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Some Properties of the Interval Quadripartitioned Neutrosophic Sets

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ABSTRACT: The Interval Quadripartitioned Neutrosophic Set (IQNS) is the combination of the quadripartitioned neutrosophic set and interval neutrosophic set. IQNS plays an important role as a mathematical tool to deal with real-life problems involving uncertainty and indeterminacy. In this paper, We define some set-theoretic operations of IQNSs namely, the symmetric difference, and prove some of their properties. We also define the convexity criteria of IQNSs and prove their properties.

Keywords: Neutrosophic set, single valued neutrosophic set, interval neutrosophic set, quadripartitioned neutrosophic set, fuzzy set, interval quadripartitioned neutrosophic set, convexity.

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1. Introduction

Neutrosophic Sets (NSs) [1] and Single Valued NSs(SVNSs) [2] are proposed to capture uncertainty convincingly by extending the Fuzzy Sets (FSs)[3] and intuitionstic FSs [4]. NS [1] was further extended to Quadripatitioned NS (QNS)[5] and Pentapatitioned NS (PNS)[6]. Combining Interval NS (INS) [7] with QNS [5], Interval QNS (IQNS)[8] was proposed by Pramanik. Similarly, combining INS [7] with PNS [6], Interval PNS (IPNS) [9] was grounded by Pramanik. Overview of NSs and their extensions were depicted in several studies [10,11, 12, 13, 14, 15, 16, 17].
Decision making has been extensively studied in SVNS environments [18, 19], QNS environments [20, 21, 22] and PNS environments [24, 25, 26, 27,28, 29, 30, 31]. SVNSs have been used in water quality assessment [32]. Graph theory [33] in PNS was introduced by Das et al. NSs and their extensions have been explored in algebra [34, 35, 36], hiperalgebra [37] and topological spcae [38, 39, 40, 41, 42, 43]. Pramanik [8] presented some basic properties of IQNSs. Further improvements such as convexity, truth favorite, false favorite, and their properties were not studied in IQNS environment.

1.1. Research Gap

Stduies relating to convexity, truth favorite, false favorite, and their properties are not reported in the literature in IQNS environments.

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1.2. Motivation

To fill the research gap, we initiate to explore convexity, truth favorite, false favorite, and their properties for IQNSs.
Rest of the paper is organized as follows:
Section 2 presents preliminaries of IQNSs.
Section 3 presents the properties of IQNSs and their proofs.
Section 4 presents future research directions. Section 5 concludes the paper.

2. Preliminaries

In this section, we recall some basic definitions related to NSs, SVNSs, INSs, and IQNSs.

Definition 2.1 ***NS*** [1] *Let* H *be a NS in a universe of discourse* $V = (v_1, v_2, \ldots, v_n)$. H *can be defined as*
$H = \{\langle T_H(v), I_H(v), F_H(v)\rangle \mid v \in V\}$.
where the functions $T_H(v)$, $I_H(v)$ *and* $F_H(v)$ *respectively denote the truth, indeterminacy and falsity membership functions of* $v \in V$ *to the set* H *satisfy the conditions*
$^-0 \leq T_H(v) \leq 1^+$, $^-0 \leq I_H(v) \leq 1^+$, $^-0 \leq F_H(v) \leq 1^+$, $^-0 \leq T_H(v) + I_H(v) + F_H(v) \leq 3^+$.

Definition 2.2 ***SVNS*** [2] *Let* V *be a fixed set. A SVNS* H *over* V *is defined as:*
$H = \{\langle v, (T_H(v), I_H(v), F_H(v)\ \rangle : v \in V\}$ *where* $T_H(v)$, $I_H(v)$, $F_H(v)$ *refer to the degree of the truth, indeterminancy and falsity membership functions respectively.* $T_H(v) : V \to [0,1]$, $I_H(v) : V \to [0,1]$, $F_H(v) : V \to [0,1]$ *and* $0 \leq T_H(v) + I_H(v) + F_H(v) \leq 3$.

Definition 2.3 ***INS*** [7] *Let* V *be a fixed set. An INS* H *over* V *is defined as:*
$H = \{\langle v, (T_H(v), I_H(v), F_H(v))\rangle \mid v \in V\}$
$=\{\langle v, ([\inf T_H(v), \sup T_H(v)], [\inf I_H(v), \sup I_H(v)\ ], [\inf F_H(v), \sup F_H(v)])\rangle \mid v \in V\}$
where each $v \in V$, $T_H(v)$, $I_H(v)$, $F_H(v)$ *denote the degree of the truth, indeterminancy, and falsity membership functions respectively.*
$T_H(v) = [\inf T_H(v), \sup T_H(v)] \subseteq [0,1]$, $I_H(v) = [\inf I_H(v), \sup I_H(v)] \subseteq [0,1]$,
$F_H(v) = [\inf F_H(v), \sup F_H(v)] \subseteq [0,1]$, *and* $0 \leq \sup I_H(v) + \sup I_H(v) + \sup F_H(v) \leq 3$.

Definition 2.4 ***QSVNS*** [5] *Let* V *be a fixed set on which a QSVNS* H *is defined. Then* H *can be expressed as:*
$H = \{\langle v, (T_H(v), C_H(v), U_H(v), F_H(v))\rangle \mid v \in V\}$, *where* $T_H(v)$, $C_H(v)$, $U_H(v)$, *and* $F_H(v)$ *denote the degree of truth, contradiction, ignorance and falsity membership functions respectively and*
$0 \leq \sup T_H(v) + \sup C_H(v) + \sup U_H(v) + \sup F_H(v) \leq 4$.

Definition 2.5 ***IQNS*** [8] *Let* H *be a IQNS in a universe of discourse and* $V = (v_1, v_2, \ldots, v_n)$ *then* H *is defined and symbolized as :*
$H=\{\langle v, (T_H(v), C_H(v), U_H(v), F_H(v))\rangle \mid v \in V\}$ *where* $T_H(v)$, $C_H(v)$, $U_H(v)$, $F_H(v)$ *are the degree of truth, contradiction, ignorance and falsity of membership function respectively and*
$T_H(v) = [\inf T_H(v), \sup T_H(v)] \subseteq [0,1]$, $C_H(v) = [\inf C_H(v), \sup C_H(v)] \subseteq [0,1]$,
$U_H(v) = [\inf U_H(v), \sup U_H(v)] \subseteq [0,1]$, $F_H(v) = [\inf F_H(v), \sup F_H(v)] \subseteq [0,1]$,
$0 \leq \sup T_H(v) + \sup C_H(v) + \sup U_H(v) + \sup F_H(v) \leq 4$.

Definition 2.6 ***Containment*** [8] *Let* J *and* H *be any two IQNSs over a universe of discourse* V*.Then* J *is said to be contained in* H *and symbolized as* $J \subseteq H$ *iff for any* $v \in V$, *then*

$$\begin{gathered}
\inf T_J(v) \leq \inf T_H(v),\ \sup T_J(v) \leq \sup T_H(v),\\
\inf C_J(v) \leq \inf C_H(v),\ \sup C_J(v) \leq \sup C_H(v),\\
\inf U_J(v) \geq \inf U_H(v),\ \sup U_J(v) \geq \sup U_H(v),\\
\inf F_J(v) \geq \inf F_H(v),\ \sup F_J(v) \geq \sup F_H(v).
\end{gathered}$$

Definition 2.7 ***Intersection*** [8] *Let J and H be any two IQNSs over V. Then $W = J \cap H$ is also an IQNS over V, and defined as* $W = J \cap H = \{\langle v$, *[inf* $T_W(v)$, *sup* $T_W(v)]$, *[inf* $C_W(v)$, *sup* $C_W(v)]$, *[inf* $U_W(v)$, *sup* $U_W(v)]$, *[inf* $F_W(v)$, *sup* $F_W(v)] \rangle \mid v \in V\}$ *where,*
inf $T_W(v) = min($*inf* $T_J(v)$, *inf* $T_H(v))$, *sup* $T_W(v) = min($*sup* $T_J(v)$, *sup* $T_H(v))$,
inf $C_W(v) = min($*inf* $C_J(v)$, *inf* $C_H(v))$, *sup* $C_W(v) = min($*sup* $C_J(v)$, *sup* $C_H(v))$,
inf $U_W(v) = max($*inf* $T_J(v)$, *inf* $T_H(v))$, *sup* $U_W(v) = max($*sup* $U_J(v)$, *sup* $U_H(v))$,
inf $F_W(v) = max($*inf* $F_J(v)$, *inf* $F_H(v))$, *sup* $F_W(v) = max($*sup* $F_J(v)$, *sup* $F_H(v))$,
for every $v \in V$.

Definition 2.8 ***Union*** [8] *Let J and H be any two IQNSs over V. Then $W^* = J \cup H$ is also an IQNS over V, and defined as* $W^* = J \cup H$
$= \{\langle v$, *[inf* $T_W*(v)$, *sup* $T_W*(v)]$, *[inf* $C_W*(v)$, *sup* $C_W*(v)]$, *[inf* $U_W*(v)$, *sup* $U_W*(v)]$, *[inf* $F_W*(v)$, *sup* $F_W*(v)] \rangle \mid v \in V\}$
where inf $T_W*(v) = max($*inf* $T_J(v)$, *inf* $T_H(v))$, *sup* $T_W*(v) = max($*sup* $T_J(v)$, *sup* $T_H(v))$,
inf $C_W*(v) = max($*inf* $C_J(v)$, *inf* $C_H(v))$, *sup* $C_W(v) = max($*sup* $C_J(v)$, *sup* $C_H(v))$,
inf $U_W*(v) = min($*inf* $T_J(v)$, *inf* $T_H(v))$, *sup* $U_W(v) = min($*sup* $U_J(v)$, *sup* $U_H(v))$,
inf $F_W*(v) = min($*inf* $F_J(v)$, *inf* $F_H(v))$, *sup* $F_W(v) = min($*sup* $F_J(v)$, *sup* $F_H(v))$,
for every $v \in V$.

3. Basic theories of IQNSs

Theorem 3.1 *Let H_1 and H_2 be any two IQNSs. Then $H_1 \cup H_2$ is the smallest set containing both H_1 and H_2.*

Proof: : Let $H = H_1 \cup H_2$.
So Inf $T_H = \max($Inf T_{H_1}, Inf $T_{H_2}) \implies$ Inf $T_H \geq$ Inf T_{H_1} and Inf $T_H \geq$ Inf T_{H_2},
Sup $T_H = \max($Sup T_{H_1}, Sup $T_{H_2}) \implies$ Sup $T_H \geq$ Sup T_{H_1}, and Sup $T_H \geq$ Sup T_{H_2},
Inf $C_H = \max($Inf C_{H_1}, Inf $C_{H_2}) \implies$ Inf $C_H \geq$ Inf C_{H_1} and Inf $C_H \geq$ Inf C_{H_2},
Sup $C_H = \max($Sup C_{H_1}, Sup $C_{H_2}) \implies$ Sup $C_H \geq$ Sup C_{H_1}, and Sup $C_H \geq$ Sup C_{H_2},
Inf $U_H = \min($Inf U_{H_1}, Inf $U_{H_2}) \implies$ Inf $U_H \leq$ Inf U_{H_1} and Inf $U_H \leq$ Inf U_{H_2},
Sup $U_H = \min$ (Sup U_{H_1}, Sup $U_{H_2}) \implies$ Sup $U_H \leq$ Sup U_{H_1}, and Sup $U_H \leq$ Sup U_{H_2},
Inf $F_H = \min($Inf F_{H_1}, Inf $F_{H_2}) \implies$ Inf $F_H \leq$ Inf F_{H_1} and Inf $F_H \leq$ Inf F_{H_2}.
Furthermore if J be any set containing both H_1 and H_2 then
Inf $T_J \geq$ Inf T_{H_1} and Inf $T_J \geq$ Inf T_{H_2},
Inf $T_J \geq \max($ Inf T_{H_1}, Inf T_{H_2} $) =$ Inf T_H, So Inf $T_J \geq$ Inf T_H.
Sup $T_J \geq$ Sup T_{H_1} and Sup $T_J \geq$ Sup T_{H_2}. Sup $T_J \geq \max($ Sup T_{H_1}, Sup T_{H_2} $) =$ Sup T_H. Sup $T_J \geq$ Sup T_H.
Inf $C_J \geq$ Inf C_{H_1} and Inf $C_J \geq$ Inf C_{H_2}, Inf $C_J \geq \max($ Inf C_{H_1}, Inf C_{H_2} $) =$ Inf C_H.
So Inf $C_J \geq$ Inf C_H.
Sup $C_J \geq$ Sup C_{H_1} and Sup $C_J \geq$ Sup C_{H_2}. Sup $C_J \geq \max($ Sup C_{H_1}, Sup C_{H_2} $) =$ Sup C_H. Sup C_J $\geq$ Sup C_H.
Inf $U_J \leq$ Inf U_{H_1} and Inf $U_J \leq$ Inf U_{H_2}, Inf $U_J \leq \min($ Inf U_{H_1}, Inf $U_{H_2}) =$ Inf U_H,
So Inf $U_J \leq$ Inf C_H.
Sup $U_J \leq$ Sup U_{H_1} and Sup $U_J \leq$ Sup U_{H_2}. Sup $U_J \leq \min($ Sup U_{H_1}, Sup $U_{H_2}) =$ Sup U_H. Sup $U_J \leq$ Sup U_H.
Inf $F_J \leq$ Inf F_{H_1} and Inf $F_J \leq$ Inf F_{H_2}, Inf $F_J \leq \min($ Inf F_{H_1}, Inf $F_{H_2}) =$ Inf F_H,
So Inf $F_J \leq$ Inf F_H.
Sup $F_J \leq$ Sup F_{H_1} and Sup $F_J \leq$ Sup F_{H_2}. Sup $F_J \leq \min($ Sup F_{H_1}, Sup F_{H_2} $) =$ Sup F_H. Sup $F_J \leq$ Sup F_H
That implies $H \subseteq J$.
Q.E.D □

Theorem 3.2 *Let H_1 and H_2 be two IQNSs. Then $H_1 \cap H_2$ is the largest set containing both H_1 and H_2.*

Proof: Let $H = H_1 \cap H_2$. So Inf $T_H = \min(\text{Inf } T_{H_1}, \text{Inf } T_{H_2})$.
Therefore Inf $T_H \leq$ Inf T_{H_1} and Inf $T_H \leq$ Inf T_{H_2}.
Sup $T_H = \min(\text{Sup } T_{H_1}, \text{Sup } T_{H_2})$. So Sup $T_H \leq$ Sup T_{H_1} and Sup $T_H \leq$ Sup T_{H_2}.
Inf $C_H = \min(\text{Inf } C_{H_1}, \text{Inf } C_{H_2})$. Therefore Inf $C_H \leq$ Inf C_{H_1} and Inf $C_H \leq$ Inf C_{H_2}.
Sup $C_H = \min(\text{Sup } C_{H_1}, \text{Sup } C_{H_2})$.
Therefore Sup $C_H \leq$ Sup C_{H_1}, and Sup $C_H \leq$ Sup C_{H_2}.
Inf $U_H = \max(\text{Inf } U_{H_1}, \text{Inf } U_{H_2})$. Therefore Inf $U_H \geq$ Inf U_{H_1} and Inf $U_H \geq$ Inf U_{H_2}.
Sup $U_H = \max(\text{Sup } U_{H_1}, \text{Sup } U_{H_2})$. Therefore Sup $U_H \geq$ Sup U_{H_1} and Sup $U_H \geq$ Sup U_{H_2}.
Inf $F_H = \max(\text{Inf } F_{H_1}, \text{Inf } F_{H_2})$. Therefore Inf $F_H \geq$ Inf F_{H_1} and Inf $F_H \geq$ Inf F_{H_2}.
Sup $F_H = \max(\text{Sup } F_{H_1}, \text{Sup } F_{H_2})$. Therefore Sup $F_H \geq$ Sup F_{H_1} and Sup $F_H \geq$ Sup F_{H_2}.
Furthermore if J be any set contained in both H_1 and H_2 then
Inf $T_J \leq$ Inf T_{H_1} and Inf $T_J \leq$ Inf T_{H_2},
Inf $T_J \leq \min(\text{Inf } T_{H_1}, \text{Inf } T_{H_2}) = \text{Inf } T_H$, So Inf $T_J \leq$ Inf T_H.
Sup $T_J \leq$ Sup T_{H_1} and Sup $T_J \leq$ Sup T_{H_2}. Sup $T_J \leq \min(\text{Sup } T_{H_1}, \text{Sup } T_{H_2}) = \text{Sup } T_H$. Sup $T_J \leq$ Sup T_H.
Inf $C_J \leq$ Inf C_{H_1} and Inf $C_J \leq$ Inf C_{H_2}, Inf $C_J \leq \min(\text{Inf } C_{H_1}, \text{Inf } C_{H_2}) = \text{Inf } C_H$.
So Inf $C_J \leq$ Inf C_H.
Sup $C_J \leq$ Sup C_{H_1} and Sup $C_J \leq$ Sup C_{H_2}. Sup $C_J \leq \min(\text{Sup } C_{H_1}, \text{Sup } C_{H_2}) = \text{Sup } C_H$. Sup $C_J \leq$ Sup C_H.
Inf $U_J \geq$ Inf U_{H_1} and Inf $U_J \geq$ Inf U_{H_2}, Inf $U_J \geq \max(\text{Inf } U_{H_1}, \text{Inf } U_{H_2}) = \text{Inf } U_H$.
So Inf $U_J \geq$ Inf C_H.
Sup $U_J \geq$ Sup U_{H_1} and Sup $U_J \geq$ Sup U_{H_2}. Sup $U_J \geq \max(\text{Sup } U_{H_1}, \text{Sup } U_{H_2}) = \text{Sup } U_H$. Sup $U_J \geq$ Sup U_H.
Inf $F_J \geq$ Inf F_{H_1} and Inf $F_J \geq$ Inf F_{H_2}, Inf $F_J \geq \max(\text{Inf } F_{H_1}, \text{Inf } F_{H_2}) = \text{Inf } F_H$.
So Inf $F_J \geq$ Inf F_H.
Sup $F_J \geq$ Sup F_{H_1} and Sup $F_J \geq$ Sup F_{H_2}. Sup $F_J \geq \max(\text{Sup } F_{H_1}, \text{Sup } F_{H_2}) = \text{Sup } F_H$. Sup $F_J \geq$ Sup F_H. That implies $J \subseteq H$.
Q.E.D. □

Definition 3.1 *Let J be an IQNS. Then J is said to be* ***convex*** *if and only if*

$$\begin{gathered}
inf\, T_J(\lambda x + (1\text{-}\lambda)y) \geq min\ (inf\, T_J(x),\ inf\, T_J(y)), \\
sup\ T_J(\lambda x + (1\text{-}\lambda)y) \geq min\ (sup\ T_J(x),\ sup\ T_J(y)), \\
inf\, C_J(\lambda x + (1\text{-}\lambda)y) \geq min\ (inf\, C_J(x),\ inf\, C_J(y)), \\
sup\ C_J(\lambda x + (1\text{-}\lambda)y) \geq min\ (sup\ C_J(x),\ sup\ C_J(y)), \\
inf\, U_J(\lambda x + (1\text{-}\lambda)y) \leq max\ (inf\, U_J(x),\ inf\, U_J(y)), \\
sup\ U_J(\lambda x + (1\text{-}\lambda)y) \leq max\ (supU_J\ \ (x),\ sup\ U_J(y)), \\
inf\, F_J(\lambda x + (1\text{-}\lambda)y) \leq max\ (inf\, F_J(x),\ inf\, F_J(y)), \\
sup\ F_J(\lambda x + (1\text{-}\lambda)y) \leq max\ (sup\ F_J(x),\ sup\ F_J(y)),
\end{gathered}$$

$\forall\ x,\ y \in X$ where X is a real Euclidian space E^n for correctness and all $\lambda \in$ (0, 1).

Definition 3.2 *An IQNS J is said to be* ***STRONGLY CONVEX*** *if for any two points x and y ($x \neq y$), and any $\lambda \in$ (0,1),*

$$\begin{gathered}
inf\, T_J(\lambda x + (1\text{-}\lambda)y) > min\ (inf\, T_J(x),\ inf\, T_J(y)), \\
sup\ T_J(\lambda x + (1\text{-}\lambda)y) > min\ (sup\ T_J(x),\ sup\ T_J(y)), \\
inf\, C_J(\lambda x + (1\text{-}\lambda)y) > min\ (inf\, C_J(x),\ inf\, C_J(y)), \\
sup\ C_J(\lambda x + (1\text{-}\lambda)y) > min\ (sup\ C_J(x),\ sup\ C_J(y)), \\
inf\, U_J(\lambda x + (1\text{-}\lambda)y) < max\ (inf\, U_J(x),\ inf\, U_J(y)), \\
sup\ U_J(\lambda x + (1\text{-}\lambda)y) < max\ (sup\ U_J(x),\ sup\ U_J(y)), \\
inf\, F_J(\lambda x + (1\text{-}\lambda)y) < max\ (inf\, F_J(x),\ inf\, F_J(y)), \\
sup\ F_J(\lambda x + (1\text{-}\lambda)y) < max\ (sup\ F_J(x),\ sup\ F_J(y)),
\end{gathered}$$

$\forall\ x,\ y \in X$ where X is a real Euclidian space E^n for correctness and all $\lambda \in$ (0, 1).

Theorem 3.3 *If J and H are any two convex IQNSs, then $J \cap H$ is convex.*

Proof: Let $V = J \cap H$, then
inf $T_V(\lambda x + (1\text{-}\lambda)y)$ = min(inf $T_J(\lambda x + (1\text{-}\lambda)y)$, inf $T_H(\lambda x + (1\text{-}\lambda)y)$),
sup $T_V(\lambda x + (1\text{-}\lambda)y)$ = min(sup$T_J(\lambda x + (1\text{-}\lambda)y)$, sup$T_H(\lambda x + (1\text{-}\lambda)y)$),
inf $C_V(\lambda x + (1\text{-}\lambda y)$ = min(inf $C_J(\lambda x + (1\text{-}\lambda)y)$, inf $C_H(\lambda x + (1\text{-}\lambda)y)$),
sup $C_V(\lambda x + (1\text{-}\lambda)y)$ = min(sup $C_J(\lambda x + (1\text{-}\lambda)y)$, sup $C_H(\lambda x + (1\text{-}\lambda)y)$),
inf $I_V(\lambda x + (1\text{-}\lambda)y)$ = max(inf $I_J(\lambda x + (1\text{-}\lambda)y)$, inf $I_H(\lambda x + (1\text{-}\lambda)y)$),
sup $I_V(\lambda x + (1\text{-}\lambda y)$ = max(sup $I_J(\lambda x + (1\text{-}\lambda y)$, sup $I_H(\lambda x + (1\text{-}\lambda y)$),
inf $F_V(\lambda x + (1\text{-}\lambda)y)$ = max(inf $F_J(\lambda x + (1\text{-}\lambda)y)$, inf $F_H(\lambda x + (1\text{-}\lambda)y)$),
sup $F_V(\lambda x + (1\text{-}\lambda)y)$ = max(sup$F_J(\lambda x + (1\text{-}\lambda)y)$, sup $F_H(\lambda x + (1\text{-}\lambda)y)$).
Since J is convex, it follows that
inf $T_J(\lambda x + (1\text{-}\lambda)y) \geq$ min(inf $T_J(x)$, inf $T_J(y)$), sup $T_J(\lambda x + (1\text{-}\lambda)y) \geq$ min(sup $T_J(x)$, sup $T_J(y)$), inf $C_J(\lambda x + (1\text{-}\lambda)y) \geq$ min(inf $C_J(x)$, inf $C_J(y)$), sup $C_J(\lambda x + (1\text{-}\lambda)y) \geq$ min(sup $C_J(x)$, sup $C_J(y)$), inf $U_J(\lambda x + (1\text{-}\lambda)y) \leq$ max(inf $U_J(x)$, inf $U_J(y)$), sup $U_J(\lambda x + (1\text{-}\lambda)y) \leq$ max(sup $U_J(x)$, sup $U_J(y)$), inf $F_J(\lambda x + (1\text{-}\lambda)y) \leq$ max (inf $F_J(x)$, inf $F_J(y)$), sup $F_J(\lambda x + (1\text{-}\lambda)y) \leq$ max (sup $F_J(x)$, sup $F_J(y)$).
H is also convex. Therefore,
inf $T_H(\lambda x + (1\text{-}\lambda)y) \geq$ min(inf $T_H(x)$, inf $T_H(y)$), sup $T_H(\lambda x + (1\text{-}\lambda)y) \geq$ min(sup $T_H(x)$, sup $T_H(y)$),
inf $C_H(\lambda x + (1\text{-}\lambda)y) \geq$ min(inf $C_H(x)$, inf $C_H(y)$), sup $C_H(\lambda x + (1\text{-}\lambda)y) \geq$ min(sup $C_H(x)$, sup $C_H(y)$),
inf $U_H(\lambda x + (1\text{-}\lambda)y) \leq$ max(inf $U_H(x)$, inf $U_H(y)$), sup $U_H(\lambda x + (1\text{-}\lambda)y) \leq$ max(sup $U_H(x)$, sup $U_H(y)$),
inf $F_H(\lambda x + (1\text{-}\lambda y) \leq$ max(inf $F_H(x)$, inf $F_H(y)$), sup $F_H(\lambda x + (1\text{-}\lambda y) \leq$ max(sup $F_H(x)$, sup $F_H(y)$).
Therefore,
inf $T_V(\lambda x + (1\text{-}\lambda)y) \geq$ min(min (inf $T_J(x)$, inf $T_J(y)$), min(inf $T_H(x)$, inf $T_H(y)$)) = min(min(inf $T_J(x)$, inf $T_H(x)$), min(inf$T_J(y)$, inf $T_H(y)$)) = min(inf $T_V(x)$, inf $T_V(y)$) .
sup $T_V(\lambda x + (1\text{-}\lambda)y) \geq$ min(min(sup $T_J(x)$, sup $T_J(y)$), min(sup $T_H(x)$, sup $T_H(y)$)) = min(min(sup $T_J(x)$, sup $T_H(x)$), min(sup $T_J(y)$, sup $T_H(y)$)) = min(sup $T_V(x)$, sup $T_V(y)$) .
inf $C_V(\lambda x + (1\text{-}\lambda)y) \geq$ min(min (inf $C_J(x)$, inf $C_J(y)$), min(inf $C_H(x)$, inf $C_H(y)$)) = min(min(inf $C_J(x)$, inf $C_H(x)$), min(inf $C_J(y)$, inf $C_H(y)$)) = min(inf $C_V(x)$, inf $C_V(y)$) .
sup $C_V(\lambda x + (1\text{-}\lambda)y) \geq$ min(min(sup $C_J(x)$, sup $C_J(y)$), min(sup $C_H(x)$, sup $C_H(y)$)) = min(min(sup $C_J(x)$, sup $C_H(x)$), min(sup $C_J(y)$, sup $C_H(y)$)) = min(sup $C_V(x)$, sup $C_V(y)$).
inf $U_V(\lambda x + (1\text{-}\lambda)y) \leq$ max(max(inf $U_J(x)$, inf $U_J(y)$), max(inf $U_H(x)$, inf $U_H(y)$)) = max(max(inf $U_J(x)$, inf $U_H(x)$), min(inf $U_J(y)$, inf $U_H(y)$)) = max(inf $U_V(x)$, inf $U_V(y)$).
sup $U_V(\lambda x + (1\text{-}\lambda)y) \leq$ max(max(sup $U_J(x)$, sup $U_J(y)$), max(sup $U_H(x)$, sup$U_H(y)$)) = max(max(sup $U_J(x)$, sup $U_H(x)$), max(sup $U_J(y)$, sup $U_H(y)$)) = max(sup $U_V(x)$, sup $U_V(y)$).
inf $F_V(\lambda x + (1\text{-}\lambda)y) \leq$ max(max(inf $F_J(x)$,inf $F_J(y)$), max(inf $F_H(x)$, inf $F_H(y)$)) = max(max(inf $F_J(x)$, inf $F_H(x)$), min(inf $F_J(y)$, inf $F_H(y)$)) = max(inf $F_V(x)$, inf $F_V(y)$).
sup $F_V(\lambda x + (1\text{-}\lambda)y) \leq$ max(max(sup$F_J(x)$, sup $F_J(y)$), max(sup $F_H(x)$, sup $F_H(y)$)) = max(max(sup $F_J(x)$, sup $F_H(x)$), max(sup $F_J(y)$, sup $F_H(y)$)) = max(sup $F_V(x)$, sup $F_V(y)$).
Therefore, V is convex.
This completes the proof of the theorem. □

Theorem 3.4 *If J and H are any two strongly convex IQNSs, then $J \cap H$ is also strongly convex.*

Proof: Let $V = J \cap H$ then
inf $T_V(\lambda x + (1\text{-}\lambda)y)$ = min (inf T_J $(\lambda x + (1\text{-}\lambda)y)$, inf $T_H(\lambda x + (1\text{-}\lambda)y)$),
sup $T_V(\lambda x + (1\text{-}\lambda)y)$ = min (sup $T_J(\lambda x + (1\text{-}\lambda)y)$, sup $T_H(\lambda x + (1\text{-}\lambda)y)$),
inf $C_V(\lambda x + (1\text{-}\lambda)y)$ = min (inf $C_J(\lambda x + (1\text{-}\lambda)y)$, inf $C_H(\lambda x + (1\text{-}\lambda)y)$),
sup $C_V(\lambda x + (1\text{-}\lambda)y)$ = min (sup $C_J(\lambda x + (1\text{-}\lambda)y)$, sup C_H $(\lambda x + (1\text{-}\lambda)y)$),
inf $I_V(\lambda x + (1\text{-}\lambda)y)$ = max (inf $I_J(\lambda x + (1\text{-}\lambda)y)$, inf $I_H(\lambda x + (1\text{-}\lambda)y)$),
sup $I_V(\lambda x + (1\text{-}\lambda)y)$ = max (sup $I_J(\lambda x + (1\text{-}\lambda)y)$, sup $I_H(\lambda x + (1\text{-}\lambda)y)$),
inf $F_V(\lambda x + (1\text{-}\lambda)y)$ = max (inf $F_J(\lambda x + (1\text{-}\lambda)y)$, inf $F_H(\lambda x + (1\text{-}\lambda)y)$),
sup $F_V(\lambda x + (1\text{-}\lambda y)$ = max (sup $F_J(\lambda x + (1\text{-}\lambda)y)$, sup $F_H(\lambda x + (1\text{-}\lambda)y)$),
Since J is strongly convex, we have

$\inf T_J(\lambda x + (1-\lambda)y) > \min(\inf T_J(x), \inf T_J(y))$, $\sup T_J(\lambda x + (1-\lambda)y) > \min(\sup T_J(x), \sup T_J(y))$,
$\inf C_J(\lambda x + (1-\lambda)y) > \min(\inf C_J(x), \inf C_J(y))$, $\sup C_J(\lambda x + (1-\lambda)y) > \min(\sup C_J(x), \sup C_J(y))$,
$\inf U_J(\lambda x + (1-\lambda)y) < \max(\inf U_J(x), \inf U_J(y))$, $\sup U_J(\lambda x + (1-\lambda)y) < \max(\sup U_J(x), \sup U_J(y))$,
$\inf F_J(\lambda x + (1-\lambda)y) < \max(\inf F_J(x), \inf F_J(y))$, $\sup F_J(\lambda x + (1-\lambda)y) < \max(\sup F_J(x), \sup F_J(y))$.
Since H is also convex, we have
$\inf T_H(\lambda x + (1-\lambda)y) > \min(\inf T_H(x), \inf T_H(y))$, $\sup T_H(\lambda x + (1-\lambda)y) > \min(\sup T_H(x), \sup T_H(y))$,
$\inf C_H(\lambda x + (1-\lambda)y) > \min(\inf C_H(x), \inf C_H(y))$, $\sup C_H(\lambda x + (1-\lambda)y) > \min(\sup C_H(x), \sup C_H(y))$,
$\inf U_H(\lambda x + (1-\lambda)y) < \max(\inf U_H(x), \inf U_H(y))$, $\sup U_H(\lambda x + (1-\lambda)y) < \max(\sup U_H(x), \sup U_H(y))$,
$\inf F_H(\lambda x + (1-\lambda)y) < \max(\inf F_H(x), \inf F_H(y))$, $\sup F_H(\lambda x + (1-\lambda)y) < \max(\sup F_H(x), \sup F_H(y))$.
Therefore, $\inf T_V(\lambda x + (1-\lambda)y) > \min(\min(\inf T_J(x), \inf T_J(y)), \min(\inf T_H(x), \inf T_H(y))) = \min(\min(\inf T_J(x), \inf T_H(x)), \min(\inf T_J(y), \inf T_H(y))) = \min(\inf T_V(x), \inf T_V(y))$.
$\sup T_V(\lambda x + (1-\lambda)y) > \min(\min(\sup T_J(x), \sup T_J(y)), \min(\sup T_H(x), \sup T_H(y))) = \min(\min(\sup T_J(x), \sup T_H(x)), \min(\sup T_J(y), \sup T_H(y))) = \min(\sup T_V(x), \sup T_V(y))$.
$\inf C_V(\lambda x + (1-\lambda)y) > \min(\min(\inf C_J(x), \inf C_J(y)), \min(\inf C_H(x), \inf C_H(y))) = \min(\min(\inf C_J(x), \inf C_H(x)), \min(\inf C_J(y), \inf C_H(y))) = \min(\inf C_V(x), \inf C_V(y))$.
$\sup C_V(\lambda x + (1-\lambda)y) > \min(\min(\sup C_J(x), \sup C_J(y)), \min(\sup C_H(x), \sup C_H(y))) = \min(\min(\sup C_J(x), \sup C_H(x)), \min(\sup C_J(y), \sup C_H(y))) = \min(\sup C_V(x), \sup C_V(y))$.
$\inf U_V(\lambda x + (1-\lambda)y) \leq \max(\max(\inf U_J(x), \inf U_J(y)), \max(\inf U_H(x), \inf U_H(y))) = \max(\max(\inf U_J(x), \inf U_H(x)), \min(\inf U_J(y), \inf U_H(y))) = \max(\inf U_V(x), \inf U_V(y))$.
$\sup U_V(\lambda x + (1-\lambda)y) < \max(\max(\sup U_J(x), \sup U_J(y)), \max(\sup U_H(x), \sup U_H(y))) = \max(\max(\sup U_J(x), \sup U_H(x)), \max(\sup U_J(y), \sup U_H(y))) = \max(\sup U_V(x), \sup U_V(y))$.
$\inf F_V(\lambda x + (1-\lambda)y) < \max(\max(\inf F_J(x), \inf F_J(y)), \max(\inf F_H(x), \inf F_H(y))) = \max(\max(\inf F_J(x), \inf F_H(x)), \min(\inf F_J(y), \inf F_H(y))) = \max(\inf F_V(x), \inf F_V(y))$.
$\sup F_V(\lambda x + (1-\lambda)y) < \max(\max(\sup F_J(x), \sup F_J(y)), \max(\sup F_H(x), \sup F_H(y))) = \max(\max(\sup F_J(x), \sup F_H(x)), \max(\sup F_J(y), \sup F_H(y))) = \max(\sup F_V(x), \sup F_V(y))$.
Therefore V is convex.
Therefore, the theorem is proved. □

Definition 3.3 *The* ***Truth-favorite*** *of IQNS J is also an IQNS H, denoted by $H = \Delta J$, whose truth-membership and falsity-membership function are related to those of J by*
$\inf T_H(x) = \min(\inf T_J(x) + \inf U_J(x), 1)$,
$\sup T_H(x) = \min(\sup T_J(x) + \sup U_J(x), 1)$,
$\inf C_H(x) = \sup C_H(x) = \inf U_H(x) = \sup U_H(x) = 0$,
$\inf F_H(x) = \inf F_J(x)$, $\sup F_H(x) = \sup F_J(x)$, $\forall\, x \in X$.

Definition 3.4 *The* ***False-favorite*** *of IQNS J is also an IQNS H, denoted by $H = \nabla J$, whose truth-membership and falsity-membership function are related to those of J by*
$\inf T_H(x) = \inf T_J(x)$, $\sup T_H(x) = \sup T_J(x)$,
$\inf C_H(x) = \sup C_H(x) = 0$,
$\inf U_H(x) = \sup U_H(x) = 0$,
$\inf F_H(x) = \min(\inf F_J(x) + \inf U_J(x), 1)$,
$\sup F_H(x) = \min(\sup F_J(x) + \sup U_J(x), 1)$,
$\forall\, x \in X$.

Theorem 3.5 *For every IQNSs J and H*
(a) $\Delta(J \cup H) \subseteq \Delta J \cup \Delta H$.
(b) $\Delta J \cap \Delta H \subseteq \Delta(J \cap H)$.
(c) $\nabla J \cup \nabla H \subseteq \nabla(J \cup H)$.
(d) $\nabla(J \cap H) \subseteq \nabla J \cap \nabla H$.

Proof: (a) Let $V = J \cup H$
$\inf T_V(x) = \max(\inf T_J(x), \inf T_H(x))$, $\sup T_V(x)) = \max(\sup T_J(x), \sup T_H(x))$,
$\inf C_V(x) = \max(\inf C_J(x), \inf C_H(x))$, $\sup C_V(x)) = \max(\sup C_J(x), \sup C_H(x))$,

inf $U_V(x)$ = min(inf $U_J(x)$, inf $U_H(x)$), sup $U_V(x)$) = min(sup $U_J(x)$, sup $U_H(x)$),
inf $F_V(x)$ = min(inf $F_J(x)$, inf $F_H(x)$), sup $F_V(x)$) = min(sup $F_J(x)$, sup $F_H(x)$),
inf $T_{\Delta V}(x)$ = min(inf $T_V(x)$ + inf $U_V(x)$, 1)
sup $T_{\Delta V}(x)$ = min(sup $T_V(x)$ + sup $U_V(x)$, 1),
inf $C_{\Delta V}(x)$ = sup $C_{\Delta V}(x) = 0$,
inf $U_{\Delta V}(x)$ = sup $U_{\Delta V}(x) = 0$,
inf $F_{\Delta V}(x)$ = inf $F_V(x)$, sup $F_{\Delta V}(x)$ = sup $F_V(x)$,
inf $T_{\Delta J}(x)$ = min(inf $T_J(x)$ + inf $U_J(x)$, 1)
sup $T_{\Delta J}(x)$ = min(sup $T_J(x)$ + sup $U_J(x)$, 1),
inf $C_{\Delta J}(x)$ = sup $C_{\Delta J}(x) = 0$,
inf $U_{\Delta J}(x)$ = sup $U_{\Delta J}(x) = 0$,
inf $F_{\Delta J}(x)$ = inf $F_J(x)$, sup $F_{\Delta J}(x)$ = sup $F_J(x)$,
inf $T_{\Delta H}(x)$ = min (inf $T_H(x)$ + inf $U_H(x)$, 1)
sup $T_{\Delta H}(x)$ = min (sup $T_H(x)$ + sup $U_H(x)$, 1),
inf $C_{\Delta H}(x)$ = sup $C_{\Delta H}(x) = 0$,
inf $U_{\Delta H}(x)$ = sup $U_{\Delta H}(x) = 0$,
inf $F_{\Delta H}(x)$ = inf $F_H(x)$, sup $F_{\Delta H}(x)$ = sup $F_H(x)$,
inf $T_{\Delta J \cup \Delta H}(x)$ = max(inf $T_{\Delta J}(x)$, inf $T_{\Delta H}(x)$), sup $T_{\Delta J \cup \Delta H}(x)$ = max(sup $T_{\Delta J}(x)$, sup $T_{\Delta H}(x)$).
inf $C_{\Delta J \cup \Delta H}(x)$ = sup $C_{\Delta J \cup \Delta H}(x) = 0$.
inf $U_{\Delta J \cup \Delta H}(x)$ = sup $U_{\Delta J \cup \Delta H}(x) = 0$.
inf $F_{\Delta J \cup \Delta H}(x)$ = min(inf $F_{\Delta J}(x)$, inf $F_{\Delta H}(x)$),
sup $F_{\Delta J \cup \Delta H}(x)$ = min(sup $F_{\Delta J}(x)$, sup $F_{\Delta H}(x)$).
Now inf $T_{\Delta J \cup H}(x)$ = min(inf $T_{J \cup H}(x)$ + inf $U_{J \cup H}(x)$, 1) = min(max(inf $T_J(x)$, inf $T_H(x)$) + min(inf $U_J(x)$, inf $U_H(x)$), 1) ≤ 1 .
inf $T_{\Delta J \cup \Delta H}(x)$ = max(min(inf $T_J(x)$ + inf $U_J(x)$, 1), min(inf $T_H(x)$ + inf $U_H(x)$, 1))
Now if inf $T_J(x)$ + inf $U_J(x) \geq 1$ or inf $T_H(x)$ + inf $U_H(x) \geq 1$ or both then inf $T_{\Delta J \cup \Delta H}(x) = 1 \geq$ inf $T_{\Delta J \cup H}$.
If inf $T_J(x)$ + inf $U_J(x) < 1$ and inf $T_H(x)$ + inf $U_H(x) < 1$, then inf $T_{\Delta J \cup \Delta H}(x)$ = max(inf $T_J(x)$ + inf $U_J(x)$, (inf $T_H(x)$ + inf $U_H(x)$).
inf $T_{\Delta(J \cup H)}(x)$ = inf $T_J(x)$ + inf $U_J(x) \leq$ max(inf $T_J(x)$ + inf $U_J(x)$, inf $T_H(x)$ + inf $U_H(x)$) = inf $T_{\Delta J \cup \Delta H}(x)$.
or
inf $T_{\Delta(J \cup H)}(x)$ = inf $T_J(x)$ + inf $U_H(x) \leq$ max(inf $T_J(x)$ + inf $U_J(x)$, inf $T_H(x)$ + inf $U_H(x)$) = inf $T_{\Delta J \cup \Delta H}(x)$
or
inf $T_{\Delta(J \cup H)}(x)$ = inf $T_H(x)$ + inf $U_J(x) \leq$ max(inf $T_J(x)$ + inf $U_J(x)$, inf $T_H(x)$ + inf $U_H(x)$) = inf $T_{\Delta J \cup \Delta H}(x)$.
or
inf $T_{\Delta(J \cup H)}(x)$ = inf $T_H(x)$ + inf $U_H(x) \leq$ max(inf $T_J(x)$ + inf $U_J(x)$, inf $T_H(x)$ + inf $U_H(x)$) = inf $T_{\Delta J \cup \Delta H}(x)$.
So inf $T_{\Delta(J \cup H)}(x) \leq$ inf $T_{\Delta J \cup \Delta H}(x)$.
Similarly, sup $T_{\Delta(J \cup H)}(x) \leq$ sup $T_{\Delta J \cup \Delta H}(x)$.
inf $C_{\Delta(J \cup H)}(x)$ = inf $C_{\Delta J \cup \Delta H}(x)$= 0, sup $C_{\Delta(J \cup H)}(x)$ = sup $C_{\Delta J \cup \Delta H}(x) = 0$,
inf $U_{\Delta(J \cup H)}(x)$ = inf $U_{\Delta J \cup \Delta H}(x) = 0$, sup $U_{\Delta(J \cup H)}(x)$ = sup $U_{\Delta J \cup \Delta H}(x) = 0$,
inf $F_{\Delta(J \cup H)}(x)$ = inf $F_{\Delta J \cup \Delta H}(x)$,
sup $F_{\Delta(J \cup H)}(x)$ = sup $F_{\Delta J \cup \Delta H}(x)$
So $\Delta(J \cup H) \subseteq \Delta J \cup \Delta H$.
Hence the identity is proved. □

Proof: (b) Let J and H be two IQNSs, and $V = J \cap H$. Then
inf $T_V(x)$ = min(inf $T_J(x)$, inf $T_H(x)$), sup $T_V(x)$ = min(sup $T_J(x)$, sup $T_H(x)$),
inf $C_V(x)$ = min(inf $C_J(x)$, inf $C_H(x)$), sup $C_V(x)$ = min(sup $C_J(x)$, sup $C_H(x)$),
inf $U_V(x)$ = max(inf $U_J(x)$, inf $U_H(x)$), sup $U_V(x)$ = max(sup $U_J(x)$, sup $U_H(x)$),

$\inf F_V(x) = \max(\inf F_J(x), \inf F_H(x))$, $\sup F_V(x) = \max(\sup F_J(x), \sup F_H(x))$,
$\inf T_{\Delta V}(x) = \min(\inf T_V(x) + \inf U_V(x), 1) = \min(\min(\inf T_J(x), \inf T_H(x)) + \max(\inf U_J(x), \inf U_H(x)), 1)$
$\sup T_{\Delta V}(x) = \min(\sup T_V(x) + \sup U_V(x), 1) = \min(\min(\sup T_J(x), \sup T_H(x)) + \max(\sup U_J(x), \sup U_H(x)), 1)$
$\inf C_{\Delta V}(x) = \sup C_{\Delta V}(x) = 0$,
$\inf U_{\Delta V}(x) = \sup U_{\Delta V}(x) = 0$,
$\inf F_{\Delta V}(x) = \inf F_V(x)$, $\sup F_{\Delta V}(x) = \sup F_V(x)$.
$\inf T_{\Delta J\cap\Delta H}(x) = \min(\inf T_{\Delta J}(x), \inf T_{\Delta H}(x)) = \min(\min(\inf T_J(x) + \inf U_J(x), 1), \min(\inf T_H(x) + \inf U_H(x), 1))$.
$\inf T_{\Delta(J\cap H)}(x) = \min(\inf T_{J\cap H}(x) + \inf U_{J\cap H}(x), 1) = \min(\min(\inf T_J(x), \inf T_H(x)) + \max(\inf U_J(x), \inf U_H(x)), 1)$.
$\inf T_{\Delta J\cap\Delta H}(x) = \min(\inf T_J(x) + \inf U_J(x), \inf T_H(x) + \inf U_H(x), 1) \leq \min(\min(\inf T_J(x), \inf T_H(x)) + \max(\inf U_J(x), \inf U_H(x)), 1) = \inf T_{\Delta(J\cap H)}(x)$.
$\sup T_{\Delta J\cap\Delta H}(x) = \min(\sup T_{\Delta J}(x), \sup T_{\Delta H}(x)) = \min(\min(\sup T_J(x) + \sup U_J(x), 1), \min(\sup T_H(x) + \sup U_H(x), 1))$.
$\sup T_{\Delta(J\cap H)}(x) = \min(\sup T_{J\cap H}(x) + \sup U_{J\cap H}(x), 1) = \min(\min(\sup T_J(x), \sup T_H(x)) + \max(\sup U_J(x), \sup U_H(x)), 1)$.
$\sup T_{\Delta J\cap\Delta H}(x) = \min(\sup T_J(x) + \sup U_J(x), \sup T_H(x) + \sup U_H(x), 1) \leq \min(\min(\sup T_J(x), \sup T_H(x)) + \max(\sup U_J(x), \sup U_H(x)), 1) = \sup T_{\Delta(J\cap H)}(x)$.
$\inf C_{\Delta J\cap\Delta H}(x) = \inf T_{\Delta(J\cap H)}(x) = 0$.
$\sup C_{\Delta J\cap\Delta H}(x) = \sup T_{\Delta(J\cap H)}(x) = 0$.
$\inf U_{\Delta J\cap\Delta H}(x) = \inf U_{\Delta(J\cap H)}(x) = 0$.
$\sup U_{\Delta J\cap\Delta H}(x) = \sup U_{\Delta(J\cap H)}(x) = 0$.
$\inf F_{\Delta J\cap\Delta H}(x) = \max(\inf F_{\Delta J}(x), \inf F_{\Delta H}(x)) = \max(\inf F_J(x), \inf F_H(x) = \inf F_{\Delta(J\cap H)}(x)$.
$\sup F_{\Delta J\cap\Delta H}(x) = \max(\sup F_{\Delta J}(x), \sup F_{\Delta H}(x)) = \max(\sup F_J(x), \sup F_H(x) = \sup F_{\Delta(J\cap H)}(x)$.
So, $\inf T_{\Delta J\cap\Delta H}(x) \leq \inf T_{\Delta(J\cap H)}(x)$.
$\inf T_{\Delta J\cap\Delta H}(x) \leq \inf T_{\Delta(J\cap H)}(x)$.
$\inf C_{\Delta J\cap\Delta H}(x) = \inf T_{\Delta(J\cap H)}(x) = 0$.
$\sup C_{\Delta J\cap\Delta H}(x) = \sup T_{\Delta(J\cap H)}(x) = 0$.
$\inf U_{\Delta J\cap\Delta H}(x) = \inf U_{\Delta(J\cap H)}(x) = 0$.
$\sup U_{\Delta J\cap\Delta H}(x) = \sup U_{\Delta(J\cap H)}(x) = 0$.
$\inf F_{\Delta J\cap\Delta H}(x) = \inf F_{\Delta(J\cap H)}(x)$.
$\sup F_{\Delta J\cap\Delta H}(x) = \sup F_{\Delta(J\cap H)}(x)$.
Therefore, $\Delta J \cap \Delta H \subseteq \Delta(J \cap H)$.
Therefore, the identity is proved. □

Proof: (c) Let $V = J \cup H$
$\inf T_{\nabla V}(x) = \inf T_V(x)$, $\sup T_{\nabla V}(x) = \sup T_V(x)$,
$\inf C_{\nabla V}(x) = \sup C_{\nabla V}(x) = 0$,
$\inf U_{\nabla V}(x) = \sup U_{\nabla V}(x) = 0$,
$\inf F_{\nabla V}(x) = \min(\inf F_V(x) + \inf U_V(x), 1)$,
$\sup F_{\nabla V}(x) = \min(\sup F_V(x) + \sup U_V(x), 1)$,
for all x in X.
Now, $\inf T_{\nabla V}(x) = \inf T_{\nabla(J\cup H)}(x) = \inf T_{J\cup H}(x) = \max(\inf T_J(x), \inf T_H(x)) = \max(\inf T_{\nabla J}(x), \inf T_{\nabla H}(x)) = \inf T_{\nabla J\cup\nabla H}(x)$,
$\sup T_{\nabla V}(x) = \sup T_{\nabla(J\cup H)}(x) = \sup T_{J\cup H}(x) = \max(\sup T_J(x), \sup T_H(x)) = \max(\sup T_{\nabla J}(x), \sup T_{\nabla H}(x)) = \sup T_{\nabla J\cup\nabla H}(x)$,
$\inf C_{\nabla V}(x)) = \inf C_{\nabla J\cup\nabla H}(x) = 0$,
$\sup C_{\nabla V}(x)) = \sup C_{\nabla J\cup\nabla H}(x) = 0$,
$\inf U_{\nabla V}(x)) = \inf U_{\nabla J\cup\nabla H}(x) = 0$,
$\sup U_{\nabla V}(x)) = \sup U_{\nabla J\cup\nabla H}(x) = 0$,
$\inf F_{\nabla V}(x) = \inf F_{\nabla(J\cup H)}(x) = \min(\inf F_{J\cup H}(x) + \inf U_{J\cup H}(x), 1) = \min(\min(\inf F_J(x), \inf F_H(x)) +$

min(inf$U_J(x)$, inf $U_H(x)$), 1),
inf $F_{\nabla J \cup \nabla H}(x)$ = min(inf $F_{\nabla J}(x)$, inf $F_{\nabla H}(x)$) = min(min (inf $F_J(x)$ + inf $U_J(x)$, 1), min(inf $F_H(x)$ + inf $U_H(x)$, 1)). So, inf $F_{\nabla(J\cup H)}(x) \leq$ inf $F_{\nabla J\cup\nabla H}(x)$,
sup $F_{\nabla(J\cup H)}(x)$ = min(inf $F_{J\cup H}(x)$ + sup $F_{J\cup H}(x)$, 1) = min(min(sup $F_J(x)$, sup $F_H(x)$) + min(sup $F_J(x)$, sup $F_H(x)$), 1),
sup $F_{\nabla J\cup\nabla H}(x)$ = min(sup $F_{\nabla J}(x)$, sup $F_{\nabla H}(x)$) = min(min (sup $F_J(x)$ + sup $U_J(x)$, 1), min(sup $F_H(x)$ + sup $U_H(x)$, 1)). So sup $F_{\nabla(J\cup H)}(x) \leq$ sup$F_{\nabla J\cup\nabla H}(x)$.
Therefore,
inf $T_{\nabla(J\cup H)}(x)$ = inf $T_{\nabla J\cup\nabla H}(x)$,
sup $T_{\nabla(J\cup H)}(x)$ = sup $T_{\nabla J\cup\nabla H}(x)$,
inf $C_{\nabla V}(x))$ = inf $C_{\nabla J\cup\nabla H}(x) = 0$,
sup $C_{\nabla V}(x))$ = sup $C_{\nabla J\cup\nabla H}(x) = 0$,
inf $U_{\nabla V}(x))$ = inf $U_{\nabla J\cup\nabla H}(x) = 0$,
sup $U_{\nabla V}(x))$ = sup $U_{\nabla J\cup\nabla H}(x) = 0$,
inf $F_{\nabla(J\cup H)}(x) \leq$ inf $F_{\nabla J\cup\nabla H}(x)$,
sup $F_{\nabla(J\cup H)}(x) \leq$ sup $F_{\nabla J\cup\nabla H}(x)$.
So, $\nabla J \cup \nabla H \subseteq \nabla(J \cup H)$.
Therefore, the identity is proved. □

Proof: (d) Let $V_2 = J \cap H$
inf $T_{\nabla V_2}(x)$ = inf $T_{V_2}(x)$, sup $T_{\nabla V_2}(x)$ = sup $T_{V_2}(x)$,
inf $C_{\nabla V_2}(x)$ = sup $C_{\nabla V_2}(x) = 0$,
inf $U_{\nabla V_2}(x)$ = sup $U_{\nabla V_2}(x) = 0$,
inf $F_{\nabla V_2}(x)$ = min(inf $F_{V_2}(x)$ + inf $U_{V_2}(x)$, 1),
sup $F_{\nabla V_2}(x)$ = min(sup $F_{V_2}(x)$ + sup $U_{V_2}(x)$,1), for all x in X .
Now, inf $T_{\nabla V_2}(x)$ = inf $T_{\nabla(J\cap H)}(x)$ = inf $T_{J\cap H}(x)$ = max(inf $T_J(x)$, inf $T_H(x)$) = max(inf $T_{\nabla J}(x)$, inf $T_{\nabla H}(x)$) = inf $T_{\nabla J\cap\nabla H}(x)$,
sup $T_{\nabla V_2}(x)$ = sup $T_{\nabla(J\cap H)}(x)$ = sup $T_{J\cap H}(x)$ = max(sup $T_J(x)$, sup $T_H(x)$) = max(sup $T_{\nabla J}(x)$, sup $T_{\nabla H}(x)$) = sup $T_{\nabla J\cap\nabla H}(x)$
inf $C_{\nabla V_2}(x))$ = inf $C_{\nabla J\cap\nabla H}(x) = 0$,
sup $C_{\nabla V_2}(x))$ = sup $C_{\nabla J\cap\nabla H}(x) = 0$,
inf $U_{\nabla V_2}(x))$ = inf $U_{\nabla J\cap\nabla H}(x) = 0$,
sup $U_{\nabla V_2}(x))$ = sup $U_{\nabla J\cap\nabla H}(x) = 0$,
inf $F_{\nabla V_2}(x))$ = inf $F_{\nabla(J\cap H)}(x)$ = min(inf $F_{J\cap H}(x)$ + inf $U_{J\cap H}(x)$, 1) = min(max(inf $F_J(x)$, inf $F_H(x)$) + max(inf $F_J(x)$, inf $F_H(x)$), 1)
inf $F_{\nabla J\cap\nabla H}(x)$ = max(inf $F_{\nabla J}(x)$, inf $F_{\nabla H}(x)$) = max(min (inf $F_J(x)$ + inf $U_J(x)$, 1), min(inf $F_H(x)$ + inf $U_H(x)$, 1)).So inf $F_{\nabla(J\cap H)}(x) \geq$ inf $F_{\nabla J\cap\nabla H}(x)$,
sup $F_{\nabla(J\cap H)}(x)$ = min(sup $F_{J\cap H}(x)$ + sup $U_{J\cap H}(x)$, 1) = min(max(sup $F_J(x)$, sup $F_H(x)$) + max(sup $U_J(x)$, sup $U_H(x)$), 1),
sup $F_{\nabla J\cap\nabla H}(x)$ = max(sup $F_{\nabla J}(x)$, sup $F_{\nabla H}(x)$) = max(min(sup $F_J(x)$ + sup $U_J(x)$, 1), min(sup $F_H(x)$ + sup $U_H(x)$, 1)). So sup $F_{\nabla(J\cap H)}(x) \geq$ sup $F_{\nabla J\cap\nabla H}(x)$.
Therefore, inf $T_{\nabla(J\cap H)}(x)$ = inf $T_{\nabla J\cap\nabla H}(x)$,
sup $T_{\nabla(J\cap H)}(x)$ = sup $T_{\nabla J\cap\nabla H}(x)$,
inf $C_{\nabla(J\cap H)}(x)$ = inf $C_{\nabla J\cap\nabla H}(x) = 0$,
sup $C_{\nabla(J\cap H)}(x)$ = sup $C_{\nabla J\cap\nabla H}(x) = 0$,
inf $U_{\nabla(J\cap H)}(x)$ = inf $U_{\nabla J\cap\nabla H}(x) = 0$,
sup $U_{\nabla(J\cap H)}(x)$ = sup $U_{\nabla J\cap\nabla H}(x) = 0$,
inf $F_{\nabla(J\cap H)}(x) \geq$ inf $F_{\nabla J\cap\nabla H}(x)$,
sup $F_{\nabla(J\cap H)}(x) \geq$ sup $F_{\nabla J\cap\nabla H}(x)$.
So, $\nabla J \cup \nabla H \supseteq \nabla(J \cup H)$.
Therefore, the identity is proved. □

4. Future Research Direction

The concept of IQNS is an advanced extension of NSs, used to handle uncertainty, indeterminacy, and inconsistency in complex system. Since IQNS is still a relatively emerging area, several future research directions can be pursued to enhance its theoretical foundations and practical applications.
Algebraic structures: Investigate algebraic operations and explore lattice, ring, and group-theoretic properties.
Axiomatic foundations: Establish a robust axiomatic framework for IQNS and compare it with classical and other NS variants.
Similarity/Dissimilarity Measures: Define and refine measures of similarity, distance, and entropy specifically for IQNSs.
Topological Structures: Develop IQNS-based topologies and explore continuity, compactness, and convergence within such spaces.
Decision-Making Models: Extend existing Multi-Criteria Decision-Making (MCDM) methods (e.g., TOPSIS [44], TODIM [45], VIKOR [46]) to handle IQNS-based information.
Multi-Criteria Group Decision-Making (MCGDM): Develop models where multiple experts provide input in IQNS environment, including consensus and conflict resolution techniques.
Water quality evaluation: Develop water quality evaluation models [47, 48] in IQNS environment.

5. Conclusions

This paper presents the convexity properties of IQNSs and also establishes the properties of truth favorites and false favorites. In the future, the logic and relational data model of IQNSs can be explored.

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