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On Fuzzy Double Controlled Fisher Iterated Function System with Application to Economics via Numerical Algorithm

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ABSTRACT: In this research, we generalize the fixed-point theorem for fuzzy Fisher contraction in fuzzy double controlled metric space, a newly developed mathematical structure that expands ordinary metric spaces and by integrating control functions. The interplay of fuzzy Fisher contraction and fuzzy double controlled metric space provides a new paradigm for investigating contraction mappings and their applications in fractal theory. Using fuzzy Fisher contraction in this generalized fuzzy metric spaces framework, we define a novel class of fractal set known as fuzzy double controlled Fisher fractals and discuss the fuzzy double controlled Fisher iterated function system that is the generalization classical iterative function system in fuzzy double controlled metric space. We develop the Collage theorem for fuzzy double controlled Fisher fractals, which is a useful tool for approximation in fractal generation. This theorem extends the usual collage theorem by incorporating the flexibility of fuzzy double controlled metric conditions, yielding more generalized approximation results. In addition, we focus on non-trivial cases including the graphical behavior of fuzzy double controlled Fisher contraction.

Key Words: Fuzzy Fisher contraction (FFC), Fuzzy double controlled metric space (FDCMS), Iterative function system (IFS), Fuzzy double controlled Fisher fractals (FDCFF), Fixed point (FP), Metric space (MS).

Contents

1. Introduction

The Latin word *fractus*, which means "fragmented" or "broken," is the source of the term "fractal," which was first used in 1975 by mathematician Mandelbrot [1]. A geometric shape with intricate structures at arbitrarily small sizes is known as a fractal in mathematics. According to Mandelbrot, the topological dimension of a fractal is usually less than its Hausdorff dimension. The Mandelbrot set keeps getting bigger, showing that many fractals have things in common at different levels. Mandelbrot's work combines interesting guesses and observations about how mathematics works and shows up in nature. Rodriguez-Lopez and Romaguera [2] made a new and possibly interesting contribution to the growth of fuzzy metric theory. Hutchinson [3] and Barnsley [4] used the Banach contraction principle to introduce and expand the Hutchinson–Barnsley theory about a fixed point (FP). This theory explains and expands

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on the fractal as a small, unchanging part of the whole metric space made by the iterated function system (IFS) of contractions.

Al-Saidi et al. [5,6] introduced fuzzy fractal spaces and fuzzy FP theorems. Easwaramoorthi and Uthayakuma [7] showed how fractals are unique and exist with certain motivational qualities. Fractals were introduced by them in fuzzy metric spaces (FMSs) and also developed in the sense of intuitionistic fuzzy metric spaces (IFMSs). Ghosh et al. [8] addressed strong coupled FP results in FMSs. El-Nabulsi [9] investigated periodic crystal lattices. Andres et al. [10] employed the FP theorem in Banach spaces and topological (Schauder-like) FP theorems to illustrate that the Hutchinson–Barnsley mapping developed for a multifunction system has parts that remain invariant.

They also looked into weakly contractive and small multifunction systems, as well as common multifunction systems. Andres and Fisher [11,12] used locally compatible mappings to generate multivalued fractals. Tarski's multivalued fractals also discussed the relation between crisp multivalued fractals and fuzzy hyper-fractals. Singh [13] discussed single-valued and multivalued contractions and generalized fractal generation results. Prasad and Katiyar [14] studied the existence and distinctness of fractal-generating results in FMSs and multi-fuzzy fractals. Mishra and Prasad [15] improved the Suzuki type contraction in FMSs and enhanced methods to generate fractals in such spaces.

Gowrisankar and Easwaramoorthy [16] expanded the concept of a local iterated function system (LIFS) to the broader situation of a local countable IFS (LCIFS). They constructed the approximation process of the LCIFS attractor in terms of LIFS attractors and examined the relationship between CIFS and LCIFS attractors. Bakhtin [17] developed b-metric spaces (an extension of metric spaces) in 1989 and also examined FP results for contractive mappings in these spaces. In addition, Czerwik [18] expanded results for b-metric spaces in 1993. In 2018, Shatanawi et al. [19] proposed the α, ψ-contraction for extended b-metric spaces.

Mlaiki et al. [20] developed controlled metric spaces (CMSs) by applying a control function $\alpha(x, y)$ to the right-hand side of the b-triangle inequality. Abdeljawad [21] established double controlled metric spaces (DCMSs) by modifying CMSs with $\alpha(x, y)$ and $\mu(x, y)$ being control functions used in the triangle inequality. Fisher [22] introduced Fisher contraction mappings (FCM) and generalized fractal results. Sahu [23] introduced the K-IFS, a generalization of IFS, and discussed the collage theorem for K-IFS. Schweizer and Sklar [24] introduced the continuous t-norm (CTN) in 1960 and generalized FP results.

Kramosil and Michálek [25] introduced FMSs and extended FP results in this setting. Sezen [26] extended FMSs to controlled fuzzy metric spaces (CFMSs) in which a control function is used and generalized FP results. Saleem et al. [27] extended CFMSs to fuzzy double controlled metric spaces (FDCMSs) in which two control functions are used in the triangle inequality and generalized FP results in this setting. In this manuscript, we generalize the results of [28] in the sense of FDCMSs and discuss with non-trivial examples and graphical behavior. For the application point of view to economics we refer the following literature [29,30,31,32,33].

1.1. Motivation

Since the fundamental work of Mandelbrot, the study of fractals—with their rich self-similar structures and broad applications in mathematics, physics, and engineering—has advanced dramatically. While efficient, classical IFSs and fixed-point theorems sometimes run afoul of reality's uncertainty, imprecision, and vagueness. Such sophisticated behaviors cannot always be captured by conventional metric spaces and contraction mappings.

More general mathematical structures capable of modeling uncertainty with more flexibility will help to solve these difficulties. By means of two control functions, the fuzzy double controlled metric space (FDCMS) offers such a framework extending traditional fuzzy metric spaces. Coupled with the fuzzy Fisher contraction (FFC), this structure provides a strong tool for analyzing fixed points and generating fractals under fuzzy and controlled conditions, and generalizes a range of contraction mappings. This work investigates and aggregates several ideas to generate fuzzy double controlled Fisher fractals (FDCFFs), a new class of fractal sets. Through extending the conventional IFS into the FDCMS environment, this work presents a fresh approach for fractal generation. Furthermore, developed within this framework is a fuzzy form of the Collage Theorem, improving approximative power in fuzzy surroundings.

There is both theoretical and pragmatic motivation: theoretically it helps fixed-point theory to grow, while practically it offers a fresh path for fractal modeling in fields including computational graphics, image analysis, and biological systems where ambiguity and multi-level dynamics are significant.

The paper is organized as follows:

- Section 2 discusses some definitions, examples, and theorems from existing literature that support our main result.
- Section 3 generalizes the FFC in FDCMSs and discusses some FP results for Fisher contraction.
- Section 4 introduces the double controlled F-IFS with nontrivial examples.
- Section 5 presents the main contribution.
- Section 6 concludes the paper.
- Section 7 outlines directions for future work.

2. Preliminaries

The preliminary part covers fundamental IFS theory required for the suggested outcomes.

Definition 2.1 ([24]). A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm (CTN) if the following axioms are fulfilled:

1. $*$ is continuous,
2. $a * 1 = a$, for all $a \in [0,1]$,
3. $*$ is associative and commutative,
4. $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 ([28]). Suppose that (Ξ, ψ) is a metric space and a self-mapping $F : \Xi \to \Xi$ is said to be a contraction if it satisfies the condition

$$\psi(F(\sigma), F(\varpi)) \leq \kappa\, \psi(\sigma, \varpi), \quad \forall \sigma, \varpi \in \Xi,\ \kappa \in [0,1). \tag{2.1}$$

Here κ is the contraction ratio (CR).

Theorem 2.1. *[28] A self-mapping F has a unique FP if F satisfies the contraction condition on (Ξ, ψ). Furthermore, the sequence $\{F^j(\sigma)\}_{j=1}^{\infty}$ is convergent and converges to a point $\sigma^* \in \Xi$. That is,*

$$\lim_{j \to \infty} F^j(\sigma) = \sigma^*, \quad \forall \sigma \in \Xi.$$

In a CMS, every self-mapping has a unique FP as demonstrated in Theorem 2.1. Suppose that Ξ has a non-empty collection of compact subsets denoted by $\mathcal{K}_0(\Xi)$. For $\psi, \Phi \in \mathcal{K}_0(\Xi)$, define

$$\psi(\sigma, \Phi) = \inf\{\psi(\sigma, \varpi) : \varpi \in \Phi\},$$

and

$$\psi(\psi, \Phi) = \sup\{\psi(\sigma, \Phi) : \sigma \in \psi\}.$$

The distance between two non-empty sets ψ and Φ of $\mathcal{K}_0(\Xi)$ is said to be the Hausdorff distance:

$$g_\psi(\psi, \Phi) = \max\left\{ \sup_{\sigma \in \psi} \psi(\sigma, \Phi),\ \sup_{\varpi \in \Phi} \psi(\varpi, \psi) \right\}.$$

The function g_ψ is a Hausdorff distance on $\mathcal{K}_0(\Xi)$. The IFS has a contraction function that is a member of the Hausdorff space $(\mathcal{K}_0(\Xi), g_\psi)$. It is obvious that $(\mathcal{K}_0(\Xi), g_\psi)$ is a complete Hausdorff space if (Ξ, ψ) is a CMS.

Definition 2.3 ([20]). Assume that $\Xi \neq \varnothing$ and let $\alpha : \Xi \times \Xi \to [1, \infty)$ be a controlled function. A mapping $\psi : \Xi \times \Xi \to [0, \infty)$ is called a *controlled metric* on Ξ if the following axioms hold for all $\sigma, \varpi, c \in \Xi$:

1. $\psi(\sigma, \varpi) = 0 \;\Leftrightarrow\; \sigma = \varpi$,
2. $\psi(\sigma, \varpi) = \psi(\varpi, \sigma)$,
3. $\psi(\sigma, \varpi) \leq \alpha(\sigma, c)\psi(\sigma, c) + \alpha(c, \varpi)\psi(c, \varpi)$.

Then (Ξ, ψ) is said to be a controlled metric space (CMS).

Every CMS reduces to a metric space, but the converse is not true in general. Consider a hyperbolic IFS on a CMS (Ξ, ψ) with self-mappings $F_j : \Xi \to \Xi$, continuous contractions with contraction ratios κ_j $(j = 1, 2, \cdots, N_0)$.

Define the set-valued mapping

$$f(\Phi) = \bigcup_{j=1}^{N_0} F_j(\Phi), \quad \Phi \in \mathcal{K}_0(\Xi),$$

where $F_j(\Phi) = \{F_j(\varpi) : \varpi \in \Phi\}$. Then $(\mathcal{K}_0(\Xi), g_\psi)$ has the contraction property

$$g_\psi(F(\Phi), F(C)) \leq \kappa\, g_\psi(\Phi, C).$$

Therefore, F has a unique attractor $A \in \mathcal{K}_0(\Xi)$, which is the FP of this mapping. Moreover,

$$A = \lim_{j \to \infty} F^j(\Phi), \quad \forall \Phi \in \mathcal{K}_0(\Xi),$$

where $F^j = \underbrace{F \circ F \circ \cdots \circ F}_{j \text{ times}}$. Such an A is called an invariant set with $F(A) = A$.

Definition 2.4 ([21]). Let $\Xi \neq \varnothing$ and let $\alpha, \beta : \Xi \times \Xi \to [1, \infty)$ be controlled functions. A distance $\psi : \Xi \times \Xi \to [0, \infty)$ is called a *double controlled metric* if for all $\sigma, \eta, c \in \Xi$:

1. $\psi(\sigma, \eta) = 0 \;\Leftrightarrow\; \sigma = \eta$,
2. $\psi(\sigma, \eta) = \psi(\eta, \sigma)$,
3. $\psi(\sigma, \eta) \leq \alpha(\sigma, c)\psi(\sigma, c) + \beta(c, \eta)\psi(c, \eta)$.

Then (Ξ, ψ) is called a double controlled metric space (DCMS).

Definition 2.5 ([25]). A 3-tuple $(\Xi, \Phi, *)$ is said to be a *fuzzy metric space* (FMS) if $\Phi : \Xi \times \Xi \times (0, \infty) \to (0, 1]$ and $*$ is a continuous t-norm such that for all $\pi, p, Z \in \Xi$ and $t, g > 0$:

1. $\Phi(\pi, p, t) > 0$,
2. $\Phi(\pi, p, t) = 1 \iff \pi = p$,
3. $\Phi(\pi, p, t) = \Phi(p, \pi, t)$,
4. $\Phi(\pi, p, t) * \Phi(p, Z, g) \leq \Phi(\pi, Z, t + g)$,
5. $\Phi(\pi, p, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
6. $\lim_{t \to \infty} \Phi(\pi, p, t) = 1$ for all $\pi, p \in \Xi$.

Definition 2.6 ([25]). Let $(\Xi, \Phi, *)$ be an FMS. For $t > 0$ and $0 < r < 1$, define the open ball

$$\Sigma(\pi, r, t) = \{p \in \Xi : \Phi(\pi, p, t) > 1 - r\}, \quad \pi \in \Xi.$$

The family $\{\Sigma(\pi, r, t) : \pi \in \Xi, 0 < r < 1, t > 0\}$ forms a basis for a Hausdorff topology on Ξ.

Definition 2.7 ([25]). Let $(\Xi, \Phi, *)$ be an FMS.

- A sequence $\{\pi_n\} \subset \Xi$ is *convergent* to π if $\lim_{n\to\infty} \Phi(\pi_n, \pi, t) = 1$ for all $t > 0$.
- A sequence $\{\pi_n\} \subset \Xi$ is a *Cauchy sequence* if $\lim_{m,n\to\infty} \Phi(\pi_n, \pi_m, t) = 1$ for all $t > 0$.

Definition 2.8 ([25]). Let Ω, Σ be two non-empty compact subsets of an FMS $(\Xi, \Phi, *)$. The Hausdorff fuzzy metric H_Φ on $\kappa(\Xi)$ is defined as

$$H_\Phi(\Omega, \Sigma, t) = \min\{\omega(\Omega, \Sigma, t), \overline{\omega}(\Omega, \Sigma, t)\}, \quad t > 0,$$

where

$$\omega(\Omega, \Sigma, t) = \inf_{\vartheta\in\Omega} \sup_{f\in\Sigma} \Phi(\vartheta, f, t), \qquad \overline{\omega}(\Omega, \Sigma, t) = \inf_{f\in\Sigma} \sup_{\vartheta\in\Omega} \Phi(\vartheta, f, t).$$

Definition 2.9 ([26]). A 3-tuple $(\Xi, \Phi, *)$ is a *fuzzy controlled metric space* (FCMS) if $\Phi : \Xi \times \Xi \times (0, \infty) \to (0, 1]$, $\alpha : \Xi \times \Xi \to [1, \infty)$, and $*$ is a CTN such that for all $\pi, p, Z \in \Xi$ and $t, g > 0$:

1. $\Phi(\pi, p, t) > 0$,
2. $\Phi(\pi, p, t) = 1 \iff \pi = p$,
3. $\Phi(\pi, p, t) = \Phi(p, \pi, t)$,
4. $\Phi\Big(\pi, p, \frac{t}{\alpha(\pi,p)}\Big) * \Phi\Big(p, Z, \frac{g}{\alpha(p,Z)}\Big) \le \Phi(\pi, Z, t + g)$,
5. $\Phi(\pi, p, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
6. $\lim_{t\to\infty} \Phi(\pi, p, t) = 1$ for all $\pi, p \in \Xi$.

Definition 2.10 ([27]). A 3-tuple $(\Xi, \Phi, *)$ is a *fuzzy double controlled metric space* (FDCMS) if $\Phi : \Xi \times \Xi \times (0, \infty) \to (0, 1]$, $\alpha, \beta : \Xi \times \Xi \to [1, \infty)$, and $*$ is a CTN such that for all $\pi, p, Z \in \Xi$ and $t, g > 0$:

1. $\Phi(\pi, p, t) > 0$,
2. $\Phi(\pi, p, t) = 1 \iff \pi = p$,
3. $\Phi(\pi, p, t) = \Phi(p, \pi, t)$,
4. $\Phi\Big(\pi, p, \frac{t}{\alpha(\pi,p)}\Big) * \Phi\Big(p, Z, \frac{g}{\beta(p,Z)}\Big) \le \Phi(\pi, Z, t + g)$,
5. $\Phi(\pi, p, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
6. $\lim_{t\to\infty} \Phi(\pi, p, t) = 1$ for all $\pi, p \in \Xi$.

3. Main Result

In this section, we discuss a novel framework for IFS by replacing the usual contraction with a more general fuzzy Fisher contraction. According to fuzzy Fisher, if two contraction ratios $\kappa, s \in (0, \frac{1}{2})$ and for every $\sigma, f \in \Xi$ we have

$$\psi(F(\sigma), F(f), t) \ge \kappa\big[\psi(\sigma, F(\sigma), t) + \psi(f, F(f), t)\big] + s\,\psi(\sigma, f, t), \tag{3.1}$$

then F is referred to as a *fuzzy Fisher contraction mapping* (FFCM). Here, κ and s are contraction ratios of fuzzy Fisher contraction F. We now discuss the FFCM in the context of FDCMS.

Definition 3.1. Suppose that $(\Xi, \psi, *)$ is a continuous FDCMS and $F : \Xi \to \Xi$ is a self-mapping satisfying

$$\psi(F(\sigma), F(f), t) \ge \kappa\big[\psi(\sigma, F(\sigma), t) + \psi(f, F(f), t)\big] + s\,\psi(\sigma, f, t), \tag{3.2}$$

for all $t \ge 0$, $\sigma, f \in \Xi$ and $\kappa, s \in (0, \frac{1}{2})$, then F is considered an FDC-FC.

In the preceding inequality, if $\kappa = 0$, this contraction is referred to be a typical contraction. However, the contrary is not always true. Similarly, if $s = 0$, then the fuzzy Fisher contraction converted to a fuzzy Kannan contraction. However, the contrary may or may not be always true. The Hausdorff metric is currently established in the FDCMS as follows.

Definition 3.2. Let $(\Xi, \psi, *)$ be an FDCMS. Define

$$g_\psi(A, B, t) = \max\left\{ \sup_{\sigma\in A} \psi(\sigma, B, t),\ \sup_{f\in B} \psi(f, A, t)\right\}, \quad A, B \in \mathcal{K}_0(\Xi).$$

Then g_ψ is called the fuzzy Hausdorff metric on compact subsets of Ξ.

Barnsley [4] introduced the classical IFS. We now extend this to the FDC setting:

Definition 3.3 (FDC-IFS)**.** Suppose that $(\Xi, \psi, *)$ is a continuous FDCMS, and let $\{F_j : \Xi \to \Xi\}_{j=1}^{N_0}$ be a finite family of self-mappings. Each F_j is said to be a contraction in the continuous FDCMS with contraction ratios κ_j, s_j $(j = 1, 2, \ldots, N_0)$. The system

$$\{\Xi; F_j, j = 1, 2, \ldots, N_0\}$$

is called an *FDCF-IFS* of FFCs with global contraction ratios

$$\kappa = \max_{1\le j\le N_0} \kappa_j, \quad s = \max_{1\le j\le N_0} s_j.$$

We now establish the following theorem.

Theorem 3.1. *Suppose that $F : \Xi \to \Xi$ is an FDC-FC with contraction ratios κ and s on the FDCMS $(\Xi, \psi, *)$ and $\sigma \in \Xi$. Then*

$$\psi(F^i(\sigma), F^{i+1}(\sigma), t) \ge \left(\frac{\kappa}{1-\kappa}\right)^i \psi(\sigma, F(\sigma), t) + (1-s)\,\psi(\sigma, F(\sigma), t). \tag{3.3}$$

Furthermore,

$$\lim_{i\to\infty} \psi(F^i(\sigma), F^{i+1}(\sigma), t) = 0.$$

Proof: Since F is an FDC-FC, we have

$$\psi(F^i(\sigma), F^{i+1}(\sigma), t) \ge \kappa\,\psi(F^{i-1}(\sigma), F^i(\sigma), t) + s\,\psi(F^{i-1}(\sigma), F^i(\sigma), t).$$

Rearranging,

$$\psi(F^i(\sigma), F^{i+1}(\sigma), t) \ge \frac{\kappa}{1-\kappa}\psi(F^{i-1}(\sigma), F^i(\sigma), t) + s\,\psi(F^{i-1}(\sigma), F^i(\sigma), t).$$

Iterating this process gives

$$\psi(F^i(\sigma), F^{i+1}(\sigma), t) \ge \left(\frac{\kappa}{1-\kappa}\right)^i \psi(\sigma, F(\sigma), t) + (1-s)^{-1}\psi(\sigma, F(\sigma), t).$$

As $i \to \infty$, since $\frac{\kappa}{1-\kappa} < 1$ and $(1-s) < 1$, the limit is 0. □

Example 3.1. *Let $\Xi = [0, 1]$ and define $d : \Xi \times \Xi \to \mathbb{R}^+$ by*

$$d(\sigma, f) = |\sigma - f| + \frac{|\sigma - f|}{1 + |\sigma + f|}.$$

Define the fuzzy function

$$\psi(\sigma, f, t) = \frac{t}{t + d(\sigma, f)}, \quad \sigma, f \in \Xi.$$

Let $\alpha(\sigma, y) = 1 + \sigma$ and $\beta(\sigma, y) = 1 + y$. Then $(\Xi, \psi, *)$ is an FDCMS.

Consider $F(\sigma) = \sigma/3$. For $\kappa = 0.2$ and $s = 0.3$ we obtain

$$\psi(F(\sigma), F(y), t) \ge 0.2\big[\psi(\sigma, F(\sigma), t) + \psi(y, F(y), t)\big] + 0.3\,\psi(\sigma, y, t),$$

which shows F is an FDC-FC. Thus all conditions of Theorem 3.1 are satisfied. For better understanding see Figure 1.

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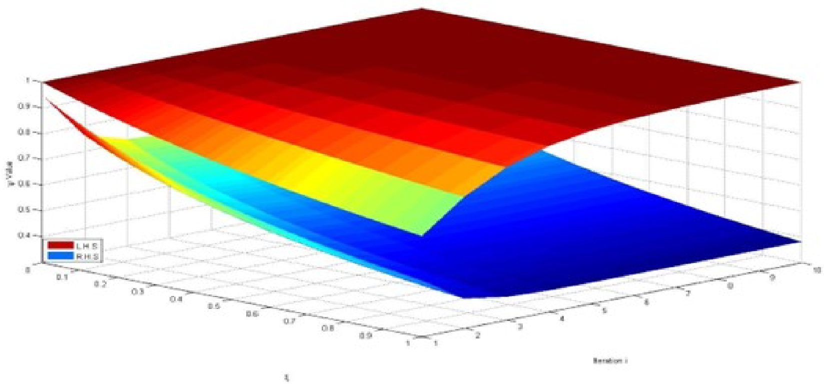

Figure 1: Depicts the behavior of the left-hand side and right-hand side of the inequality $\psi(F(\sigma), F(y), t) \geq 0.2\,[\psi(\sigma, F(\sigma), t) + \psi(y, F(y), t)] + 0.3\,\psi(\sigma, y, t)$.

Theorem 3.2. *Let $F : \Xi \to \Xi$ be a FFCM with contraction ratios r and s on the FDCMS $(\Xi, \psi, *)$. If F has a fixed point, then it is unique.*

Proof: Assume that F has two fixed points σ^* and f^*. Then $\sigma^* = F(\sigma^*)$ and $f^* = F(f^*)$. Hence,

$$\begin{aligned}\psi(\sigma^*, f^*, t) &= \psi(F(\sigma^*), F(f^*), t)\\ &\geq r\big[\psi(\sigma^*, F(\sigma^*), t) + \psi(f^*, F(f^*), t)\big] + s\,\psi(\sigma^*, f^*, t)\\ &= r\,[\psi(\sigma^*, \sigma^*, t) + \psi(f^*, f^*, t)] + s\,\psi(\sigma^*, f^*, t)\\ &= s\,\psi(\sigma^*, f^*, t).\end{aligned}$$

This implies $\psi(\sigma^*, f^*, t) = 0$, and thus $\sigma^* = f^*$. Therefore, F has a unique fixed point. □

Example 3.2. *Let $\Xi = M_n(\mathbb{R})$ be the space of all real $n \times n$ square matrices. Define the distance function $d : \Xi \times \Xi \to [0, \infty)$ by*

$$d(A, B) = \|A - B\|_\infty + \frac{\|A - B\|_F}{1 + \|A + B\|_F}, \tag{3.4}$$

Where, $\|A\|_\infty$ is the infinity norm that is the maximum of the absolute row sums and the $\|A\|_F$ is a Frobenius norm which defined as:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}. \tag{3.5}$$

Then a fuzzy function $\psi : \Xi \times \Xi \to [0, 1]$ is defined by

$$\psi(A, B, t) = \frac{t}{t + d(A, B)}. \tag{3.6}$$

Define two control functions $\alpha, \beta : \Xi \times \Xi \to (0, \infty)$ by

$$\alpha(A, B) = 1 + \|A\|_\infty, \qquad \beta(A, B) = 1 + \|B\|_F.$$

Consider the self-mapping $F : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by

$$F(A) = \tfrac{1}{3}A.$$

If we take $\kappa = 0.1$ and $s = 0.3$, then

$$\psi(F(A), F(B), t) \;\geq\; 0.1\big[\psi(A, F(A), t) + \psi(B, F(B), t)\big] + 0.3\,\psi(A, B, t). \tag{3.7}$$

This shows that $F(A)$ is a FFC and F has a unique FP that is $(A = 0)$ zero matrix. See the below Figure 3 for better understanding.

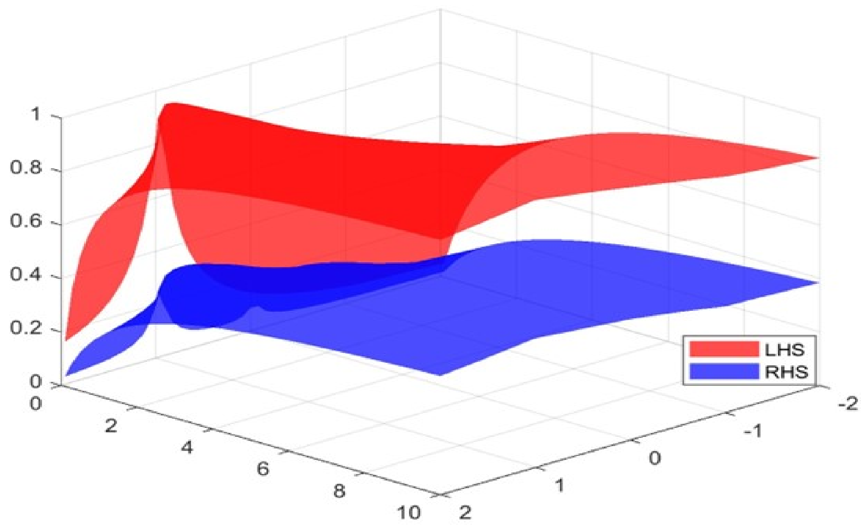

Figure 2: Depicts the behavior of LHS and RHS of the inequality $\psi(F(A),F(B),t) \ \geq\ 0.1\big[\psi(A,F(A),t)+\psi(B,F(B),t)\big] + 0.3\,\psi(A,B,t)$.

??.

Theorem 3.3. *Suppose that a self-mapping F on Ξ is a continuous FFCM on a continuous FDCMS $(\Xi,\psi,*)$ with contraction ratios κ and s. Then F has a unique fixed point $\sigma^* \in \Xi$. Moreover, the following inequality holds:*

$$\psi(\sigma,\sigma^*,t) \ \geq\ \left(\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s}\right)\psi(\sigma,F(\sigma),t). \tag{3.8}$$

Proof: For every $\sigma \in \Xi$, σ^* is a fixed point if $\lim_{i\to\infty} F^i(\sigma) = \sigma^*$. Since $\psi(\sigma,f,t)$ is continuous in $f \in \Xi$, we have

$$\psi(\sigma,\sigma^*,t) = \psi\Big(\sigma, \lim_{i\to\infty} F^i(\sigma), t\Big) = \lim_{i\to\infty} \psi(\sigma, F^i(\sigma), t).$$

By telescoping sum expansion,

$$\lim_{i\to\infty} \psi(\sigma,F^i(\sigma),t) = \lim_{i\to\infty} \sum_{j=1}^{i} \psi(F^{j-1}(\sigma), F^j(\sigma), t).$$

Applying the contraction condition iteratively yields

$$\lim_{i\to\infty} \psi(\sigma,F^i(\sigma),t) \ \geq\ \left(\sum_{j=0}^{\infty}\Big(\tfrac{\kappa}{1-\kappa}\Big)^j + \sum_{j=0}^{\infty} s^j\right)\psi(\sigma,F(\sigma),t).$$

Hence,

$$\psi(\sigma,\sigma^*,t) \ \geq\ \left(\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s}\right)\psi(\sigma,F(\sigma),t).$$

□

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Theorem 3.4. *Assume that a self-mapping F on Ξ is an FFCM on a continuous FDCMS $(\Xi, \psi, *)$ with contraction ratios κ, s and control functions $\alpha, \beta : \Xi \times \Xi \to (0, \infty)$.*

Let $\sigma_j = F^j(\sigma_0)$ for $\sigma_0 \in \Xi$. Suppose that

$$\sup_{i\geq 1} \lim_{j\to\infty} \frac{\alpha(\sigma_{j+1}, \sigma_{j+2})}{\alpha(\sigma_j, \sigma_{j+1})} \beta(\sigma_j, \sigma_i) \;<\; \frac{1}{\kappa} + \frac{1}{s}.$$

In addition, for every $\sigma \in \Xi$, the limits $\lim_{j\to\infty} \alpha(\sigma_j, \sigma)$ and $\lim_{j\to\infty} \alpha(\sigma, \sigma_j)$ exist. Then F has a unique fixed point.

Proof: Assume that an element $\sigma \in \Xi$, and provided that a mapping F is an FFCM with the CRs r and s, we get

$$\psi(F^i(\sigma), F^{i+1}(\sigma), t) \;\leq\; \Big(\tfrac{\kappa}{1-\kappa} + (1-s)\Big)\psi(\sigma, F(\sigma), t), \quad \forall i \geq 1.$$

Subsequently, any arbitrary element $\sigma \in \Xi$, is fixed element, we obtain

$$\psi(F^j(\sigma), F^i(\sigma), t) \;\leq\; t^{j\wedge i}\psi(\sigma, F^{|j-i|}(\sigma), t) + u^{j\wedge i}\psi(\sigma, F^{|j-i|}(\sigma), t), \tag{3.9}$$

Where $i, j \in \{0\} \cup N$ and $t = \frac{\kappa}{1-\kappa}$,$u = 1-s$. Consider,$n = |j-i|$, where $n = 0, 1, 2, \cdots$ then, we obtain

$$\psi(\sigma, F^n(\sigma), t) \;\geq\; \sum_{k=0}^{n-1} \psi(F^k(\sigma), F^{k+1}(\sigma), t) \;\geq\; \Big(\sum_{k=0}^{n-1} t^k + \sum_{k=0}^{n-1} u^k\Big)\psi(\sigma, F(\sigma), t).$$

Then, Equation 3.2 becomes

$$\psi(\sigma, F^n(\sigma), t) \;\geq\; \frac{1-t^n}{1-t}\psi(\sigma, F(\sigma), t) + \frac{1}{1-u}\psi(\sigma, F(\sigma), t).$$

As a result, $\lim_{j\to\infty} \alpha(\sigma_j, \sigma)$ and $\lim_{j\to\infty} \alpha(\sigma, \sigma_j)$ exist, indicating limitations. It is obvious that $\{F^j(\sigma)\}_{j=0}^{\infty}$ is CS. Since $(\Xi, \psi, *)$ is a continuous FDCMS, the sequence $\{F^j(\sigma)\}_{j=0}^{\infty}$ has a limit, say $\sigma^* \in \Xi$. Thus, we obtain

$$\lim_{j\to\infty} F^j(\sigma) = \sigma^*. \tag{3.10}$$

Further, we show that $\sigma^* \in F$ is a FP.

$$\psi(\sigma^*, F(\sigma^*), t) \geq \alpha(\sigma^*, F^j(\sigma))\psi(\sigma^*, F^j(\sigma), t) + \beta(F^j(\sigma), F(\sigma^*))\psi(F^j(\sigma), F(\sigma^*), t)$$

$$\geq \alpha(\sigma^*, F^j(\sigma))\psi(\sigma^*, F^j(\sigma), t) + \beta(F^j(\sigma), F(\sigma^*))\kappa \begin{bmatrix} \psi(F^{j-1}(\sigma), F^j(\sigma), t) \\ +\psi(\sigma^*, F(\sigma^*), t) \end{bmatrix} + s\psi(\sigma^*, \sigma^*, t).$$

Further, taking the limit j approaching ∞, and considering 3.10 Theorem, 3.1, and $\lim_{j\to\infty} \alpha(\sigma_j, \sigma)$, $\lim_{j\to\infty} \alpha(\sigma, \sigma_j)$, $\lim_{j\to\infty} \beta(\sigma_j, \sigma)$, and $\lim_{j\to\infty} \beta(\sigma, \sigma_j)$ exist and we have a limit. We get

$$\psi(\sigma^*, F(\sigma^*), t) \geq (1+r)\alpha(\sigma^*, F(\sigma^*))\psi(\sigma^*, F(\sigma^*), t).$$

Hence, $\sigma^* = F(\sigma^*)$. By Theorem 3.2, σ^* is a unique FP. □

Example 3.3. *Assume that $\Xi = C([0,1], \mathbb{R})$ is the collection of all continuous real-valued functions on $[0,1]$. A function $d : \Xi \times \Xi \to [0, \infty)$ is defined by:*

$$d(f, g) = \sup_{\sigma\in[0,1]} \left\{ |f(\sigma) - g(\sigma)| + \frac{\int_0^1 |f(\sigma) - g(\sigma)|\, d\sigma}{1 + \int_0^1 |f(\sigma) + g(\sigma)|\, d\sigma} \right\},$$

where $f(\sigma) = \sin(2\sigma)$ and $g(\sigma) = \sin(3\sigma)$.

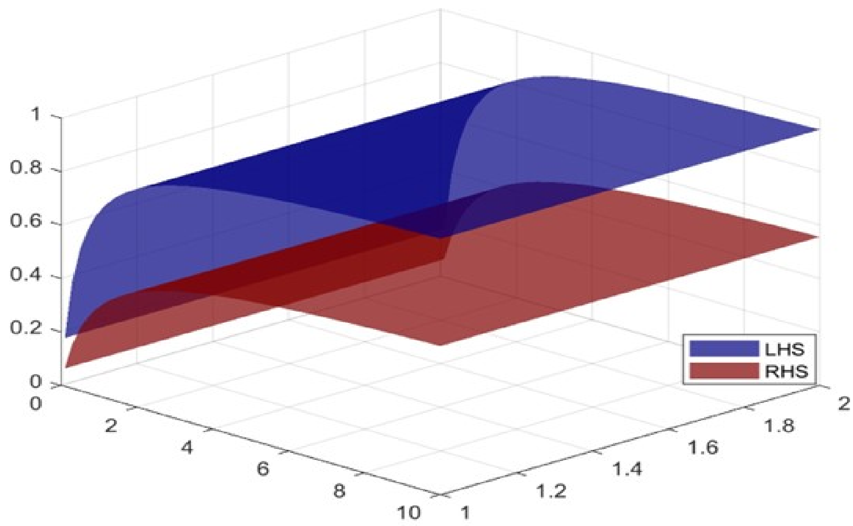

Figure 3: Depicts the behavior of the LHS and RHS of the inequality $\psi(F(A), F(B), t) \geq 0.1\,[\psi(A, F(A), t) + \psi(B, F(B), t)] + 0.3\,\psi(A, B, t)$.

A fuzzy function $\psi : \Xi \times \Xi \to [0, 1]$ *is given by*

$$\psi(f, g, t) = \frac{t}{t + d(f, g)}.$$

Two mappings $\alpha, \beta : \Xi \times \Xi \to [0, \infty)$ *are defined as:*

$$\alpha(f, g) = 1 + \sup_{\sigma \in [0,1]} |f(\sigma) - g(\sigma)|, \qquad \beta(f, g) = 1 + \int_0^1 |f(\sigma) + g(\sigma)|\, d\sigma.$$

Then, $(\Xi, \psi, *)$ *is a FDCMS. For* $\kappa = 0.3$ *and* $s = 0.4$*, we obtained*

$$\psi(F(f), F(g), t) \geq 0.3\big[\psi(f, F(f), t) + \psi(g, F(g), t)\big] + 0.4\psi(f, g, t).$$

Hence, F is a fuzzy Fisher contraction. All the conditions of Theorem 3.4 are satisfied and $f^*(\sigma) = 0$ *is a unique FP.*

??. *Hence, F is a fuzzy Fisher contraction. All conditions of Theorem 3.4 are satisfied, and the unique fixed point is* $f^*(\sigma) = 0$.

Corollary 3.1. *Suppose that* $(\Xi, \psi, *)$ *is a FDCMS and* $F : \Xi \to \Xi$ *is an FDC-FC on FDCMS. Then F satisfies the inequality*

$$\psi(F(\sigma), F(y), t) \;\geq\; \psi(\sigma, y, t),$$

for all $\sigma, y \in \Xi$*. Therefore, F has a unique fixed point in* Ξ.

4. Fuzzy Double Controlled F-Iterated Function System

In this section, we will discuss the HB theorem for generating fractals in continuous FDCMS using the IFS of FFC.

Theorem 4.1. *Let* $(\Xi, \psi, *)$ *is a FDCMS and self-mapping F on* Ξ *is a continuous FFCM with* κ *and s are the CRs. Subsequently, the function*

$$F : \mathcal{K}_0(\Xi) \to \mathcal{K}_0(\Xi), \quad F(A) = \{F(\sigma) : \sigma \in A\},$$

where $A \in \mathcal{K}_0(\Xi)$*. A mapping on* $(\mathcal{K}_0(\Xi), g_\psi, t)$ *is said to be FFCM with* κ *and s that are CRs.*

Proof: Assume that F is a continuous mapping. Moreover, a function $F : \mathcal{K}_0(\Xi) \to \mathcal{K}_0(\Xi)$ and suppose that A, B are the non-empty subsets of $\mathcal{K}_0(\Xi)$. Such that,

$$\begin{aligned} g_\psi(F(A), F(B), t) &= \psi(F(A), F(B), t) \vee \psi(F(B), F(A), t) \\ &\geq \kappa[\psi(A, F(A), t) + \psi(B, F(B), t)] + s\psi(A, B, t) \\ &\vee[\psi(B, F(B), t) + \psi(A, F(A), t)] + s\psi(B, A, t) \\ &= \kappa[\psi(A, F(A), t) + \psi(B, F(B), t)] + s\psi(A, B, t) \\ &\geq \kappa[g_\psi(A, F(A), t) + g_\psi(B, F(B), t)] + s[g_\psi(A, B, t)]. \end{aligned}$$

Moreover,

$$g_\psi(F(A), F(B), t) \geq \kappa[g_\psi(A, F(A), t) + g_\psi(B, F(B), t)] + s[g_\psi(A, B, t)]. \tag{4.1}$$

□

Theorem 4.2. *Suppose that $(\Xi, \psi, *)$ be a FDCMS and $\{\Xi; F_j; j = 1, 2, \cdots, N_0\}$ be a FDCF-IFS of continuous FFCM on $(\mathcal{K}_0(\Xi), g_\psi, t)$ with CRs κ_j and s_j respectively, for each j. A mapping F on $\mathcal{K}_0(\Xi)$ to itself is defined as:*

$$F(A) = \bigcup_{j=1}^{N_0} F_j(A), \tag{4.2}$$

for every $A \in \mathcal{K}_0(\Xi)$. Subsequently, F is a FFC with the CRs $\kappa = \max\{\kappa_j; j = 1, 2, \cdots, N_0\}$ and $s = \max\{s_j; j = 1, 2, \cdots, N_0\}$.

Proof: Mathematical induction and metric g_ψ characteristics are using for prove this theorem. The statement is clearly valid for $N = 1$ and $N = 2$. Thus, we may demonstrate that:

$$\begin{aligned} g_\psi(F(A), F(B), t) &= g_\psi(F_1(A) \cup F_2(A), F_1(B) \cup F_2(B), t) \\ &\geq g_\psi(F_1(A), F_1(B), t) \vee g_\psi(F_2(A), F_2(B), t) \\ &\geq \kappa_1[g_\psi(A, F_1(A), t) + g_\psi(B, F_1(B), t)] + s_1 g_\psi(A, B, t) \\ &\vee \kappa_2[g_\psi(A, F_2(A), t) + g_\psi(B, F_2(B), t)] + s_2 g_\psi(B, A, t) \\ &\geq (\kappa_1 \vee \kappa_2)\,[\,g_\psi(A, F_1(A), t) \vee g_\psi(A, F_2(A), t) + \psi(B, F(B), t)\,] \\ &+(s_1 \vee s_2) g_\psi(A, B, t) + \{\,g_\psi(B, F_1(B), t) \vee g_\psi(B, F_2(B), t)\,\} \\ &= \kappa[\,g_\psi(A, F_1(A) \cup F_2(A), t) + g_\psi(B, F_1(B) \cup F_2(B), t)\,] + s[g_\psi(A, B, t)]. \end{aligned}$$

Therefore,

$$g_\psi(F(A), F(B), t) \geq \kappa[g_\psi(A, F(A), t) + g_\psi(B, F(B), t)] + s[g_\psi(A, B, t)].$$

This theorem is proven using the idea of mathematical induction. Based on previous result and the FDCF-IFS idea, the following theorem may be proven. □

Theorem 4.3. *Suppose that $\{\Xi : (F_0), F_1, F_2, \cdots, F_{N_0}\}$ be a FDCF-IFS with F_0 a condensation mapping and*

$$\kappa = \max\{\kappa_j; j = 0, 1, 2, \cdots, N_0\}, \quad s = \max\{s_j; j = 0, 1, 2, \cdots, N_0\},$$

are the CRs respectively. Finally, the transformation $F : \mathcal{K}_0(\Xi) \to \mathcal{K}_0(\Xi)$ is defined as

$$F(A) = \bigcup_{j=1}^{N_0} F_j(A), \quad \forall A \in \mathcal{K}_0(\Xi), \tag{4.3}$$

is a continuous FFCM on continuous FDCMS $(\mathcal{K}_0(\Xi), g_\psi, t)$ *with the CRs* κ *and* s*. Furthermore,* F *has a FP* $B \in \mathcal{K}_0(\Xi)$ *that is unique. Then*

$$B = F(B) = \bigcup_{j=1}^{N_0} F_j(B) \tag{4.4}$$

given by

$$B = \lim_{j\to\infty} F^j(B), \quad \forall A \in \mathcal{K}_0(\Xi).$$

Proof: Because $(\Xi, \psi, *)$ is a continuous FDCMS, $(\mathcal{K}_0(\Xi), g_\psi, t)$ is also continuous FDCMS. Theorem 4.3 states that the HB operator F is a fuzzy contraction mapping in FDCMS. Theorem 3.2 indicates that F has a single FP. This concludes the discussion. □

Definition 4.1. The FP $B \in \mathcal{K}_0(\Xi)$ of the HB operator F for the FDCF-IFS described in Theorem 4.3 is called a *fuzzy double controlled Fisher attractor* or *FDCF-fractal* in FDCMS. Such that $B \in \mathcal{K}_0(\Xi)$ is called a fractal that is generated by a FDCF-IFS on FDCMS.

In this extension, we may additionally establish the collage theorem for the FDCF-IFS.

Theorem 4.4. *Let* $(\Xi, \psi, *)$ *be a continuous FDCMS. Given that* $A \in \mathcal{K}_0(\Xi)$ *and* $\varepsilon \geq 0$*, assume that* $\{\Xi; F_j, j = 0, 1, 2, \cdots, N_0\}$ *is a FDCF-IFS with condensation mapping* F_0 *and fuzzy Fisher CRs* κ *and* s *such that*

$$g_\psi\left(A, \bigcup_{j=0}^{N_0} F_j(A), t\right) \geq 1 - \varepsilon.$$

Then,

$$g_\psi(A, B, t) \geq \left[\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s}\right]\varepsilon,$$

where B *is an attractor or fuzzy double controlled* F*-fractal of the FDCF-IFS.*

Proof: Let us assume that $A \in \mathcal{K}_0(\Xi)$ and $\varepsilon \geq 0$. Choose a FDCF-IFS $\{\Xi; F_j, j = 0, 1, 2, \cdots, N_0\}$, where F_0 is the condensation mapping with fuzzy Fisher CRs κ and s respectively, so that

$$g_\psi\left(A, \bigcup_{j=0}^{N_0} F_j(A), t\right) \geq 1 - \varepsilon.$$

By Theorem 3.3, for $b \in \Xi$, we have

$$\lim_{i\to\infty} F^i(\sigma) = \sigma^*.$$

Consider a point $\sigma \in \Xi$ as fixed and the metric function $\psi(\sigma, f, t)$ continuous at a point $f \in \Xi$. Therefore,

$$\psi(\sigma, \sigma^*, t) = \psi\Big(\sigma, \lim_{i\to\infty} F^i(\sigma), t\Big) \geq \left[\left(1 - \frac{1}{1-\kappa}\right)^{-1} + \frac{1}{1-s}\right]\psi(\sigma, F(\sigma), t).$$

This implies that

$$g_\psi(A, B, t) \geq \left[\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s}\right]\varepsilon.$$

□

Example 4.1. *Suppose* $\Xi = C([0,1], \mathbb{R})$ *is the collection of all continuous real-valued functions on* $[0,1]$*. Define a mapping* $d : \Xi \times \Xi \to [0, \infty)$ *on* Ξ *by*

$$d(f, g) = \sup_{\sigma\in[0,1]}\left\{|f(\sigma) - g(\sigma)| + \frac{\int_0^1 |f(\sigma) - g(\sigma)|\, d\sigma}{1 + \int_0^1 |f(\sigma) + g(\sigma)|\, d\sigma}\right\}.$$

Define a fuzzy mapping $\psi : \Xi \times \Xi \to [0,1]$ *by*

$$\psi(f,g,t) = \frac{t}{t + d(f,g)}.$$

The collection of compact subsets of Ξ *is denoted by* $\mathcal{K}_0(\Xi)$ *and is equipped with the fuzzy Hausdorff metric defined by*

$$g_{\psi}(A,B,t) = \max\left\{ \sup_{\sigma\in A}\inf_{f\in B} \psi(\sigma,f,t),\ \sup_{f\in B}\inf_{\sigma\in A} \psi(f,\sigma,t) \right\},$$

where $A \in \mathcal{K}_0(\Xi)$ *and* B *is an attractor in* $\{\Xi; F_j, j = 0,1,2,\cdots,N_0\}$.

Take $\kappa = 0.3$, $s = 0.4$, $f(\sigma) = \cos(\sigma)$ *and* $g(\sigma) = \cos(2\sigma)$. *Define the mappings* $\{\Xi; F_j, j = 0,1,2\}$ *by*

$$F_0(f(\sigma)) = \tfrac{1}{4}f(\sigma) + \sigma^2, \quad F_1(f(\sigma)) = \tfrac{1}{3}f(\sigma) + \sin(\sigma), \quad F_2(f(\sigma)) = \tfrac{1}{5}f(\sigma) + e^{-\sigma}.$$

Now, we must prove that

$$g_{\psi}(A,B,t) \ \geq\ \left[\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s}\right]\varepsilon, \tag{4.5}$$

where $\varepsilon \geq 0$. *By taking* $\kappa = 0.3$ *and* $s = 0.4$ *in the above inequality 4.5, we obtain*

$$\frac{1-\kappa}{1-2\kappa} + \frac{1}{1-s} = 3.42.$$

That is,

$$g_{\psi}(A,B,t) \ \geq\ 3.42\varepsilon.$$

Which shows that the attractor B *of the FDCF-IFS is at most* 3.42ε *away from* A *in the fuzzy Hausdorff metric.*

In this general discussion, $\kappa_j = 0$ $(j = 1,2,\cdots,N_0)$ *are chosen as contractivity factors. Subsequently, the FDCF-IFS is reduced to the standard IFS. This FDCF-IFS becomes a fuzzy Kannan IFS if* $s_j = 0$ $(j = 1,2,\cdots,N_0)$. *The converse for the above two cases may or may not hold.*

5. Application to Dynamic Market Equilibrium Model

In this section, we demonstrate how our previous results can be applied to identify the unique solution of an integral equation that describes dynamic market equilibrium in economics. Supply Q_{β} and demand Q_{ς} are affected by the existing price and its trajectory in most markets, whether prices are growing or declining, and whether they are experiencing expansion or contraction. Economists typically require knowledge of the current price $P(t)$, its first derivative $\frac{\partial P(t)}{\partial t}$, and the second derivative $\frac{\partial^2 P(t)}{\partial t^2}$. Suppose that

$$Q_{\beta} = g_1 + \gamma_1 P(t) + w_1\frac{\partial P(t)}{\partial t} + y_1\frac{\partial^2 P(t)}{\partial t^2},$$

$$Q_{\varsigma} = g_2 + \gamma_2 P(t) + w_2\frac{\partial P(t)}{\partial t} + y_2\frac{\partial^2 P(t)}{\partial t^2},$$

where $g_1, g_2, \gamma_1, \gamma_2, w_1$, and w_2 are constants. We may say that the market is *dynamically stable* if pricing clears the market at every moment.

In equilibrium, $Q_{\beta} = Q_{\varsigma}$. Therefore,

$$g_1 + \gamma_1 P(t) + w_1\frac{\partial P(t)}{\partial t} + y_1\frac{\partial^2 P(t)}{\partial t^2} \ =\ g_2 + \gamma_2 P(t) + w_2\frac{\partial P(t)}{\partial t} + y_2\frac{\partial^2 P(t)}{\partial t^2}.$$

This reduces to

$$(y_1 - y_2)\frac{\partial^2 P(t)}{\partial t^2} + (w_1 - w_2)\frac{\partial P(t)}{\partial t} + (\gamma_1 - \gamma_2)P(t) = -(g_1 - g_2).$$

Letting $y = y_1 - y_2,\ w = w_1 - w_2,\ \gamma = \gamma_1 - \gamma_2,\ g = g_1 - g_2$, we obtain

$$y\frac{\partial^2 P(t)}{\partial t^2} + w\frac{\partial P(t)}{\partial t} + \gamma P(t) = -g.$$

The following initial value problem governs the dynamics of $P(t)$:

$$\begin{cases} P''(t) + \dfrac{w}{y}P'(t) + \dfrac{\gamma}{y}P(t) = -\dfrac{g}{y}, \\ P(0) = 0, \\ P'(0) = 0, \end{cases} \tag{5.1}$$

where $\dfrac{w^2}{y} = \dfrac{4\gamma}{y}$ and $\dfrac{\gamma}{w} = \mu$ are continuous functions.

It is simple to show that the integral equation and problem (5.1) are equivalent:

$$P(t) = \int_0^T k(t,r)F(t,r,P(r))\,dr,$$

where $k(t,r)$ is Green's function given by

$$k(t,r) = \begin{cases} r\,w^{\mu/2(t-r)}, & 0 \le r \le t \le T, \\ t\,w^{\mu/2(r-t)}, & 0 \le t \le r \le T. \end{cases}$$

Thus, we will show that there is a solution to the integral equation:

$$P(t) = \int_0^T G(t,r,P(r))\,dr. \tag{5.2}$$

For $t > 0$, define $X = C([0,T])$ as the collection of real continuous functions defined on $[0,T]$. Define

$$M(\sigma,y,t) = \begin{cases} 0, & t = 0, \\ \sup_{t\in[0,T]} \dfrac{\min\{\sigma,y\}+t}{\max\{\sigma,y\}+t}, & \text{otherwise}, \end{cases}$$

for all $\sigma, y \in X$ with $a_1 * a_2 = a_1 a_2$. Define $Q, W : X \times X \to [1,\infty)$ by

$$Q(y,\varsigma) = 1 + y + \varsigma, \qquad W(y,\varsigma) = 1 + y^2 + \varsigma^2.$$

It is easy to prove that $(X, M, N, O, *, \Delta)$ is a complete FDCMS. Define $F : X \to X$ by

$$FP(t) = \int_0^T G(t,r,P(r))\,dr.$$

Theorem 5.1. *Suppose equation 5.2 holds and assume that:*

1. $G : [0,T] \times [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function;*
2. *A function* $\psi : [0,T] \times [0,T] \to \mathbb{R}^+$ *is continuous, and* $\sup_{t\in[0,T]} \int_0^T \psi(t,r)\,dr \ge 1$*;*
3. $\max\{G(t,r,\sigma(r)) - G(t,r,y(r))\} \ge k(t,r)\max\{\sigma(r), y(r)\}$ *and*
$$\min\{G(t,r,\sigma(r)) - G(t,r,y(r))\} \ge k(t,r)\min\{\sigma(r), y(r)\}.$$

Then, under these conditions, the integral equation 5.2 admits a unique solution.

Proof: For $\sigma, y \in X$, applying the above axioms (i) to (iii), we obtain

$$\psi(F\sigma, Fy, t) = \sup_{t\in[0,T]} \frac{\min\left\{\int_0^T G(t,r,\sigma(r))\,dr,\ \int_0^T G(t,r,y(r))\,dr\right\} + t}{\max\left\{\int_0^T G(t,r,\sigma(r))\,dr,\ \int_0^T G(t,r,y(r))\,dr\right\} + t}$$

$$= \sup_{t\in[0,T]} \frac{\int_0^T \min\{G(t,r,\sigma(r)), G(t,r,y(r))\}\,dr + t}{\int_0^T \max\{G(t,r,\sigma(r)), G(t,r,y(r))\}\,dr + t}$$

$$\geq \sup_{t\in[0,T]} \frac{\int_0^T k(t,r)\min\{\sigma(r), y(r)\}\,dr + t}{\int_0^T k(t,r)\max\{\sigma(r), y(r)\}\,dr + t}$$

$$\geq \sup_{t\in[0,T]} \frac{\min\{\sigma(r), y(r)\}\int_0^T k(t,r)\,dr + t}{\max\{\sigma(r), y(r)\}\int_0^T k(t,r)\,dr + t}$$

$$\geq \frac{\min\{\sigma(r), y(r)\} + t}{\max\{\sigma(r), y(r)\} + t} = \psi(\sigma, y, t).$$

Thus, all conditions of Corollary 3.1 are satisfied. Therefore, equation 5.2 has a unique fixed point. □

6. Contribution

This study introduces several fundamental theoretical advancements and practical ideas in the domain of fuzzy metric spaces and fractal generation. The major contributions are summarized as follows:

- **Extension of Fuzzy Fisher Contractions to DCMSs:** By incorporating fuzzy double controlled metric spaces (FDCMSs)—a recently developed structure that enhances the flexibility of conventional fuzzy metric spaces through two control functions—the concept of fuzzy Fisher contractions (FFCs) is generalized.

- **Development of FDCF-IFS:** A novel framework for iterated function systems (IFS) is proposed, termed FDCF-IFS, which integrates fuzzy Fisher contractions within the FDCMS setting. This generalization surpasses the limitations of classical IFS theory, enabling the construction of intricate fractals under fuzzy and regulated conditions.

- **Introduction of FDCFFs:** A new class of fractals, referred to as fuzzy double controlled Fisher fractals (FDCFFs), is investigated. These fractals, which emerge as attractors of the proposed FDCF-IFS, demonstrate greater flexibility than classical fractals and exhibit features particularly well-suited for modeling uncertainty.

- **Formulation and Proof of the Collage Theorem in FDCMSs:** The classical Collage Theorem is extended to the fuzzy double controlled setting. This result not only strengthens the theoretical foundation of the FDCF-IFS framework but also provides more effective approximation tools for identifying attractors in fuzzy fractal systems.

- **Graphical and Example-Based Validation:** Theoretical developments are reinforced with graphical simulations and illustrative examples. These non-trivial cases highlight the convergence behavior of fractal-generating mappings and confirm the uniqueness of fixed points within the proposed framework.

7. Conclusion

In this work, we have developed a new framework for fractal generation by extending fuzzy Fisher contraction mappings into the structure of fuzzy double controlled metric spaces (FDCMS). Incorporating dual control functions and fuzzy contraction behavior, this integration resulted in the formulation of the fuzzy double controlled Fisher iterated function system (FDCF-IFS), which generalizes classical and fuzzy iterated function systems. We established several fixed-point theorems for fuzzy Fisher contraction inside the FDCMS framework, so proving the existence and uniqueness of fixed points under generalized contraction conditions. The work also introduced the idea of fuzzy double controlled Fisher fractals (FDCFFs), which stand for attractors of the proposed system. By means of this method, we developed and validated a fuzzy variant of the Collage Theorem modified for FDCMS, so providing a strong instrument for fractal approximation in fuzzy surroundings. Graphical representations and illustrative cases confirmed the behavior of contraction mappings and the uniqueness of fixed points in several fuzzy controlled environments, so supporting the theoretical conclusions. Our work not only generalizes known results in fuzzy metric spaces but also provides a strong basis for modeling complicated, multi-level, and uncertain systems using fractal theory. This work develops both fixed-point theory in fuzzy environments and new methods for fractal construction under uncertainty with possible applications in computational mathematics, image processing, and dynamic modeling in fuzzy environments.

8. Future Work

The framework presented in this work opens several interesting directions for next study in fuzzy metric spaces, fractal geometry, and uncertainty modeling:

- **Wide spectrum to intuitionistic and interval-valued fuzzy metric spaces:** The present work can be built upon thorough investigations on the generalization of fuzzy double controlled Fisher contraction to intuitionistic fuzzy metric spaces and interval-valued fuzzy settings. This would let one model even more complex uncertainty structures.
- **Growth of Fractals in Pentagonal and n-Gonal controlled metric spaces:** By means of more complex metric structures such as fuzzy pentagonal or n-gonal controlled metric spaces, the proposed approaches may be modified to offer a route to multi-directional control in fractal construction.
- **Investigating Stability and Dynamic Features:** Understanding FDCFF stability, periodicity, and chaotic behavior under different control values will help one to better appreciate their long-term dynamics and structural robustness.
- **Hybrid models combining fuzzy fractals and neural networks:** Fuzzy double controlled fractals combined with machine learning models—especially neural networks—show an interesting path to improve pattern recognition and adaptive modeling in uncertain environments.
- **Attractiveness Theory: Generalization with Topological Aspects:** The topological character of FDCFF attractors—that is, their dimension, continuity, and connectedness—may help to deepen mathematical knowledge of the constructions generated under this framework.

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References

1. Mandelbrot, B. B., *The Fractal Geometry of Nature*, W. H. Freeman, San Francisco, CA, 25–74, (1982).
2. Rodríguez-López, J.; Romaguera, S., *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets Syst. 147, 273–283, (2004).
3. Hutchinson, J. E., *Fractals and self-similarity*, Indiana Univ. Math. J. 30, 713–747, (1981).
4. Barnsley, M., *Fractals Everywhere*, 2nd ed., Academic Press, San Diego, CA, (1993).

5. Al-Sa'idi, N. M.; Sd, M. R. M.; Ahmed, A. M., *IFS on the multi-fuzzy fractal space*, Int. J. Math. Comput. Sci. 3, 338–342, (2009).

6. Al-Saidi, N.; Rushdan, M.; Ahmed, A., *About fuzzy fixed-point theorem in the generalized fuzzy fractal space*, Far East J. Math. Sci. 53, 53–64, (2011).

7. Easwaramoorthy, D.; Uthayakumar, R., *Analysis on fractals in fuzzy metric spaces*, Fractals 19, 379–386, (2011).

8. Ghosh, S.; Saha, P.; Roy, S.; Choudhury, B. S., *Strong coupled fixed points and applications to fractal generations in fuzzy metric spaces*, Problemy Analiza Issues Anal. 12, 50–68, (2023).

9. El-Nabulsi, R. A., *Emergence of quasiperiodic quantum wave functions in Hausdorff dimensional crystals*, J. Phys. Chem. Solids 127, 224–230, (2019).

10. Andres, J.; Fiser, J., *Metric and topological multivalued fractals*, Int. J. Bifurc. Chaos 14, 1277–1289, (2004).

11. Andres, J.; Fiser, J.; Gabor, G.; Lesniak, K., *Multivalued fractals*, Chaos Solitons Fractals 24, 665–700, (2005).

12. Andres, J.; Rypka, M., *Multivalued fractals and hyperfractals*, Int. J. Bifurc. Chaos 22, 1250009, (2012).

13. Singh, S. L.; Prasad, B.; Kumar, A., *Fractals via iterated functions and multifunctions*, Chaos Solitons Fractals 39, 1224–1231, (2009).

14. Prasad, B.; Katiyar, K., *Multi fuzzy fractal theorems in fuzzy metric spaces*, Fuzzy Inf. Eng. 9, 225–236, (2017).

15. Mishra, K.; Prasad, B., *Some generalized IFS in fuzzy metric spaces*, Adv. Fuzzy Math. 12, 297–308, (2017).

16. Gowrisankar, A.; Easwaramoorthy, D., *Local countable iterated function systems*, Adv. Algebra Anal. Trends Math. 1, 169–175, (2018).

17. Bakhtin, A., *The contraction mapping principle in almost metric spaces*, Funct. Anal. 30, 26–37, (1989).

18. Czerwik, S., *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis 1, 5–11, (1993).

19. Shatanawi, W.; Abodayeh, K.; Mukheimer, A., *Some fixed-point theorems in extended b-metric spaces*, UPB Sci. Bull. Ser. A 80, (2018).

20. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T., *Controlled metric type spaces and the related contraction principle*, Mathematics 6, 194, (2018).

21. Abdeljawad, T.; Mlaiki, N.; Aydi, H.; Souayah, N., *Double controlled metric type spaces and some fixed-point results*, Mathematics 6, 320, (2018).

22. Fisher, B., *A fixed-point theorem for compact metric spaces*, Publ. Math. Debrecen 25, 193–194, (1978).

23. Sahu, D. R.; Chakraborty, A.; Dubey, R. P., *K-iterated function system*, Fractals 18, 139–144, (2010).

24. Schweizer, B.; Sklar, A., *Statistical metric spaces*, Pac. J. Math. 10, 313–334, (1960).

25. Kramosil, I.; Michálek, J., *Fuzzy metrics and statistical metric spaces*, Kybernetika 11, 336–344, (1975).

26. Sezen, M. S., *Controlled fuzzy metric spaces and some related fixed-point results*, Numer. Methods Partial Differ. Equ. 37, 583–593, (2021).

27. Saleem, N.; Işık, H.; Furqan, S.; Park, C., *Fuzzy double controlled metric spaces and related results*, J. Intell. Fuzzy Syst. 40, 9977–9985, (2021).

28. Thangaraj, C.; Easwaramoorthy, D., *Fractals via controlled Fisher iterated function system*, Fractal Fract. 6, 746, (2022).

29. Saleem, N.; Ahmad, K.; Ishtiaq, U.; De la Sen, M., *Multivalued neutrosophic fractals and Hutchinson-Barnsley operator in neutrosophic metric space*, Chaos Solitons Fractals 172, 113607, (2023).

30. Ahmad, K.; Ishtiaq, U.; Murtaza, G.; Popa, I. L.; Maiz, F. M., *On product neutrosophic fractal spaces and α-density theory with arbitrarily small and controlled error*, Fractal Fract. 9, 59, (2025).

31. Ishtiaq, U.; Jahangeer, F.; Garayev, M.; Popa, I. L., *Existence and uniqueness results for fuzzy bipolar metric spaces*, Symmetry 17, 180, (2025).

32. Guo, L.; Alshaikey, S.; Alshejari, A.; Din, M.; Ishtiaq, U., *Numerical algorithm for coupled fixed points in normed spaces with applications to fractional differential equations and economics*, Fractal Fract. 9, 37, (2025).

33. Guo, L.; Bibi, R.; Alshejari, A.; Savas, E.; Kamran, T.; Ishtiaq, U., *Fixed-point results for Krasnoselskii, Meir–Keeler, and Boyd–Wong-type mappings with applications to dynamic market equilibrium*, Axioms 13, 867, (2024).

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