3.2) and equate coefficient of the same power of ρ, we get,

$$\pounds(q_1(h)) = c_1\mu(q_0(h)) \tag{3.9}$$

$$\pounds(q_m(h) - q_{m-1}(h)) = c_m\mu(q_0(h)) + \sum_{j=1}^{m-1} c_j[\pounds(q_{m-j}(h) + \mu_{m-j}(q_0(h) + q_1(h) + ... + q_{m-1}(h)))] \tag{3.10}$$

Where, $m = 2, 3, ...$ and $\mu_m(q_0(h), q_1(h), ..., q_m)(h))$ are the coefficient of ρ^m in extension of $\mu(q(h, \rho))$ about ρ.

$$\mu(q(h, \rho, c_j)) = \mu_0(q_0(h)) + \sum_{m=1}^{\infty} \mu_m(q_0(h), q_1(h), ...q_m(h))\rho^m \tag{3.11}$$

The m^{th} order approximations result are;

$$q^m(h, c_{i,j}) = q_0(h) + \sum_{k=1}^{m} q_k(h, c_j) \tag{3.12}$$

$j = 1, 2, ...m$.
Replacing equation (3.12) into (3.1), we acquire residual equation as;

$$\Re(h, c_j) = \pounds(q^m(h, c_j)) + f(h) + \mu(q^m(h, c_j)) \tag{3.13}$$

If $\Re(h, c_j) = 0$ then $q^m(h, c_j)$, will be an exact solution. Using Least squares method for outcome the constants, $C_j, j = 1, 2, 3, ..$ first of all suppose,

$$J(C_j) = \int_a^b \Re^2(H, C_j)dh \tag{3.14}$$

Where $C_j, j = 1, 2, 3, ...$ are the constants and it is known as follow;

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \frac{\partial J}{\partial C_3} ... = \frac{\partial J}{\partial C_m} = 0 \tag{3.15}$$

If we know the values of, $C_j, j = 1, 2, 3, ...,$ then solution is obtained.

4. Adomian Decomposition Method (ADM)

This method is actually introduce by Adomian in his latest work [15], that is why it is called Adomian Decomposition Method or just Decomposition method. It is a stable and efficient method for a variety of equations, differential and integral equations, linear and nonlinear equations. The method is usually decomposes the unknown ξ into addition of uncountable components that can be easily calculate through various iterations.

This is called the Adomian Polynomial $A_n, n \geq= 0$ Adomian provides an appropriate process to find a reliable representation for all form of nonlinear terms [7].

4.1.The Adomian Polynomials Calculation [7]

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Let suppose the unknown non-linear function $\xi(h)$ can be denoted by an boundless decomposition series such as;

$$\xi(h)=\sum_{n=0}^{\infty}\xi_n(h) \tag{4.1}$$

The term $G(\xi(h))$, is expressed by as infinite series which is called Adomian Polynomials A_n,

$$A_n=\frac{1}{\vartheta!}\frac{d^{\vartheta}}{d\kappa^{\vartheta}}G(\sum\nolimits_{i=0}^{\vartheta}\kappa^i\xi_i)_{\kappa=0} \tag{4.2}$$

Where, $\vartheta=1,2,3,...$

A completely nonlinear form of the so-called Adomian polynomials A_n can be deliberate. The general formula of eq. (4.2) can be used very simply as follow. Assuming that the nonlinear function is, $G(\xi(h))$ therefore, by using (4.2), Adomian polynomial is given by;

$$\begin{aligned} A_0 &= G(\xi_0),\\ A_1 &= \xi_1 G'(\xi_0),\\ A_2 &= \xi_2 G'(\xi_0)+\frac{1}{2!}\xi_1^2 G''(\xi_0),\\ A_3 &= \xi_3 G'(\xi_0)+\xi_1\xi_2 G''(\xi_0)+\frac{1}{3!}\xi_1^3 G'''(\xi_0). \end{aligned} \tag{4.3}$$

Here, we can made two very important observations. Firstly, A_0 depends only on ξ_0, A_1 depends only on ξ_0 and ξ_1, A_2 depends only on ξ_0, ξ_1 and ξ_3, and son on.Secondly, substituting eq. (4.3) into eq. (4.2) gives;

$$\begin{aligned} G(\xi) &= A_0+A_1+A_2+A_3+...\\ & G(\xi_0)+(\xi_1+\xi_2+\xi_3+...)G'(\xi_0)\\ &+\frac{1}{2!}(\xi_1^2+2\xi_1\xi_2+2\xi_1\xi_3+\xi_2^2+...)G''(\xi_0)+...\\ &+\frac{1}{3!}(\xi_1^3+3\xi_1^2\xi_2+3\xi_1^2\xi_3+6\xi_1\xi_2\xi_3+...)G'''(\xi_0)+...\\ &= G(\xi_0)+(\xi-\xi_0)G'(\xi_0)+\frac{1}{2!}(\xi-\xi_0)^2G''(\xi_0)+... \end{aligned} \tag{4.4}$$

The last expansion approved the fact that the sequence in A_n, the polynomial is about the Taylor series of the function ξ_0, not the point in the standard series. The few Adomian polynomials given in (4.3) above clearly show that the sum of the subscripts of the components of ξ_h in each term of A_n is equal to n.

5. Numerical Examples

Here we use the Optimal Homotopy asymptotic method and Adomian Decomposition method for solving some nonlinear Volterra integral equations. Also the exact solutions of the problems are known to us. We find the solution of the problems by two methods i.e. OHAM and ADM.
Let see which method have the best solution of the given problems.

Model 1. Let consider a nonlinear Volterra integral equation with exact solution $\xi(h)=tan(h)$ [7].

$$\xi(h)=h+\int_0^h \xi^2(t)dt \tag{5.1}$$

Let find out the OHAM Solution;
Used OHAM for finding solution. First we find zero order solution;

$$-h + \xi_0(h) = 0. \tag{5.2}$$

$$\xi_0(h) = h. \tag{5.3}$$

$$\xi_1(h) = -h - hc_1 + \xi_0 + c_1\xi_0 - hc_1\xi_0^2. \tag{5.4}$$

Similarly we proceed,

$$\xi_1(h) = -h^3 c_1. \tag{5.5}$$

$$\xi_2(h) = -hc_2 + c_2\xi_0 - hc_2\xi_0^2 + \xi_1 + c_1\xi_1 - 2hc_1\xi_0\xi_1. \tag{5.6}$$

$$\xi_2(h) = -h^3c_1 - h^3c_1^2 + 2h^5c_1^2 - h^3c_2. \tag{5.7}$$

$$\xi_3(h) = -hc_3 + c_3(\xi_0) - hc_3\xi_0^2 + c_2(\xi_1) - 2hc_2\xi_0(\xi_1) - h(c_1)(\xi_1^2) + \xi_2 + c_1(\xi_2) - 2hc_1(\xi_0)\xi_2 \tag{5.8}$$

$$\xi_3(h) = -h^3\left(\left(2-4h^2\right)c_1^2 + \left(1-4h^2+5h^4\right)(c_1^3) + c_2 + c_1\left(1+\left(2-4h^2\right)c_2\right) + c_3\right) \tag{5.9}$$

The series solution is given by;

$$\xi(h) = \xi_0(h) + \xi_1(h) + \xi_2(h) + \xi_3(h). \tag{5.10}$$

$$\xi(h) = -h\left(-1 + \left(3h^2 - 6h^4\right)c_1^2 + \left(h^2 - 4h^4 + 5h^6\right)c_1^3 + 2h^2c_2 + h^2c_1\left(3 + \left(2-4h^2\right)c_2\right) + h^2c_3\right) \tag{5.11}$$

Used least square method for finding the values of C_i,

$$c_1 = -0.3042906945, c_2 = 0.3071758553, c_3 = -0.1105674314$$

$$\xi(h) = h + 0.346426h^3 + 0.0689734h^5 + 0.140876h^7. \tag{5.12}$$

Now let solve the same problem by Adomian Decomposition Method

Now we find out the solution by ADM;

$$\xi(h) = h + \int_0^h \xi^2(t)dt. \tag{5.13}$$

$f(h) = h, \lambda = 1, k(h,t) = 1$. For finding solution $\xi(h)$ by ADM, we have to find $\xi_0(h), \xi_1(h), \xi_2(h), \xi_3(h), ...$

Select the function for zeroth component $\xi_0(h)$ as;

$$\xi_0(h) = h. \tag{5.14}$$

$$\xi_1(h) = \int_0^h \xi_0^2(t)dt. \tag{5.15}$$

By solving $\xi_1(h)$, we get

$$\xi_1(h) = \frac{1}{3}h^3. \tag{5.16}$$

$$\xi_2(h) = \int_0^h 2[\xi_0(t)\xi_1(t)dt.] \tag{5.17}$$

By solving $\xi_2(h)$, we get

$$\xi_2(h) = \frac{2}{15}h^5. \tag{5.18}$$

$$\xi_3(h) = \int_0^h [2\xi_0(t)\xi_2(t) + \xi_1^2(t)]dt. \tag{5.19}$$

By solving $\xi_3(h)$, we get

$$\xi_3(h) = \frac{17}{315}h^7. \tag{5.20}$$

Thus the series solution is;

$$\xi(h) = \xi_0 + \xi_1 + \xi_2 + \xi_3. \tag{5.21}$$

$$\xi(h) = h + \frac{h^3}{3} + \frac{2h^5}{15} + \frac{17h^7}{315}. \tag{5.22}$$

Table 1: Comparison of Results of the eq. 5.1.

h	OHAM solution	ADM solution [7]	Exact solution	γ^*	γ^{**}
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.100347	0.100335	0.100335	0.0000124577	2.19585×10^{-11}
0.2	0.202795	0.20271	0.20271	0.0000852469	1.13817×10^{-8}
0.3	0.309552	0.309336	0.309336	0.000215666	4.46752×10^{-7}
0.4	0.423108	0.422787	0.422793	0.000315142	6.13048×10^{-6}
0.5	0.546559	0.546255	0.546302	0.000256766	0.0000475295
0.6	0.684135	0.683879	0.684137	$1.81018\times^{-6}$	0.000258043
0.7	0.842018	0.841187	0.842288	0.000270196	0.0011012
0.8	1.02952	1.02568	1.02964	0.000123486	0.00396326
0.9	1.26065	1.24754	1.26016	0.000494817	0.0126134
1.0	1.55628	1.52063	1.55741	0.00113268	0.0367728

From table 1. We can see that the error in the OHAM solution is very less as compared to ADM solution for the eq.5.1. Thus from table 1. We can say that OHAM have best solution and very near to exact solution. In the last row of table, we see that OHAM solution is very near approaching to exact solution of the problem.

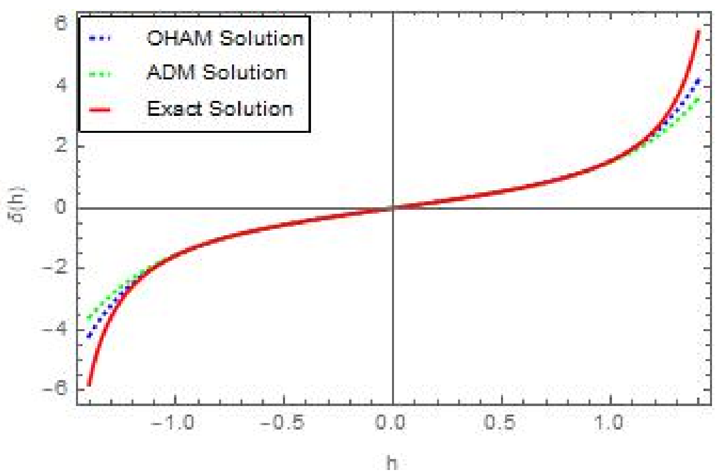

Figure 1: Comparison of OHAM, ADM and the exact solution of eq. 5.1.

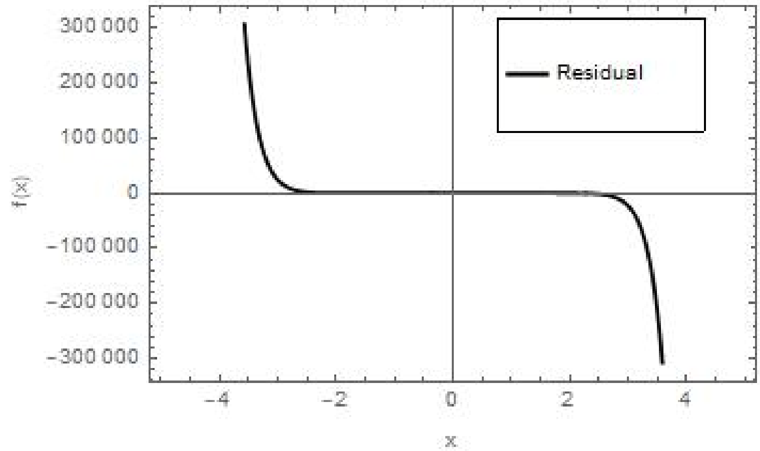

Figure 2: This figure indicates the residual solution of eq.5.1.

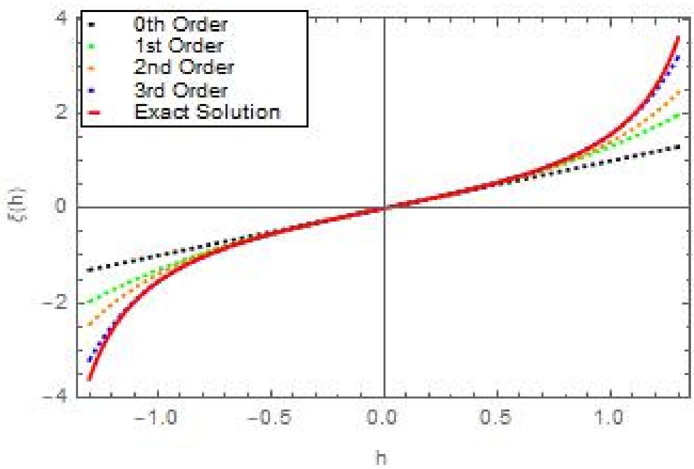

Figure 3: Show orders of solution i.e. zero, first, second and third order of the eq. 5.1.

Model 2. Let consider we have a nonlinear VIE with exact solution $\xi(h) = h$ [16].

$$\xi(h) = h + \frac{h^5}{5} - \int_0^h t\xi^3(t)dt \tag{5.23}$$

OHAM Solution;
By using OHAM,the solution of the given problem is determined as;

$$-h - \frac{h^5}{5} + \xi_0(h) = 0 \tag{5.24}$$

$$\xi_0(h) = h + \frac{h^5}{5} \tag{5.25}$$

$$\xi_1(h) = -h - \frac{-h^5}{5} - hc_1 - \frac{h^5 c_1}{5} + \xi_0 + c_1\xi_0 + \frac{1}{2}h^2 c_1 \xi_0^3 \tag{5.26}$$

$$\xi_1(h) = \frac{1}{250}h^5(5+h^4)^3c_1 \tag{5.27}$$

$$\xi_2(h) = -hc_2 - \frac{h^5c_2}{5} + c_2\xi_0 + \frac{1}{2}h^2c_2\xi_0^3 + \xi_1 + c_1\xi_1 + \frac{3}{2}h^2c_1\xi_0^2\xi_1 \tag{5.28}$$

$$\xi_2(h) = \frac{h^5\left(5+h^4\right)^3\left(50c_1 + \left(50+75h^4+30h^8+3h^{12}\right)c_1^2 + 50c_2\right)}{12500} \tag{5.29}$$

$$\begin{aligned}\xi_3(h) = -hc_3 - \frac{h^5c_3}{5} + c_3\xi_0 + \frac{1}{2}h^2c_3\xi_0^3 + c_2\xi_1 + \frac{3}{2}h^2c_2\xi_0^2\xi_1 + \frac{3}{2}h^2c_1\xi_0\xi_1^2 \\ +\xi_2 + c_1\xi_2 + \frac{3}{2}h^2c_1\xi_0^2\xi_2\end{aligned} \tag{5.30}$$

$$\begin{aligned}\xi_3(h) = \frac{1}{156250}h^5\left(5+h^4\right)^3(25\left(50+75h^4+30h^8+3h^{12}\right)c_1^2 + T(h)c_1^3 + \\ 25c_1\left(25+\left(50+75h^4+30h^8+3h^{12}\right)c_2\right) + 625\left(c_2+c_3\right))\end{aligned} \tag{5.31}$$

Where, $T(h) = 625 + 1875h^4 + 2625h^8 + 1575h^{12} + 450h^{16} + 60h^{20} + 3h^{24}$
Hence, series solution can be written as;

$$\xi(h) = \xi_0(h) + \xi_1(h) + \xi_2(h) + \xi_3(h) \tag{5.32}$$

$$\begin{aligned}\xi(h) = \frac{1}{312500}(62500h5 + h^4 + 1250h^55 + {h^4}^3c_1 + \\ 25h^55 + {h^4}^3 50c_1 + T(h)c_1^2 + 50c_2 + \\ 2h^5(5+h^4)^3(25T(h)c_1^2 + (625 + 1875h^4 + 2625h^8 + 1575h^{12} \\ +450h^{16} + 60h^{20} + 3h^{24})c_1^3 + \\ 25c_1(25 + T(h)c_2) + 625(c_2 + c_3)\end{aligned} \tag{5.33}$$

While, $T(h) = 50 + 75h^4 + 30h^8 + 3h^{12}$
By using Least Square Method, we can calculate values of C_i, where, $i = 1, 2, 3, ...$
$c_1 = -0.2375324889, c_2 = 0.0684463364, c_3 = 0.0657098256$

By putting values ofC_i in eq. (5.33), we obtained;

$$\begin{aligned}\xi(h) = h + 0.00667582h^5 - 0.033536h^9 + 0.0391566h^{13} + 0.00329263h^{17} \\ -0.0102898h^{21} - 0.00496917h^{25} - 0.00109938h^{29} - 0.000135092h^{33} \\ -9.00613\times10^{-6}h^{37} - 2.57318\times10^{-7}h^{41}\end{aligned} \tag{5.34}$$

Now the same problem is done by Adomian decomposition method.

Solution by ADM:

$$\xi(h) = h + \frac{h^5}{5} - \int_0^h (t)\xi^3(t)dt \tag{5.35}$$

$f(h) = h + \frac{h^5}{5}, \lambda = -1, k(h,t) = t$. For finding solution ξ(h)by ADM, we have to find, $\xi_0(h), \xi_1(h), \xi_2(h), \xi_3(h), ...$

Select the function for zeroth component ξ_0(h) as;

$$\xi_0(h) = h + \frac{h^5}{5} \tag{5.36}$$

First order problem with its solution.

$$\xi_1(h) = -\int_0^h (t)\xi_0^3(t)dt \tag{5.37}$$

$$\xi_1(h) = -\frac{h^5\left(16575 + 5525h^4 + 765h^8 + 39h^{12}\right)}{82875} \tag{5.38}$$

Second order problem with its solution.

$$\xi_2(h) = -\int_0^h t[\{3\xi_0^2(t)\xi_1(t)\}]dt \tag{5.39}$$

$$\xi_2(h) = \frac{h^9}{15} + \frac{11h^{13}}{325} + \frac{214h^{17}}{27625} + \frac{566h^{21}}{580125} + \frac{231h^{25}}{3453125} + \frac{3h^{29}}{1540625} \tag{5.40}$$

Third order problem with its solution.

$$\xi_3(h) = -\int_0^h [t\{3(\xi_0^2(t)\xi_2(t)) + 3(\xi_0(t)\xi_1^2(t))\}]dt \tag{5.41}$$

$$\begin{aligned}\xi_3(h) = -\frac{8h^{13}}{325} - \frac{464h^{17}}{27625} - \frac{9304h^{21}}{1740375} - \frac{73744h^{25}}{72515625} - \frac{1130088h^{29}}{9112796875} - \\ \frac{14490704h^{33}}{1503611484375} - \frac{691224h^{37}}{1574711328125} - \frac{48h^{41}}{5369078125}\end{aligned} \tag{5.42}$$

Thus the series solution is;

$$\xi(h) = \xi_0 + \xi_1 + \xi_2 + \xi_3 \tag{5.43}$$

$$\begin{aligned}\xi(h) = h - \frac{263h^{17}}{27625} - \frac{7606h^{21}}{1740375} - \frac{68893h^{25}}{72515625} - \frac{1112343h^{29}}{9112796875} - \\ \frac{14490704h^{33}}{1503611484375} - \frac{691224h^{37}}{1574711328125} - \frac{48h^{41}}{5369078125}\end{aligned} \tag{5.44}$$

Table 2: Comparison of Results of the eq. 5.23

h	OHAM solution	ADM solution	Exact solution	γ^*	γ^{**}
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	0.1	6.67247×10^{-8}	0.0
0.2	0.200002	0.200001	0.2	2.11912×10^{-6}	1.249×10^{-14}
0.3	0.300016	0.300015	0.3	0.0000155684	1.23404×10−11
0.4	0.40006	0.400005	0.4	0.0000598324	1.65491×10−9
0.5	0.500148	0.500147	0.5	0.000147919	7.47471×10^{-8}
0.6	0.600233	0.599998	0.6	0.000232604	1.7101×10^{-6}
0.7	0.700149	0.699975	0.7	0.000149299	0.0000247197
0.8	0.799798	0.799742	0.8	0.000202267	0.000258473
0.9	0.899913	0.89786	0.9	0.0000871239	0.00214018
1.0	0.999086	0.985027	1.0	0.000913713	0.0149729

From table 2. We can see that the error in the OHAM solution is very less as compared to ADM solution for the eq. 5.23. Thus from table 2. We can say that OHAM have best solution and very near to exact solution. In the last row of table, it is found that OHAM solution is very near to approaching the exact solution of the problem.

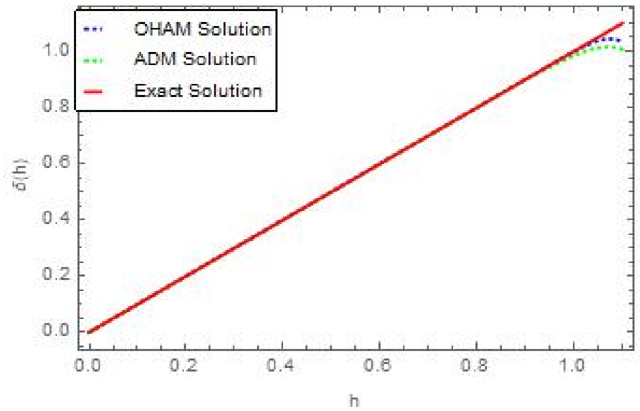

Figure 4: Comparison of OHAM, ADM and the exact solution of eq. 5.23.

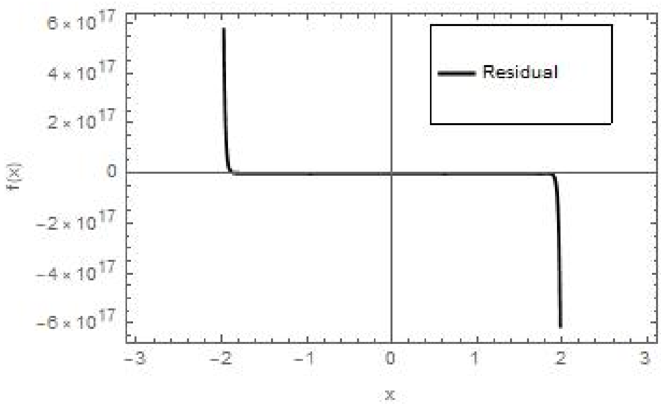

Figure 5: Shows Residual solution of eq.5.23.

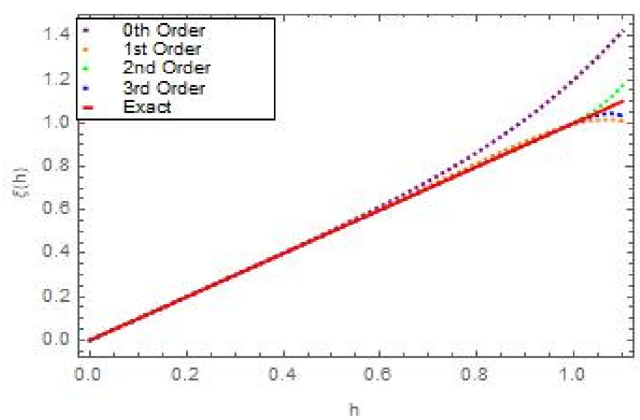

Figure 6: Shows orders of solutions of equation. 5.23.

Model 3. Consider a nonlinear Volterra integral equation with exact solution $\xi(h) = tanh(h)$ [7].

$$\xi(h) = h - \int_0^h \xi^2(t)dt \tag{5.45}$$

OHAM Solution:

The solution of the problem can be obtained as;

$$\xi_0(h) = h \tag{5.46}$$

$$\xi_0(h) = h \tag{5.47}$$

$$\xi_1(h) = -h - hc_1 + \xi_0(h) + c_1\xi_0(h) + hc_1\xi_0^2(h) \tag{5.48}$$

$$\xi_1(h) = h^3 c_1 \tag{5.49}$$

$$\xi_2(h) = -hc_2 + c_2\xi_0 + hc_2\xi_0^2 + \xi_1 + c_1\xi_1 + 2hc_1\xi_0\xi_1 \tag{5.50}$$

$$\xi_2(h) = \frac{1}{32}h^2\left(-2 + e^h\left(2 + h^2\right)\right)\left(4c_1 + \left(4 + h^2\right)c_1^2 + 4c_2\right) \tag{5.51}$$

$$\xi_3(h) = -hc_3 + c_3\xi_0 + hc_3\xi_0^2 + c_2\xi_1 + 2hc_2\xi_0\xi_1 + hc_1\xi_1^2 + \xi_2 + c_1\xi_2 + 2hc_1\xi_0\xi_2 \tag{5.52}$$

$$\xi_3(h) = h^3\left(\left(2 + 4h^2\right)c_1^2 + \left(1 + 4h^2 + 5h^4\right)c_1^3 + c_2 + c_1\left(1 + \left(2 + 4h^2\right)c_2\right) + c_3\right) \tag{5.53}$$

Hence, series solution can be written as;

$$\xi(h) = \xi_0(h) + \xi_1(h) + \xi_2(h) + \xi_3(h) \tag{5.54}$$

That is;

$$\xi(h) = h + 3\left(h^3 + 2h^5\right)c_1^2 + \left(h^3 + 4h^5 + 5h^7\right)c_1^3 + 2h^3c_2 + h^3c_1\left(3 + \left(2 + 4h^2\right)c_2\right) + h^3c_3 \tag{5.55}$$

We can calculate values of C_i, where $i = 1, 2, 3....$ using (LSM). $c_1 = -0.1762025556$, $c_2 = 0.0630065766, c_3 = 0.0059538249$

By putting values of C_i in eq. (5.55), we obtained;

$$\xi(h) = h - 0.331173h^3 + 0.119994h^5 - 0.0273531h^7 \tag{5.56}$$

Now the same problem is done by Adomian decomposition method.

Solution by ADM:

$$\xi(h) = h - \int_0^h \xi^2(t)dt \tag{5.57}$$

$f(h) = h, \lambda = -1, k(h,t) = 1.$
For finding solution ξ(h) by ADM, we have to find, $\xi_0(h), \xi_1(h), \xi_2(h), \xi_3(h), ...$
Select the function for zeroth component ξ_0(h)as;

$$\xi_0(h) = h \tag{5.58}$$

First order problem with solution.

$$\xi_1(h) = -\int_0^h \xi_0^2(t)dt \tag{5.59}$$

$$\xi_1(h) = -\frac{h^3}{3} \tag{5.60}$$

Second order problem with solution.

$$\xi_2(h) = -\int_0^h 2[\{\xi_0(t)\xi_1(t)\}]dt \tag{5.61}$$

$$\xi_2(h) = \frac{2h^5}{15} \tag{5.62}$$

Third order problem with solution.

$$\xi_3(h) = -\int_0^h [2\{\xi_0(t)\xi_2(t)\} + \xi_1^2(t)]dt \tag{5.63}$$

$$\xi_3(h) = -\frac{17h^7}{315} \tag{5.64}$$

Thus the series solution is:

$$\xi(h) = \xi_0 + \xi_1 + \xi_2 + \xi_3 \tag{5.65}$$

$$\xi(h) = h - \frac{h^3}{3} + \frac{2h^5}{15} - \frac{17h^7}{315} \tag{5.66}$$

Table 3: Comparison of Results of the eq. 5.45

h	OHAM solution	ADM solution [7]	Exact solution	γ^*	γ^{**}
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.09967	0.099668	0.099668	2.02945×10^{-6}	2.17812×10^{-11}
0.2	0.197389	0.197375	0.197375	0.0000133427	1.10186×10^{-8}
0.3	0.291344	0.291312	0.291313	0.0000313161	$4.15309\times10-7$
0.4	0.379989	0.379944	0.379949	0.0000398796	5.38384×10^{-6}
0.5	0.462139	0.462078	0.462117	0.0000223145	0.0000387842
0.6	0.537032	0.536857	0.53705	0.0000179501	0.000192333
0.7	0.604322	0.603631	0.604368	0.0000454339	0.000736295
0.8	0.664023	0.661706	0.664037	0.0000141781	0.00233073
0.9	0.716347	0.709919	0.716298	0.0000492022	0.00637872
1.0	0.761468	0.746032	0.761594	0.000126506	0.0155624

From table 3. We can see that the error in the OHAM solution is very less as compared to ADM solution for the eq.5.45. Thus from table 3. We can say that OHAM have best solution and very near to exact solution. In the last row of table, we see that OHAM solution is very near approaching to exact solution of the problem.

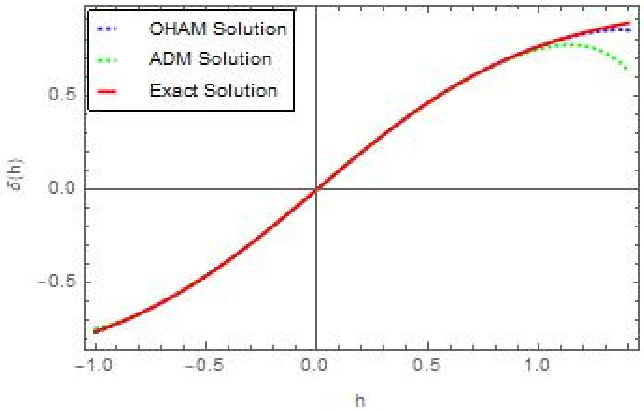

Figure 7: Comparison of OHAM, ADM and the exact solution of eq. 5.45.

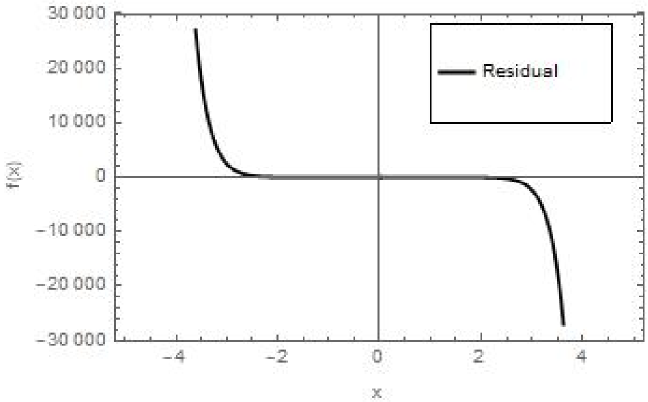

Figure 8: Shows Residual solution of eq.5.45.

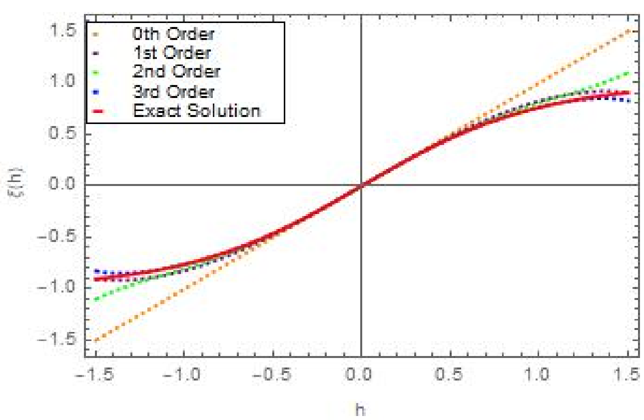

Figure 9: Shows orders of solutions of equation. 5.45.

6. Results and Discussion

In this paper, we used application of OHAM by resolving three models of nonlinear Volterra integral equations of the second kind. This method was tested on three dissimilar examples with the comparison of Adomian Decomposition method (ADM) and exact solution. This method (OHAM) proved to be a precise and efficient technique for judgment estimated solutions for the nonlinear Volterra integral equations of the second kind. Applying OHAM is not difficult at all. It was demonstrated that the OHAM can provide adequate accuracy and convergence with just a few terms. OHAM has the potential to be a useful tool for resolving intensely nonlinear models. OHAM converges quickly and flawlessly to the precise solution. Mathematica 9 generates all graphs and computational work.

7. Conclusion

In this paper we calculated the solution of the nonlinear Volterra integral equations of the second kind by OHAM and ADM. Form table1, table 2 and table 3, we concluded that OHAM have less error and quick approach to the exact solution of the problems as compared to ADM solution. We also concluded from figure 1, figure 4 and figure 7 that OHAM have more convergence to the exact solution as compared to ADM solution. Also figure 2, figure 5 and figure 8 shows the residual solutions of the given problems. From figure 3, figure 6 and figure 9, we concluded that OHAM have very quick convergence and good efficiency as compared to other analytical methods such as ADM. Thus OHAM is an efficient, quick and beneficial method for nonlinear Volterra integral equations as compared to ADM.

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