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Rings in Which the Product of Two Non-Nilpotent Elements is Non-Zero

Tikaram Subedi, Debraj Roy and Sanjiv Subba

ABSTRACT: This article studies rings wherein the product of two non nilpotent elements is non-zero. We call such rings 'nil-domains'. The class of nil-domains strictly contains the class of domains and local rings with nil Jacobson radical. We explore and study some ring extensions of nil-domain which preserves the nil-domain property. Also some consequences of nil-domain conditions on some closely related and well known classes of rings are looked into.

Keywords: Domain, nilpotent element, abelian ring, nil-domain.

Contents

1. Introduction

Throughout this paper, all rings are associative with identity and R denotes a ring. The symbols $E(R)$, $J(R)$, $N(R)$, $N^*(R)$, $P(R)$, $U(R)$ respectively stand for the set of all idempotent elements, the Jacobson radical, the set of all nilpotent elements, the upper nilradical, the prime radical and the set of all unit elements in R. We denote the left (right) annihilator of any element $a \in R$ by $l(a)$ $(r(a))$. R is *abelian* if all its idempotents are central. R is *left (right) quasi-duo* if every maximal left (right) ideal of R is an ideal and R is *quasi-duo* if it is both left and right quasi-duo. R is *reduced* if $N(R) = 0$. R is an *NI* ring if $N(R)$ is an ideal of R. R is semiprimitive if $J(R) = 0$. A left R-module M is *Wnil-injective* ([8]) if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and every left R-homomorphism from Ra^n to M extends to one from R to M. R is *left (right) Artinian* if R does not contain any infinite descending chain of left (right) ideals. R is a *left (right) SF-ring* ([5]) if its simple left (right) modules are flat. R is *strongly regular* if for any $a \in R$, there exists some $b \in R$ such that $a = a^2b$. It is known that strongly regular rings are left (right) SF-rings. It is hitherto unknown whether a left (right) SF-ring is necessarily strongly regular.

As a generalization of domains, it is quite natural to consider rings wherein the product of two non-nilpotent elements is non-zero. We call such rings 'nil-domains'. It turns out that nil-domains exhibit some interesting properties and have close relationship with some existing classes of rings. Among other results, we establish properties of nil-domains, investigate some ring extensions of nil-domains and provide conditions under which a nil-domain becomes a reduced ring or a division ring.

2. Nil-domains

Definition 2.1 *We call a ring R nil-domain if for any $a, b \in R$, $ab = 0$ implies either $a \in N(R)$ or $b \in N(R)$.*

Example 2.2

1. *Every domain is a nil-domain.*
2. *A local ring with nil Jacobson radical is a nil-domain.*
3. *Not every local ring is a nil-domain. Let $R = \mathbb{R}[x, y]/(xy)$, then $I = (\overline{x}, \overline{y})$ is a prime ideal of R. Consider the localization R_I. In R_I, $\overline{xy} = 0$ and $\overline{x} \notin N(R_I), \overline{y} \notin N(R_I)$. So R_I is not a nil-domain.*

The proof of the following proposition is trivial.

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Proposition 2.3 *Let R be a commutative ring. The following conditions are equivalent:*

1. *R is a nil-domain.*
2. *For any a_1, a_2, ...$a_n \in R$, $a_1a_2...a_n = 0$ implies $a_i \in N(R)$ for some i.*
3. *$N(R)$ is a prime ideal of R.*

Corollary 2.4 *For a positive integer n, $\mathbb{Z}_n$ is a nil-domain if and only if n is a power of a prime.*

Theorem 2.5 *Let R be a left Artinian ring. Then R is a nil-domain if and only if R is a local ring.*

Proof: Let R be nil-domain. For any $a \in R$, consider the following descending chain of left ideals:

$$Ra \supseteq Ra^2 \supseteq Ra^3 \cdots .$$

By hypothesis, there exists a positive integer n such that $Ra^n = Ra^{n+1} = \cdots$. Then $a^n \in Ra^n = Ra^{n+1}$ so that $a^n = ra^{n+1}$ for some $r \in R$ and so $(1 - ra)a^n = 0$. If $a \notin N(R)$, then $1 - ra \in N(R)$ which implies that $a \in U(R)$. Hence every non-unit of R is nilpotent so that R is a local ring. Also as $J(R)$ is nil for a left right Artinian ring, the converse follows.

□

Theorem 2.6 *Let R be a nil-domain. Then the characteristic of R is either zero or a power of a prime.*

Proof: Let $0 \neq n$ be the characteristic of R. Let p be a prime which divides n. Then $n = p^r q$ where r is a positive integer and q is a positive integer such that $p \nmid q$. Now, $n1 = 0 = (p^r1)(q1)$. As R is a nil-domain, we have $p^r1 \in N(R)$ or $q1 \in N(R)$. If $p^r1 \in N(R)$, then $n \mid p^k$ for some integer k. As $n = p^r q$ and $p \nmid q$, this leads to $q = 1$. Hence $n = p^r$. If $q1 \in N(R)$, then $n \mid q^l$ for some integer l. Therefore $\frac{q^l}{y} = p^r q$ for some integer y which implies that $q^{l-1} = p^r y$ which leads to $p \mid q^{l-1}$. Hence $p \mid 1$ or $p \mid q$ which is a contradiction. □

R is *weakly reversible* ([3]) if for any $a, b, r \in R, ab = 0$ implies $Rbra$ is a nil left ideal. R is said to be *weakly semicommutatative* ([2]) if for any $a, b \in R$, $ab = 0$ implies $arb \in N(R)$ for all $r \in R$.

Proposition 2.7 *Let R be a nil-domain. Then R is weakly reversible.*

Proof: Let $a, b, r \in R$ with $ab = 0$. For any $s \in R$, $(sbra)(bras) = 0$. By hypothesis, either $sbra \in N(R)$ or $bras \in N(R)$ which implies that $sbra \in N(R)$. Hence $Rbra$ is a nil left ideal. □

We also observe the following result which will be referred to later in our work.

Proposition 2.8 *A weakly reversible ring is weakly semicommutative.*

Proof: Let R be a weakly reversible ring and $a, b \in R$ with $ab = 0$. Then for any $r \in R$, $rba \in N(R)$ which implies that $arb \in N(R)$. □

Example 2.9 *There exists a local ring R with nil Jacobson radical such that $R[x]$ is not weakly semicommutative by ([2], Corollary 3.9). Therefore it follows that a polynomial ring over a nil-domain need not be a nil-domain.*

A non-zero, non-unit element in a commutative ring is said to be *irreducible* if it is not a product of two non-units. A non-zero non-unit element a in a commutative ring R is called *prime* if, whenever $a \mid bc$ for some $b, c \in R$, then either $a \mid b$ or $a \mid c$.

Theorem 2.10 *Let R be a commutative nil-domain and $p \in R$ be a prime element. Then p is either nilpotent or irreducible.*

Proof: Let $p \notin N(R)$ and $p = ab$ for some $a, b \in R$. By hypothesis, either $p \mid a$ or $p \mid b$. If $p \mid a$, then $a = px$ for some $x \in R$ leading to $p = pxb$, so that $p(1 - xb) = 0$. Since R is a nil-domain and $p \notin N(R)$, $1 - xb \in N(R)$ which implies that b is a unit. Similarly, $p \mid b$ implies a is a unit. Hence p is irreducible. □

Proposition 2.11 *Let R be a nil-domain which is not reduced, then R has a non zero nil one sided ideal.*

Proof: Let $0 \neq a \in N(R)$. Then there exists a positive integer $n > 1$ such that $a^n = 0$, so that $(a^{n-1})^2 = 0$. If Ra^{n-1} is not a nil left ideal, then there exists $s \in R$ such that $sa^{n-1} \notin N(R)$. For any $r \in R, sa^{n-1}a^{n-1}r = 0$. By hypothesis, $a^{n-1}r \in N(R)$, so $a^{n-1}R$ is a non-zero nil right ideal.

□

Corollary 2.12 *Let R be a nil-domain such that the Kothe's Conjencture holds in R. Then $N(R) \subseteq N^*(R)$.*

We now present an example that illustrates Theorems 2.5, 2.6, 2.10 and Proposition 2.11

Example 2.13 *Let p be a prime number and $n \geq 2$. Consider the ring $R = \mathbb{Z}_{p^n}$. Since R is finite, it is Artinian; it has a unique maximal ideal pR, hence R is local, and because every zero divisor in R is nilpotent, R is a nil-domain. The characteristic of R is $\mathrm{char}(R) = p^n$, and in particular $p \in N(R)$. Finally, the ring R is not reduced; indeed, the ideal pR is a nonzero nil ideal.*

An ideal I of a commutative ring R is *primary* if, for any a, $b \in R$, $ab \in I$ implies either $a \in I$ or $b^n \in I$ for some positive integer n.

The proof of the following proposition is trivial.

Proposition 2.14 *Let I be a primary ideal of a commutative ring R. Then R/I is a nil-domain.*

Theorem 2.15 *Let I be a nil ideal of a ring R such that R/I is a nil-domain, then R is a nil-domain.*

Proof: Let $a, b \in R$ such that $ab = 0$, then $\overline{a}\overline{b} = 0$ in R/I. By hypothesis, either $\overline{a} \in N(R/I)$ or $\overline{b} \in N(R/I)$ which implies that either $a^m \in I$ for some positive integer m or $b^n \in I$ for some positive integer n. This implies that $a \in N(R)$ or $b \in N(R)$ as I is a nil ideal. □

Corollary 2.16 *If $R/P(R)$ is a nil-domain then R is a nil-domain.*

An NI ring need not be a nil-domain.

Example 2.17 *$T_n(\mathbb{R})(n > 1)$, the ring of $n \times n$ upper triangular matrices over $\mathbb{R}$, is an NI ring but not a nil-domain.*

Theorem 2.18 *Let R be an NI ring.*

1. *If $R/N(R)$ is a nil-domain, then R is a nil-domain.*
2. *If the nilpotent elements are central, then R is a nil-domain implies $R/N(R)$ is a domain.*

Proof:

1. The result easily follows from Theorem 2.15.
2. Let R be a nil-domain and $\overline{a}$, $\overline{b} \in R/N(R)$ with $\overline{ab} = \overline{0}$. Then $ab \in N(R)$, so that there exists $n \in \mathbb{N}$ such that $(ab)^n = 0$. By hypothesis,, $a^n b^n = 0$. Therefore either $a \in N(R)$ or $b \in N(R)$. Hence $R/N(R)$ is a domain.

□

Let R be a commutative ring and M be a left R-module. Then $R\bigoplus M$ have a ring structure with the usual componentwise addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ where $r_1, r_2 \in R$, $m_1, m_2 \in M$. This extension is called the *Nagata extension*, denoted by $N(R, M)$, of R by M.

Lemma 2.19 *Let $N(R, M)$ be the Nagata extension of a commutative ring R by a left R-module M. For any $(r, m) \in N(R, M)$ and any positive integer k,*

$$(r, m)^k = (r^k, kr^{k-1}m).$$

Proof: We prove the result by induction on k, the power of (r, m). For $k = 1$, the result is true. Let the result be true for $k = n$. Now, $(r, m)^{n+1} = (r, m)^n(r, m) = (r^n, nr^{n-1}m)(r, m) = (r^{n+1},\ r^nm + nr^nm) = (r^{n+1},\ (n + 1)r^nm)$. So the result is true for $k = n + 1$ if it is true for $k = n$. □

Theorem 2.20 *A commutative ring R is a nil-domain if and only if $N(R, M)$ is a nil-domain for any left R-module M.*

Proof: By Lemma 2.19, $N(N(R, M)) = \{(r, m) \in N(R, M)|r \in N(R), m \in M\}$. Let R be a nil-domain and $\overline{(r_1, m_1)}, \overline{(r_2, m_2)} \in N(R, M)/N(N(R, M))$ such that $\overline{(r_1, m_1)(r_2, m_2)} = \overline{0}$. Then $r_1r_2 \in N(R)$, so that there exists $n \in \mathbb{N}$ such that $(r_1r_2)^n = 0$. Since R is commutative, $(r_1)^n(r_2)^n = 0$. By hypothesis, either $r_1 \in N(R)$ or $r_2 \in N(R)$. Therefore by Lemma 2.19, $N(R, M)/N(N(R, M))$ is a domain. Hence by Theorem 2.18, $N(R, M)$ is a nil-domain.
Converse is trivial. □

Theorem 2.21 *Let R be a ring and S be a multiplicatively closed subset of R consisting of central regular elements. Then R is a nil-domain if and only if $S^{-1}R$ is a nil-domain.*

Proof: If $S^{-1}R$ is a nil-domain, then it is clear that R is a nil-domain. Conversely, let R be a nil-domain and $\alpha,\ \beta \in S^{-1}R$ such that $\alpha\beta = 0$. Let $\alpha = m^{-1}a, \beta = n^{-1}b$ where $m, n \in S, a,\ b \in R$. As $S \subseteq Z(R)$, $0 = \alpha\beta = m^{-1}an^{-1}b = (m^{-1}n^{-1})ab = (mn)^{-1}ab$, so $ab = 0$. Since R is a nil-domain, either $a \in N(R)$ or $b \in N(R)$. Therefore $\alpha \in N(S^{-1}R)$ or $\beta \in N(S^{-1}R)$ which implies that $S^{-1}R$ is a nil-domain. □

R is *Armendariz* ([6]) if for any $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$, $f(x)g(x) = 0$ implies $a_ib_j = 0$ for every $i,\ j$.

A nil-domain need not be an Armendariz ring as shown by the following example:

Example 2.22 *Consider $R = N(\mathbb{Z}_8, \mathbb{Z}_8)$, the Nagata extension of $\mathbb{Z}_8$ by $\mathbb{Z}_8$. By Theorem 2.20, R is a nil-domain but by ([6], Example 3.2), R is not an Armendariz ring.*

Theorem 2.23 *Consider the following statements:*

1. *R is a nil-domain.*
2. *$R[x]$ is a nil-domain.*
3. *The ring of Laurent polynomials $R[x; x^{-1}]$ is a nil-domain.*

Then $(2) \Longrightarrow (3) \Longrightarrow (1)$. Further $(1) \Longrightarrow (2)$ if R is an Armendariz ring.

Proof: Let R be an Armendariz ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Let $f(x) \notin N(R[x])$. Since R is Armendariz, by ([1], Lemma 2.6) we have $N(R)[x] \subseteq N(R[x])$. So there exists k such that $a_k \notin N(R)$. Since R is Armendariz, $a_k b_j = 0$ for all j. As R is a nil-domain and $a_k \notin N(R)$, $b_j \in N(R)$ for all j. Hence by ([1], Lemma 2.6), $g(x) \in N(R[x])$.
(2) $\Longrightarrow$ (3). Assume $R[x]$ is a nil-domain and let $S = \{1, x, x^2, \cdots, x^n, \cdots\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Therefore by Theorem 2.21, $S^{-1}R[x]$ is a nil-domain. Since $R[x; x^{-1}] \simeq S^{-1}R[x]$, the result follows.
(3) $\Longrightarrow$ (1) is trivial. □

Corollary 2.24 *For any positive integer n, $\mathbb{Z}_n[x]$ is a nil-domain if and only if n is a power of a prime.*

Theorem 2.25 *The following conditions are equivalent for a ring R:*

1. *R is a nil-domain.*
2. $S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \dots & a_{1n} \\ 0 & a & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} : a, a_{ij} \in R, i < j \leq n \right\}$ *is a nil-domain for any $n \geq 2$.*

Proof: Let $I = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} : a_{ij} \in R,\ i < j \leq n \right\}$.
Then I is a nil ideal of $S_n(R)$.

(2) $\Longrightarrow$ (1) trivially follows.
(1) $\Longrightarrow$ (2). Since $S_n(R)/I \simeq R$, by Theorem 2.15, $S_n(R)$ is a nil-domain.

□

Remark 2.26 *It is easy to see that for a nil-domain R, $E(R) = \{0, 1\}$.*

Remark 2.27 *For any nil-domain R, $T_n(R)(n > 1)$ can never be a nil-domain as $T_n(R)$ always have non-trivial idempotents.*

It is clear that a reduced nil-domain is a domain. In the following, we see that under some significant conditions nil-domains turn out to be reduced (and hence a domain).

Theorem 2.28 *Let R be a nil-domain. Then R is reduced if R satisfies any of the following conditions:*

1. *R is semiprimitive.*
2. *Every simple singular left R-module is Wnil-injective.*
3. *R is a left SF-ring*

Proof: It is enough to prove that R is reduced. Let R satisfy condition (1). If R is not reduced, then by Proposition 2.11, $N^*(R) \neq 0$, so that $J(R) \neq 0$, a contradiction. Therefore R is reduced.

Let R satisfy condition (2) and $0 \neq a \in R$ such that $a^2 = 0$. There exists a maximal left ideal L of R such that $l(a) \subseteq L$. It follows that L is essential and there exists some $b \in R$ such that $1 - ab \in L$. Since $baab = 0$ and R is a nil-domain, $ab \in N(R)$. Therefore $1 - ab$ is a unit which is a contradiction. Hence R is reduced.

Let R be a left SF-ring and $a \in R$ such that a is not a unit. Then $Ra \neq R$, so that there exists a maximal left ideal M of R containing Ra. As $a \in M$ and R is a left SF-ring, by ([7], Lemma 3.14), there exists $b \in M$ such that $a = ab$. Then $a(1-b) = 0$. As R is a nil-domain, this leads to $a \in N(R)$ or $1-b \in N(R)$. If $1-b \in N(R)$, then b is a unit which is a contradiction, so $a \in N(R)$. Therefore R is a local ring and hence a left quasi duo ring. Since a left quasi-duo, left SF-ring is strongly regular ([7], Theorem 4.10), R is reduced.

□

Proposition 2.29 *Let R be a nil-domain which is a left SF-ring. Then R is a division ring.*

Proof: Let $0 \neq a \in R$. If a is not a unit then a is an element of some maximal left ideal M of R. As R is left SF-ring, there exists some $b \in M$ such that $a = ab$. Then $a(1-b) = 0$. In view of Theorem 2.28, $b - 1 = 0$, that is $b = 1$ a contradiction. □

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Tikaram Subedi,
Department of Mathematics,
National Institute of Technology Meghalaya,
Sohra-793108, Meghalaya, India.
E-mail address: `tsubedi2010@gmail.com`

and

Debraj Roy,
Department of Mathematics,
Budge Budge Institute of Technology,
Kolkata-700137, India.
E-mail address: `debraj.hcu@gmail.com`

and

Sanjiv Subba,
School of Applied Sciences,
UPES, Bidholi-248007,
Dehradun, India.
E-mail address: `sanjivsubba59@gmail.com`