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A New Conformal Fractional Integrals and Derivatives

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ABSTRACT: In this work, a new definition of conformal fractional integrals and derivatives are defined by another function $h(x)$, namely α-M-conformal fractional derivative and α-M-conformal fractional integral as generalized for some types of conformal fractional derivative and integral. This type of conformal fractional obeys classical properties like linearity, power rule, chain rule, product and quotient rule.

Key Words: Conformal fractional, fractional integral, fractional derivative, continuous function, integral transformation.

Contents

1. Introduction

In recent years, fractional calculus has gained significant attention from researchers [1,2,3,5] , growing into a robust area of both theoretical exploration and practical application. The origins of this field trace back to the 17th century, particularly in 1695 when Leibniz described a derivative of order $\alpha = \frac{1}{2}$ in a correspondence with L'Hospital. This intriguing concept quickly captivated mathematicians as well as professionals in physics, biology, engineering, and economics [24].

Classical fractional calculus relies on various definitions for integration and differentiation operators, including Riemann-Liouville, Hadamard, IIyas-Farid and Katugampola [4,10,11,12]. These types of fractional integrals and derivatives are not satisfy many properties as linearity, power rule, chain rule, product and quotient rule, thus, the need for a new type of fractional integrals arose. In [13], defines new type of fractional integral , namely conformal fractional integral and derivative. This type of fractional integrals and derivatives benefit for developing many papers as future works [6-9]. In this paper we define a new conformal fractional derivative as generalized some types of conformal derivatives and integrals [14]. We found and proved the relationship between a new conformal fractional derivative and regular derivative to facilitate dealing with the new type of fractional derivatives, Also proved that every function fractional differentiable at point is continuous at the same point.

2. Fundamental Concepts

Definition 2.1 *[4]: Let $\phi(x)$ be a continuous function in a Banach space $C_([a,b])$ then (left/right) Riemann-Liouville integrals defined as following:*

$$I_{RL}^{\alpha,a^+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1}\phi(u)du$$

$$I_{RL}^{\alpha,b^-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (u-x)^{\alpha-1}\phi(u)du$$

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Definition 2.2 *[12]: Let ϕ be a continuous function in a Banach space $C_([a,b])$ and $n-1<\alpha<n$ then (left/right) Riemann-Liouville derivatives as following::*

$$D_{RL}^{\alpha,a^+}\phi(x) = (\frac{d}{dx})^n I_{RL}^{n-\alpha,a^+}\phi(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n \int_a^x (x-u)^(\alpha-1)\phi(u)du, x\in[a,b]$$

$$D_{RL}^{\alpha,b^-}\phi(x) = (\frac{d}{dx})^n I_{RL}^{n-\alpha,b^-}\phi(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n \int_x^b (u-x)^(\alpha-1)\phi(u)du, x\in[a,b]$$

Proposition 2.1 *[12]*
Let ϕ be a continuous function and $0<\alpha<1$ then

$$D_{RL}^{\alpha,a^+} I_{RL}^{\alpha,a^+}\phi(x) = \phi(x)$$

Definition 2.3 *[13]:*
If $f:[0,\infty)\to R$ is a function and $0<\alpha<1$ then conformal α -fractional derivative define as follows:

$$T_{KH}^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}$$

Definition 2.4 *[13]:*
If $f:[0,\infty)\to R$ be a function and $0<\alpha<1$ then conformal α -fractional integral define as follows:

$$I_a^{\alpha}(f(x)) = \int_a^x \frac{f(t)}{t^{1-\alpha}}\,dt$$

Definition 2.5 *[14]:*
If $f:[0,\infty)\to R$ be a function and $0<\alpha<1$ then α -conformal fractional derivative define as follows:

$$T_{Kat}^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon}$$

3. Proposed Conformal Fractional

Definition 3.1 *[13]: Let $n-1<\alpha\le n$, $n\in N$ then we define a new conformal fractional derivative of order α , as following: $D_{M,h}^{\alpha}f(t) = \lim_{\varepsilon\to 0}\frac{f\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)-f(t)}{\varepsilon}$*
namely α-M-conformal fractional derivative.
If f is α-differentiable in (0,a) and $\lim_{t\to 0^+} D_{M,h}^{\alpha}f(t)$ exists for all $t>0$, then define :

$$D_{M,h}^{\alpha}f(0) = \lim_{t\to 0^+} D_{M,h}^{\alpha}f(t)$$

Theorem 3.1 *:*
Let $f:(0,-\infty)\to R$ is α-M-differentiable function at a point $t_0>0$, $\alpha\in(0,1)$, then f is continuous at t_0 .
Proof:
Let $t_0>0$ then we have

$$f\left(h^{-1}\left(h(t_0)+\varepsilon {t_0}^{-\alpha}\right)\right)-f(t_0) = \frac{f\left(h^{-1}\left(h(t_0)+\varepsilon {t_0}^{-\alpha}\right)\right)-f(t_0)}{\varepsilon}\quad\varepsilon$$

Thus

$$\lim_{\varepsilon\to 0}\left[f\left(h^{-1}\left(h(t_0)+\varepsilon {t_0}^{-\alpha}\right)\right)-f(t_0)\right] = \lim_{\varepsilon\to 0}\left[\frac{f\left(h^{-1}\left(h(t_0)+\varepsilon {t_0}^{-\alpha}\right)\right)-f(t_0)}{\varepsilon}\right]\ .\ \lim_{\varepsilon\to 0}\varepsilon$$

Let $\tau = \frac{1}{h'(t_0)}\varepsilon {t_0}^{-\alpha}+o(\varepsilon^2)$ then $\tau\to 0$ at $\varepsilon\to 0$, we get

$$\lim_{\varepsilon\to 0}\left[f\left(h^{-1}\left(h(t_0)+\varepsilon {t_0}^{-\alpha}\right)\right)-f(t_0)\right] = h'(t_0)\,{t_0}^{\alpha}\lim_{\varepsilon\to 0}\left[\frac{f(t_0+\tau)-f(t_0)}{\tau}\right]\ .\ \lim_{\varepsilon\to 0}\varepsilon\ = 0$$

Hence f(t) is continuous at t_0.

Theorem 3.2 *:*
Let f and g be -M-differentiable functions a point $t>0$, $\alpha \in (0,1)$, then

1. $D^{\alpha}_{M,h}\left(af+bg\right) = aD^{\alpha,a}_{M,h}\ f + b\ D^{\alpha,a}_{M,h}g$
2. $D^{\alpha}_{M,h}\left(f.g\right) = f\ D^{\alpha,a}_{M,h}g + g\ D^{\alpha,a}_{M,h}\ f$
3. $D^{\alpha}_{M,h}\ \left(\frac{f}{g}\right) = \frac{g\ D^{\alpha,a}_{M,h}f - f\ D^{\alpha,a}_{M,h}g}{g^2}$
4. $D^{\alpha}_{M,h}\left(C\right) = 0$
5. $D^{\alpha}_{M,h}\left(t^n\right) = \frac{n}{h'(t)}t^{n-\alpha-1}$

Proof:

$1 - D^{\alpha}_{M,h}\left(af+bg\right) = \lim_{\varepsilon\to 0} \frac{(af+bg)\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)-(af+bg)(t)}{\varepsilon}$

$$= \lim_{\varepsilon\to 0} \frac{a\left[f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)\right] + b\left[g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)\right]}{\varepsilon}$$

$$= a\lim_{\varepsilon\to 0} \frac{\left[f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)\right]}{\varepsilon} + b\lim_{\varepsilon\to 0} \frac{\left[g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)\right]}{\varepsilon}$$

$D^{\alpha}_{M,h}\left(af+bg\right) = aD^{\alpha,a}_{M,h}\ f + b\ D^{\alpha,a}_{M,h}g$

$2 - D^{\alpha}_{M,h}\left(f.g\right) = \lim_{\varepsilon\to 0} \frac{(f.g)\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)-(f.g)(t)}{\varepsilon}$

$$= \lim_{\varepsilon\to 0} \frac{f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right).g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right) - f\left(t\right).g(t)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{g\left(t\right)\left[f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)\right] + f\left(t\right)\left[g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)\right]}{\varepsilon}$$

$$= g\left(t\right)\lim_{\varepsilon\to 0} \frac{\left[f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)\right]}{\varepsilon} + f\left(t\right)\lim_{\varepsilon\to 0} \frac{\left[g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)\right]}{\varepsilon}$$

Therefore $D^{\alpha}_{M,h}\left(f.g\right) = f\ D^{\alpha,a}_{M,h}g + g\ D^{\alpha,a}_{M,h}\ f$

$3 - D^{\alpha}_{M,h}\ \left(\frac{f}{g}\right) = \lim_{\varepsilon\to 0} \frac{\left(\frac{f}{g}\right)\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)-\left(\frac{f}{g}\right)(t)}{\varepsilon} \ = \lim_{\varepsilon\to 0} \frac{\left(\frac{f\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)}{g\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)}\right)-\left(\frac{f(t)}{g(t)}\right)}{\varepsilon}$

$$= \lim_{\varepsilon\to 0} \frac{g\left(t\right)f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right) - f\left(t\right)g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right) - f\left(t\right)g\left(t\right) + f\left(t\right)g(t)}{\varepsilon\ g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)g(t)}$$

$$= \lim_{\varepsilon\to 0} \frac{g\left(t\right)\left[f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)\right] - f\left(t\right)\left[g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)\right]}{\varepsilon\ g\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)g(t)}$$

Therefore $D^{\alpha}_{M,h}\ \left(\frac{f}{g}\right) = \frac{g\ D^{\alpha,a}_{M,h}f - f\ D^{\alpha,a}_{M,h}g}{g^2}$

$4 - D^{\alpha}_{M,h}\left(C\right) == \lim_{\varepsilon\to 0}\frac{C-C}{\varepsilon}\ = \lim_{\varepsilon\to 0} 0\ = 0$

$5 - D^{\alpha}_{M,h}\left(t^n\right) = \lim_{\varepsilon\to 0} \frac{\left(h^{-1}\left(h(t)+\varepsilon t^{-\alpha}\right)\right)^n - t^n}{\varepsilon}\ = \frac{0}{0}$

Thus by using $L'H^{o}pital's$ rule we obtain

$$D_{M,h}^{\alpha}\left(t^{n}\right)=\lim_{\varepsilon\to 0}\frac{n\big(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\big)^{n-1}}{1}\;\frac{t^{-\alpha}}{h'\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)}$$

$D_{M,h}^{\alpha}\left(t^{n}\right)=\frac{nt^{n-\alpha-1}}{h'(t)}$

Theorem 3.3 *:*

Let f be α-M-differentiable functions a point $t0>0$, $\alpha\in(0,1)$ then

$$D_{M,h}^{\alpha}\left(f(t)\right)=\frac{t^{-\alpha}}{h'(t)}\;\frac{df}{dt}$$

Proof:

$$D_{M,h}^{\alpha}\left(f(t)\right)=\lim_{\varepsilon\to 0}\frac{f\left(h^{-1}\left(h\left(t\right)+\varepsilon t^{-\alpha}\right)\right)-f(t)}{\varepsilon}$$

Let $\tau=\frac{1}{h'(t_0)}\varepsilon {t_0}^{-\alpha}+o\left(\varepsilon^{2}\right)=\frac{1}{h'(t_0)}\varepsilon {t_0}^{-\alpha}\left(1+o\left(\varepsilon\right)\right)\to\varepsilon=\frac{\tau\;h'(t_0)t_0^{\alpha}}{1+o(\varepsilon)}$

$$D_{M,h}^{\alpha}\left(f(t)\right)=\lim_{\tau\to 0}\frac{f\left(t+\tau\right)-f(t)}{\frac{\tau\;h'(t_0)t_0^{\alpha}}{1+o(\varepsilon)}}\;=\frac{t^{-\alpha}}{h'(t)}\lim_{\tau\to 0}\frac{f\left(t+\tau\right)-f(t)}{\tau}\;=\frac{t^{-\alpha}}{h'(t)}\;\frac{df}{dt}$$

Corollary 3.1 *:*

Let g be α-M-differentiable functions a point $t0>0$, $\alpha\in(0,1)$ and f is differentiable function at $g\left(t_0\right)$, then

$$D_{M,h}^{\alpha}\left((fog)(t_0)\right)=f'\left(g\left(t_0\right)\right)\;D_{M,h}^{\alpha}g\left(t_0\right)$$

Proof:

$$D_{M,h}^{\alpha}\left(\left(fog\right)\left(t_0\right)\right)=\frac{{t_0}^{-\alpha}}{h'\left(t_0\right)}\;\frac{d\left(fog\right)\left(t_0\right)}{dt}=\frac{{t_0}^{-\alpha}}{h'\left(t_0\right)}\;f'\left(g\left(t_0\right)\right)\;g'\left(t_0\right)$$

$$=\;f'\left(g\left(t_0\right)\right)\;\;\frac{{t_0}^{-\alpha}}{h'\left(t_0\right)}g'\left(t_0\right)=f'\left(g\left(t_0\right)\right)\;D_{M,h}^{\alpha}g\left(t_0\right)$$

Definition 3.2 *:*

Let f be a continuous function and $\alpha\in(0,1)$, $a>0$ then:

$$I_{M,h}^{\alpha,a}\left(f\left(t\right)\right)\left(x\right)=\int_{a}^{x}t^{\alpha}\;h'\left(t\right)f\left(t\right)dt$$

Proposition 3.1 *:*

$$D_{M,h}^{\alpha}I_{M,h}^{\alpha,a}\left(f\left(t\right)\right)\left(x\right)=f(x)$$

Proof:

$$D_{M,h}^{\alpha}I_{M,h}^{\alpha,a}\left(f\left(t\right)\right)\left(x\right)=D_{M,h}^{\alpha}\int_{a}^{x}t^{\alpha}\;h'\left(t\right)f\left(t\right)dt$$

$$=\frac{x^{-\alpha}}{h'(x)}\;\frac{d}{dx}\left(\int_{a}^{x}t^{\alpha}\;h'\left(t\right)f\left(t\right)dt\right)=\frac{x^{-\alpha}}{h'(x)}x^{\alpha}\;h'\left(x\right)f\left(x\right)=f(x)$$

Proposition 3.2 *:*

$$I_{M,h}^{\alpha,a}D_{M,h}^{\alpha}\left(f\left(t\right)\right)\left(x\right)=f\left(x\right)-f(a)$$

Proof:

$$I_{M,h}^{\alpha,a}D_{M,h}^{\alpha}\left(f\left(t\right)\right)\left(x\right)=I_{M,h}^{\alpha,a}\left(\frac{x^{-\alpha}}{h'\left(x\right)}\;\frac{df}{dx}\right)=\int_{a}^{x}t^{\alpha}\;h'\left(t\right)\frac{t^{-\alpha}}{h'\left(t\right)}f^{'(t)}dt=\int_{a}^{x}f'\left(t\right)dt=\left.f(t)\right|_{a}^{x}=f\left(x\right)-f(a)$$

References

1. Abdeljawad T., AL Horani M. , and R. Khalil, *Conformable fractional semigroups of operators*, J. Semigroup Theory Appl., vol. 2015, p. Article ID 7, 2015.

2. Almeida R., A. B. Malinowska, and M. T. T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Mathematical Methods in the Applied Sciences, vol. 41, no. 1, pp. 336-352, 2018.

3. Caozong C., Ton J., *Fractional boundary value problems with Riemann-Liouville fractional derivatives*, Advances in Difference Equations, 2015, pp. 1-80.

4. Balachandran K. , M. Matar, N. Annapoorani, and D. Prabu, *Hadamard functional fractional integrals and derivatives and fractional differential equations*, Filomat, vol. 38, no. 3, pp. 779-792, 2024.

5. Balachandran K. , *An introduction to fractional differential equations.* Singapore: Springer, 2023

6. Farid G. , F. Rehman, Z. Javid, K. Shahzadi, A. Javed, and M. Ghamkhar, *Some New Fractional Integrals And Their Semigroup Properties*, Migration Letters, vol. 21, no. S11, pp. 1159-1168, 2024.

7. Hassan S.M., D.M. Hamid, M.H. Geem , *A Novel Approach using Residual Power series Method for solving nonlinear fractional partial differential equation*, submitted on journal the Bulletin of Parana´s Mathematical Society, named Boletim da Sociedade Paranaense de Matematica (BSPM), 2025.

8. Geem M. H. , *On strongly continuous ρh-semigroup*, in Journal of Physics: Conference Series, 2019, vol. 1234, no. 1: IOP Publishing.

9. Geem M. H. and A. M. Abbood, *On α-g-Transformation and its properties*, in American Institute of Physics Conference Series, 2023, vol. 2834, no. 1, p. 080103.

10. Hassan, S.M., Hamid, D.M. and Geem M.H., *A Novel Approach using Residual Power series Method for solving nonlinear fractional partial differential equation*, in Boletim Da Sociedade Paranaense De MatematicaOpen, 2025, vol. 43, no. 1, pp. 1-5.

11. Geem M.H. ,Hassan, Ahmed Raad and Neamah, Hayder Ismael,*0-Semigroup of g-transformation*,Journal of Interdisciplinary Mathematics, vol.28:1,2025, pp.311–316, DOI: 10.47974/JIM-1986

12. Iqbal S. , S. Mubeen, and M. Tomar, *On Hadamard k-fractional integrals* J. Fract. Calc. Appl, vol. 9, no. 2, pp. 255-267, 2018.

13. Ji S. and D. Yang, *Solutions to Riemann–Liouville fractional integrodifferential equations via fractional resolvents* Advances in Difference Equations, vol. 2019, no. 1, p. 524, 2019.

14. Kilbas A. A. , H. M. Srivastava, and J. J. Trujillo,*Theory and applications of fractional differential equations*. elsevier, 2006.

15. Kilbas R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative* Journal of Computational and Applied Mathematics, vol. 264, pp. 65–70, 2014.

16. KOCHUBEI A. ,*Handbook of fractional calculus with applications*. Berlin: Gruter, 2019.

17. Katugampola U. N. , *New approach to a generalized fractional integral* Applied mathematics and computation, vol. 218, no. 3, pp. 860-865, 2011.

18. Oliveira D. S. and E. Capelas de Oliveira, *On a Caputo-type fractional derivative," Advances in Pure and Applied Mathematics*, vol. 10, no. 2, pp. 81-91, 2019.

19. AL-Ameri Y., M. H. Geem, *On g-transformation*, Journal of university of Babylon,Pure and applied sciences, Vol29,2021.

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