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Comparative numerical study of the second-order boundary value problems

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ABSTRACT: In this paper, we present a comprehensive analytical investigation of second-order boundary value problems using the semi-analytical Homotopy Analysis Method (HAM). A key aspect of our approach involves determining the optimal value of the convergence control parameter $\hbar$ by analyzing the residual error associated with the approximate solution. This enables us to enhance both the accuracy and convergence of the method. To illustrate the applicability and effectiveness of HAM, several representative examples are provided, each demonstrating the method's flexibility and precision. Furthermore, a comparative study is conducted in which the results obtained via HAM are evaluated against those produced by other established numerical techniques, such as the B-spline method and finite difference approaches. The comparison clearly demonstrates the superior accuracy and robustness of HAM in solving second-order boundary value problems, thereby affirming its potential as a reliable tool for a wide range of applications, including higher-order and fractional differential equations.

Key Words: Homotopy Analysis Method (HAM), second-order boundary value problems, convergence control parameter, residual error, semi-analytical solution.

Contents

1. Introduction

In this paper, we study a class of second-order boundary value problems (BVPs) of the general form:

$$\begin{cases} u^{(2)}(x) = f(x, u, u'), & a \leq x \leq b, \\ u(a) = \alpha, & u(b) = \beta, \end{cases} \tag{1.1}$$

where f is a continuous function defined on the interval $[a, b]$, and α, β are real constants representing the prescribed boundary values.

Boundary value problems of this type play a critical role in modeling various physical, chemical, electrical, and engineering phenomena. These problems, particularly two-point BVPs, naturally arise in scenarios such as heat transfer, beam deflection, electrostatics, and fluid dynamics. Due to the prevalence and importance of such problems, a wide range of analytical and numerical techniques have been developed for their solution.

Among the classical and modern methods used to address second-order BVPs are the B-spline cubic method [2,3,4], the hybrid cubic B-spline method [5], the variational iteration method [6], and the sinc-collocation method [7]. Furthermore, the two-step method and the Runge-Kutta-Nyström method have been proposed and studied extensively by Athraa Abdulsalam [8]. While these techniques have proven effective for many linear and weakly nonlinear problems, they often encounter limitations when applied to strongly nonlinear systems, particularly with respect to convergence, stability, and accuracy.

Recent advances in polynomial-based approaches, such as truncated polynomial expansions [9], and generalized polynomial families (such as Apostol–type Frobenius–Euler polynomials [10] and discrete orthogonal U-Bernoulli Korobov-type polynomials [11]), have provided new tools to enhance accuracy

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and adaptability. Similarly, fractional calculus techniques, including those applied to epidemiological models [12] and nonlinear field equations [13], demonstrate the potential for extending classical BVP solvers to handle complex, memory-dependent, or singularly perturbed systems. However, a systematic comparison of these methods in the context of second-order BVPs remains an open area of investigation. We halso in numerical techniques for solving second-order boundary value problems have demonstrated significant improvements in accuracy and efficiency. Liu and Xu [14] established $L-2$ error estimates for the unsymmetric RBF collocation method in elliptic problems, providing a solid theoretical foundation for convergence analysis. Iqbal et al. [15] proposed a modified cubic B-spline basis with a free parameter, successfully applying it to various engineering problems. Similarly, Siraj-ul-Islam et al. [16] employed a collocation approach based on Haar wavelets, showcasing its effectiveness for a wide range of second-order boundary value problems.

The modeling of second-order boundary value problems is of central importance in scientific inquiry, providing a robust mathematical framework for describing complex dynamic systems. These models are typically governed by differential equations that encapsulate key relationships within a given system, thereby enabling deeper insights into the underlying physical phenomena. The second-order nature of these problems reflects the reality of many physical systems more accurately than first-order models, particularly in situations involving acceleration, curvature, or gradients of physical quantities.

In the context of scientific modeling, second-order BVPs form a crucial bridge between theoretical constructs and empirical data. Applications range from thermal conduction and elasticity to chemical reaction modeling and atmospheric dynamics. Their elegance lies in their ability to express a system's evolution while adhering to specified constraints at the boundaries, thereby capturing the full spectrum of system behavior within a defined domain. As such, the development of reliable and flexible solution techniques for these problems is not merely of academic interest but is fundamental to technological advancement and innovation.

In 1992, Shijun Liao introduced an innovative analytical approach known as the Homotopy Analysis Method (HAM) [17]. This method has since been successfully applied to a wide variety of nonlinear differential equations across numerous scientific and engineering fields. One of the key features of HAM is its ability to provide a convergent series solution that is independent of small parameters, which often limit the applicability of traditional perturbation methods. Furthermore, HAM introduces an auxiliary parameter, denoted by $\hbar$, that allows for the explicit control of the convergence rate and region of the solution series.

Liao and others have demonstrated the effectiveness and flexibility of HAM through its application to numerous nonlinear problems, including but not limited to ordinary differential equations (ODEs), partial differential equations (PDEs), and integral equations [18,19]. However, a challenge arises when applying HAM to BVPs, as the method typically requires initial conditions rather than boundary conditions. Specifically, HAM necessitates knowledge of the function and its derivative at the initial point, whereas boundary value problems impose constraints at two distinct points.

To address this discrepancy, we propose a modified approach for applying HAM to second-order BVPs. Consider the classical boundary condition:

$$u(a) = \alpha, \quad u(b) = \beta,$$

To apply HAM, we require an additional condition $u'(a) = \alpha'$ that is not initially given. Therefore, our approach begins by introducing the transformation:

$$\begin{aligned} u_1(x) &= u(x), \\ u_2(x) &= u'(x), \end{aligned}$$

which converts the second-order differential equation (1.1) into a first-order system:

$$\begin{cases} u_1'(x) = u_2(x), \\ u_2'(x) = f(x, u_1(x), u_2(x)), \end{cases} \tag{1.2}$$

with initial conditions given by:

$$u_1(a) = \alpha, \quad u_2(a) = \alpha'.$$

In this formulation, α' is treated as an unknown parameter. We apply the HAM with a fixed auxiliary parameter $\hbar = -1$, which yields a series solution depending on α'. Then, by enforcing the boundary condition $u(b) = \beta$, we iteratively determine the correct value of α' that ensures the series solution satisfies both boundary conditions. This effectively transforms the BVP into an initial value problem (IVP) suitable for HAM, while preserving the original boundary constraints through a shooting-like strategy.

This modification extends the applicability of HAM to a broader class of problems, including those with strong nonlinearities that challenge conventional numerical solvers. The approach maintains the key advantages of HAM—namely, analytical tractability and convergence control—while enabling its use in scenarios previously inaccessible due to boundary condition incompatibilities.

The remainder of this paper is organized as follows. In Section 2, we provide a detailed description of the modified HAM technique and the construction of the homotopy. Section 3 presents illustrative examples to demonstrate the accuracy and efficiency of the proposed method. Finally, conclusions and directions for future work are outlined in Section 4.

2. Description of method

In this section, we will describe HAM, using the transformation:

$$u^{(i)}(x) = u_{i+1}(x), \qquad i = 0, 1 \tag{2.1}$$

we rewrite the second-order initial value problem (1.1) as the system of ordinary differential equations:

$$\begin{cases} u_1'(x) &= u_2(x) \\ u_2'(x) &= f(x, u_1(x), u_2(x)) \end{cases} \tag{2.2}$$

with the initial conditions

$$u_i(a) = u^{(i-1)}(a) \qquad i = 1, 2 \tag{2.3}$$

let consider the following equation

$$\mathfrak{N}_i[u_i(x)] = g_i(x), \qquad i = 1, 2 \tag{2.4}$$

where $\mathfrak{N}_i$ is the nonlinear operators, $u_i(x)$ are unknown functions and $g_i(x)$ is a known function.
The zeroth-order deformation equation of HAM constructed by Liao [17], [18], and [19] is:

$$(1 - q)\mathfrak{L}[\phi_i(x, q) - y_{i,0}(x)] = \hbar_i q H_i(x)\{\mathfrak{N}_i[\phi_i(x, q) - g_i(x)]\}, \tag{2.5}$$

where $\mathfrak{L}_i, i = 1, 2$ is the auxiliary linear operator, $q \in [0, 1]$, $\hbar_i \neq 0$ is the convergence-control parameter, and the auxiliary function $H_i(x)$.
Obviously, when $q = 0$ and $q = 1$ one has:

$$\phi_i(x, 0) = u_{i,0}(x) \quad \textit{and} \quad \phi_i(x, 1) = U_i(x), \qquad i = 1, 2, \tag{2.6}$$

thus as q increases from 0 to 1, the solution $\phi_i(x, q)$ varies from the initial guesses $u_{i,0}(x)$ to the exact solutions $u_i(x)$.
We expand in Taylor series $\phi_i(x, q)$ with respect to q:

$$\phi_i(x, q) = u_{i,0}(x) + \sum_{m=1}^{+\infty} u_{i,m}(x) q^m, \tag{2.7}$$

where

$$u_{i,m}(x) = \frac{1}{m!} \frac{\partial^m \phi_i(x, q)}{\partial q^m}|_{q=0}, \tag{2.8}$$

if $\mathfrak{L}$, $u_{i,0}(x)$, and the auxiliary function are so properly chosen, the series solutions (2.7) converge at $q = 1$, and we have

$$\phi_i(x, 1) = u_{i,0}(x) + \sum_{i=1}^{+\infty} u_{i,m}(x),$$

the optimal value of $\hbar_i$ is determined using the residual error

$$\Delta(\hbar) = \int N(u_n(x))^2 dx \tag{2.9}$$

which must be of the solutions of (1.1).
The mth-order deformation equation is

$$\mathfrak{L}[u_{i,m}(x) - \chi_m u_{i,m-1}(x)] = \hbar_i H_i(x) R_{i,m}(\overrightarrow{u}_{i,m-1}), \tag{2.10}$$

where

$$R_{i,m}(\overrightarrow{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\phi_i(x,q)]}{\partial q^{m-1}} \tag{2.11}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

3. Numerical Simulations and Comparative Analysis

In this section, we implement the Homotopy Analysis Method (HAM) to solve a series of problems. To evaluate its efficacy, we compare the numerical results obtained from HAM with those generated by the B-spline method [1], analyzing key metrics such as convergence rate, computational efficiency, and residual error.

Example 3.1 *Let us consider the following two-order boundary value problem*

$$u^{''}(x) - u(x) = -e^{x-1} - 1, \qquad 0 \leq x \leq 1 \tag{3.1}$$

with:

$$u(0) = 0, \quad u(1) = 0 \tag{3.2}$$

The exact solution is

$$u(x) = x(1 - e^{x-1})$$

Using (2.1) we reformulate (3.1) as the following system:

$$\begin{cases} u_1'(x) &= u_2(x) \\ u_2'(x) &= u_2(x) - e^x(35 + 12x + 2x^2) \end{cases} \tag{3.3}$$

with the initial conditions

$$u_1(0) = 0, \quad u_2(0) = \alpha$$

We choose the initial guesses as

$$u_{1,0}(0) = 0 \quad u_{2,0}(0) = \alpha$$

and the linear operator

$$\mathfrak{L}[\phi_i(x,q)] = \frac{\partial \phi_i(x,q)}{\partial x}, \qquad i = 1, 2 \tag{3.4}$$

Furthermore, the operators N_i corresponding to equations (3.3) are defined as:

$$\begin{cases} \mathfrak{N}_1[\phi_1(x,q), \phi_1(x,q)] &= \frac{\partial \phi_1(x,q)}{\partial x} - \phi_2(x), \\ \mathfrak{N}_2[\phi_1(x,q), \phi_1(x,q)] &= \frac{\partial \phi_2(x,q)}{\partial x} - \phi_1(x,q) + e^{x-1} + 1. \end{cases} \tag{3.5}$$

The zeroth-order deformation equation is

$$(1-q)\mathfrak{L}[\phi_i(x,q) - u_{i,0}(x)] = \hbar_i q H(x)\{\mathfrak{N}_i[\phi_i(x,q) - g_i(x)]\}, \qquad i = 1, 2, \tag{3.6}$$

and the mth-order deformation equation is

$$\mathfrak{L}[u_{i,m}(x) - \chi_m u_{i,m-1}(x)] = \hbar_i R_{i,m}(\overrightarrow{u}_{i,m-1}) \tag{3.7}$$

with the initial conditions

$$u_{1,0}(0) = 0 \quad u_{2,0}(0) = 1 \tag{3.8}$$

where

$$\begin{cases} R_{1,m}(\overrightarrow{u}_{i,m-1}) & = & u'_{1,m-1}(x) - u_{2,m-1}(x), \\ R_{2,m}(\overrightarrow{u}_{i,m-1}) & = & u'_{2,m-1}(x) - u_{1,m-1}(x) + e^x(35 + 12x + 2x^2) \end{cases} \tag{3.9}$$

Now, the solution of the mth-order deformation Eq. (3.7) for $m \geq 1$ becomes

$$u_{i,m}(x) = \chi_m u_{i,m-1}(x) + \hbar_i \int_0^1 R_{i,m}(\overrightarrow{u}_{i,m-1})dx, \qquad i = 1, 2 \tag{3.10}$$

According to the homotopy analysis method with $\hbar = -1$ and $c = 1$, using the boundary condition 3.2 at $x = 1$, we get:

$$\alpha \simeq 0.632120546609618...$$

HAM solution is written in the form

$$\begin{aligned} u(x) &= \textstyle\sum_{i=0}^{\infty} u_i(x) = u_0(x) + u_1(x) + u_2(x) + ... \\ &= 3.48303 - 3.48303e^x + 4.11515x + 1.37364x^2 + 0.396565x^3 + 0.083813x^4 + 0.013697x^5 + ... \end{aligned}$$

To determine the valid range of the convergence control parameter $\hbar$, we plot the values of $u'_\hbar(0)$ and $u''_\hbar(0)$ as functions of $\hbar$. This graphical representation is commonly referred to as the $\hbar$-curve, which is illustrated in Figure 1.

In Table 1, we present the numerical results obtained using the Homotopy Analysis Method (HAM), along with the corresponding absolute errors. These results are compared with those derived from the B-spline method as reported in [1]. It is evident from the comparison that the HAM solution exhibits better agreement with the exact solution than the B-spline method.

Furthermore, Table 2 provides a comparative analysis of the absolute errors obtained by several numerical techniques. Specifically, we compare the performance of HAM with the New Cubic B-Spline method [1], the Finite Difference Method (FDM) [1], the Finite Element Method (FEM) [1], the Finite Volume Method (FVM) [1], and the B-Spline Interpolation method (BSI) [1].

Table 1 *HAM approximation with a comparison of the obtained error with B-spline method*

x	*Exacte solution*	*HAM solution*	*Error(HAM)*	*Error(B-spline method) [1]*
0	0	-4.623×10^{-15}	4.623×10^{-15}	0
0.1	0.059343034	0.059343032	1.285×10^{-9}	8.095×10^{-8}
0.2	0.110134207	0.110134204	2.705×10^{-9}	1.699×10^{-7}
0.3	0.151024408	0.151024404	4.275×10^{-9}	2.447×10^{-7}
0.4	0.180475345	0.180475339	6.009×10^{-9}	3.030×10^{-7}
0.5	0.196734670	0.196734662	7.927×10^{-9}	3.398×10^{-7}
0.6	0.197807972	0.197807962	1.005×10^{-8}	3.495×10^{-7}
0.7	0.181427245	0.422891026	1.239×10^{-8}	3.254×10^{-7}
0.8	0.145015397	0.356056460	1.497×10^{-8}	2.600×10^{-7}
0.9	0.085646323	0.221281999	1.783×10^{-8}	1.392×10^{-7}
1	1	-0.00014345	2.098×10^{-8}	0

Table 2 *A comparison of maximum error for different value of N*

N	*FDM*	*FEM*	*FVM*	*B-spline*	*HAM*
10	8.24×10^{-5}	6.35×10^{-5}	3.18×10^{-5}	3.50×10^{-7}	
100	8.31×10^{-7}	6.36×10^{-7}	3.18×10^{-7}	3.74×10^{-11}	1.98×10^{-21}
1000	8.31×10^{-9}	6.39×10^{-9}	3.18×10^{-9}	6.76×10^{-14}	

Example 3.2 *Now we consider the following problem :*

$$u^{''}(x) - u(x) = 2e^{x-1}, \qquad 0 \leq x \leq 1, \tag{3.11}$$

with the boundary conditions:

$$u(0) = 0 \quad u(1) = 1, \tag{3.12}$$

the exact solution :

$$u(x) = xe^{x-1}$$

Using (2.1), we obtain:

$$\begin{cases} u_1'(x) &= u_2(x) \\ u_2'(x) &= u_1(x) + 2e^{x-1}, \end{cases} \tag{3.13}$$

with the initial conditions

$$u_1(0) = 1 \quad u_2(0) = \alpha \tag{3.14}$$

Applying HAM with $\hbar = -1$, and using the boundary conditions at $x = 1$, we get:

$$\alpha \simeq 0.36787945048713....$$

Table3 show that the HAM solution coincide with the exact solution more than B-spline method with $h = 1/10$. In table 4 a comparison wth the maximum error obtained by HAM and B-spline, LSM, FDM, BSI methods.

Table 3 *Comparison between HAM and B-Spline Solutions.*

x	*Exacte solution*	*HAM solution*	*Absolute error HAM*	*B-spline [1]*
0	0	1.13173×10^{-13}	1.13173×10^{-13}	0
0.1	0.040656966	0.040656966	9.329×10^{-10}	2.281×10^{-8}
0.2	0.089865849	0.089865794	1.874×10^{-9}	5.619×10^{-8}
0.3	0.148975671	0.148975593	2.836×10^{-9}	8.063×10^{-8}
0.4	0.219524751	0.219524658	3.826×10^{-9}	9.735×10^{-8}
0.5	0.303265435	0.303265334	4.857×10^{-9}	1.052×10^{-7}
0.6	0.402192130	0.402192033	5.947×10^{-9}	1.032×10^{-7}
0.7	0.518572844	0.518572761	7.122×10^{-9}	8.983×10^{-8}
0.8	0.654984666	0.654984610	8.379×10^{-9}	6.398×10^{-8}
0.9	0.814353695	0.814353685	9.540×10^{-9}	1.887×10^{-8}
1	1	1	0	0

Table 4 *a comparison of maximum error between HAM and LSM, FDM, BSI and B-spline method*

N	*LSM*	*FDM*	*BSI*	*B-spline*	*HAM*
10	3.66×10^{-7}	2.66×10^{-4}	2.66×10^{-4}	1.05×10^{-7}	3.082×10^{-15}
100	4.01×10^{-11}	2.68×10^{-6}	2.68×10^{-6}	1.28×10^{-11}	3.082×10^{-15}

3.1. Concluding remarks

In this paper, the Homotopy Analysis Method (HAM) has been successfully applied to solve a second-order boundary value problem. The results demonstrate that HAM is not only efficient but also highly applicable for this class of differential equations.

Our analysis includes a graphical investigation which highlights the significant role played by the convergence control parameter $\hbar$. The behavior of the solution is shown to depend sensitively on the choice of this parameter.

We have also performed a comparative study between the results obtained using HAM and those derived from the B-spline method. The comparison indicates that HAM provides more accurate approximations to the exact solution.

Based on these findings, we conclude that HAM is an effective and reliable analytical technique. Moreover, its flexibility suggests that it can be extended to solve boundary value problems of higher order, as well as those involving fractional differential equations.

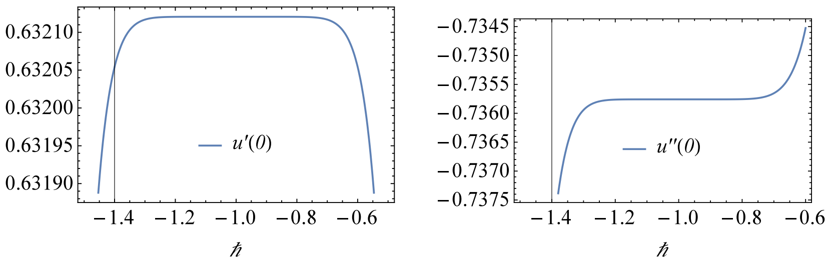

Figure 1: The $\hbar-$curve of $u_h'(0)$, and $u_h''(0)$ for (3.1)

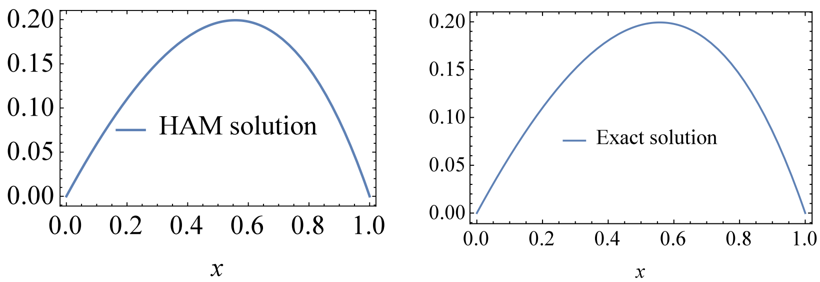

Figure 2: The HAM solution (left), and exact solution (right) for (3.1)

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