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Results of fuzzy prime near-rings with fuzzy involutions

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ABSTRACT: The main objective of this paper is to introduce the concept of fuzzy involution on fuzzy near-rings, and we prove that a fuzzy near-ring satisfying certain identities involving fuzzy involution must be a fuzzy ring or a commutative fuzzy ring. Also, an example proving the existence of this type of mapping is given.

Key Words: fuzzy group, fuzzy near-rings, involutions.

Contents

1. Introduction

Throughout the paper X will denote a nonempty classical set and θ is a fixed number in $[0,1[$. As noted in [15], a fuzzy subset T of $X \times X \times X$ is said to be a fuzzy binary operation on X if: (i) $\forall\, x, y \in X, \exists\, z \in X$ such that $T(x, y, z) > \theta$, (ii) $\forall\, x, y, t_1, t_2 \in X$, if $T(x, y, t_1) > \theta$ and $T(x, y, t_2) > \theta$, then $t_1 = t_2$. This famous definition allowed to Yuan and Lee [15] to give the definition of a fuzzy group and some of their basic results as follows: a set X equipped with a fuzzy binary operation T is called a fuzzy group if

1. $\forall a, b, c, c_1, c_2 \in X, ((a \circ b) \circ c)(c_1) > \theta$ and $(a \circ (b \circ c))(c_2) > \theta \implies c_1 = c_2$.
2. $\exists\, e \in X$ satisfies $\forall x \in X$, $(e \circ x)(x) > \theta$ and $(x \circ e)(x) > \theta$, e is called the identity element of (X, T).
3. $\forall x \in X$, $\exists\, y \in X$ such that $(x\ o\ y)(e) > \theta$ and $(y\ o\ x)(e) > \theta$, in this case y is called the inverse element of x and is denoted by x^{-1}. Moreover, (X, T) is said to be abelian if it has the property $\forall x, y, z \in X$, $T(x, y, z) > \theta \iff T(y, x, z) > \theta$.

By taking advantage of this study and by being motived by the classical theory, in [1] H. Aktaş and N. Çagman defined a fuzzy ring as a nonempty set X equipped with two fuzzy binary operations T and L which satisfy the four followings conditions:

$i)$ (X, T) is a fuzzy group abelian,

$ii)$ $(a * (b * c))(x_1) > \theta$ and $((a * b) * c)(x_2) > \theta \implies x_1 = x_2$, $\forall a, b, c, x_1, x_2 \in X$,

$iii)$ $(a * (b \circ c))(x_1) > \theta$ and $((a * b) \circ (a * c))(x_2) > \theta \implies x_1 = x_2$, $\forall a, b, c, x_1, x_2 \in X$,

$iv)$ $((b \circ c) * a)(x_1) > \theta$ and $((b * a) \circ (c * a))(x_2) > \theta \implies x_1 = x_2, \forall a, b, c, x_1, x_2 \in X$.

Noting that the two operations $\circ$ and $*$ are two mappings constructed from the fuzzy binary operations T and L, respectively, as shown in the preliminary section.
After theses studies, many researchers published enormous works (see [4,6,7,12,13,14], where further references can be found). Motivated by this theory, we defined in [5] the notion of left fuzzy near-ring (resp. right fuzzy near-ring), and we provided the definition of a prime fuzzy near-ring along with its

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multiplicative center. Additionally, some of their fundamental properties have been indicated. In this study, motivated by the notion of involution given in ordinary near-rings (see for example [11]), we will define the notion of fuzzy involution on fuzzy near-rings. For unexplained notions of near-rings employed in this work, the reader is referred to [2,3,8,9,10]. Subsequently, we will show, in the light of this new definition and some other conditions, that a fuzzy near-ring must be a fuzzy ring.

2. Preliminary

Let T and L be two fuzzy binary operations on X, as noted in [1], we can defined the following mappings:

$$\begin{array}{rcl} \circ : \mathbb{F}(X)\times\mathbb{F}(X) & \longrightarrow & \mathbb{F}(X) \\ (\mu, v) & \longmapsto & \mu\circ v \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} * : \mathbb{F}(X)\times\mathbb{F}(X) & \longrightarrow & \mathbb{F}(X) \\ (\mu, v) & \longmapsto & \mu * v \end{array},$$

where $\mathbb{F}(X) = \{\mu \,/\, \mu : X \longrightarrow [0,1]\}$, and

$$\begin{cases} \forall\, \mu, v \in \mathbb{F}(X),\ z \in X,\ (\mu * v)(z) = \bigvee_{x,y\in X}\Big(\mu(x)\wedge v(y)\wedge L(x,y,z)\Big), \\ \forall\, \mu, v \in \mathbb{F}(X),\ z \in X,\ (\mu \circ v)(z) = \bigvee_{x,y\in X}\Big(\mu(x)\wedge v(y)\wedge T(x,y,z)\Big). \end{cases}$$

Let $x, y \in X$, $\mu = \{x\}$ and $v = \{y\}$, and let $\mu\circ v$ and $\mu * v$ be denoted by $x\circ y$ and $x * y$, respectively. Then, we have

$$(x\circ y)(z) = T(x,y,z)\ \forall z \in X,$$

$$(x * y)(z) = L(x,y,z)\ \forall z \in X,$$

$$((x\circ y)\circ z)(t) = \bigvee_{h\in X}\Big(T(x,y,h)\wedge T(h,z,t)\Big)\ \forall z,t \in X,$$

$$(x\circ (y\circ z))(t) = \bigvee_{h\in X}\Big(T(y,z,h)\wedge T(x,h,t)\Big)\ \forall z,t \in X,$$

$$((x * y) * z)(t) = \bigvee_{h\in X}\Big(L(x,y,h)\wedge L(h,z,t)\Big)\ \forall z,t \in X,$$

$$(x * (y * z))(t) = \bigvee_{h\in X}\Big(L(y,z,h)\wedge L(x,h,t)\Big)\ \forall z,t \in X,$$

$$(x * (y\circ z))(t) = \bigvee_{h\in X}\Big(T(y,z,h)\wedge L(x,h,t)\Big)\ \forall z,t \in X,$$

$$((x * y)\circ (x * z))(t) = \bigvee_{d,s\in X}\Big(L(x,y,d)\wedge L(x,z,s)\wedge T(d,s,t)\Big)\ \forall z,t \in X.$$

Lemma 2.1 [15, Proposition 2.1] *Let (X, T) be a fuzzy group, then*

1- $(x\circ y)(a) > \theta$ *and* $(x\circ z)(a) > \theta \implies y = z$;

2- $(a\circ x)(y) > \theta$ *and* $(b\circ x)(y) > \theta \implies a = b$;

3- $(a\circ b)(c) > \theta$ *and* $(b^{-1}\circ a^{-1})(d) > \theta \implies d = c^{-1}$;

4- $(a\circ a)(a) > \theta \implies a = e$.

Definition 2.1 [5, Definition 1.1] *A non empty set X equipped with two fuzzy binary operations T and L is said to be left fuzzy near-ring if the following properties hold:*

(i) (X, T) *is a fuzzy group, not necessarily commutative.*

(ii) $(a * (b * c))(x_1) > \theta$ *and* $((a * b) * c)(x_2) > \theta \implies x_1 = x_2,\ \forall a, b, c, x_1, x_2 \in X$.

(iii) $(a * (b\circ c))(x_1) > \theta$ *and* $((a * b)\circ (a * c))(x_2) > \theta \implies x_1 = x_2,\ \forall a, b, c, x_1, x_2 \in X$.

Further, if we change the condition (iii) by:
(iv) $((b \circ c) * a)(x_1) > \theta$ *and* $((b * a) \circ (c * a))(x_2) > \theta \implies x_1 = x_2, \forall a, b, c, x_1, x_2 \in X,$ *then* (X, T, L) *is called right fuzzy near-ring.*

Definition 2.2 [5, Definition 1.1] *Let* (X, T, L) *be a fuzzy near-ring.* (X, T, L) *is called a prime fuzzy near-ring, if it has the property:*

$$\forall x, y, z \in X, \ ((x * y) * z)(e) > \theta \implies x = e \text{ or } z = e.$$

Lemma 2.2 [5, Lemma 3.1 & Lemma 3.2]
1- Let (X, T, L) *be a left fuzzy near-ring, then*

$$\forall a, b \in X, \quad \big((a * b) \circ (a * b^{-1})\big)(e) > \theta.$$

2- Let (X, T, L) *be a right fuzzy near-ring, then*

$$\forall a, b \in X, \quad \big((a * b) \circ (a^{-1} * b)\big)(e) > \theta.$$

3. Main results

We begin by giving the definition of the concept of a fuzzy left semigroup ideal, a fuzzy right semigroup ideal and fuzzy semigroup ideal of a fuzzy near-ring as follows:

Definition 3.1 *Let* (X, T, L) *be a fuzzy near-ring and* I *a nonempty subset of* X*,*

1. *If* $(x * s)(t) > \theta \implies t \in I, \forall x \in X, \forall s \in I$*, then* I *is called a fuzzy left semigroup ideal of* X*.*
2. *If* $(s * x)(t) > \theta \implies t \in I, \forall x \in X, \forall s \in I$*, then* I *is called a fuzzy right semigroup ideal of* X*.*
3. *If* I *is both fuzzy right semigroup ideal and fuzzy left semigroup ideal, then* I *is called a fuzzy semigroup ideal of* X*. Moreover,* I *is said to be nontrivial if* $I \neq \{e\}$*.*

Definition 3.2 *Let* (X, T, L) *be a fuzzy near-ring. A mapping* σ *defined on* X *is a called fuzzy involution on* X *if the following conditions are hold:*

1. $T(x, y, z) > \theta \implies T(\sigma(x), \sigma(y), \sigma(z)) > \theta, \forall x, y, z \in X,$
2. $L(x, y, z) > \theta \implies L(\sigma(y), \sigma(x), \sigma(z)) > \theta, \forall x, y, z \in X,$
3. $\forall x \in X, \ \sigma(\sigma(x)) = x.$

The following example shows the existence of this type of mapping.

Example 3.1 *Let* $X = \{e, a\}$ *and define the fuzzy binary operations* T *and* L *on* X*, where* $\theta = 0,5$ *by:*

$$\begin{array}{llll} T(e,e,e) = 1, & T(e,a,e) = 0, & T(a,e,e) = 0, & T(a,a,e) = 1, \\ T(e,e,a) = 0, & T(e,a,a) = 1, & T(a,e,a) = 1, & T(a,a,a) = 0, \end{array}$$

and

$$\begin{array}{llll} L(e,e,e) = 1, & L(e,a,e) = 1, & L(a,e,e) = 1, & L(a,a,e) = 1, \\ L(e,e,a) = 0, & L(e,a,a) = 0, & L(a,e,a) = 0, & L(a,a,a) = 0. \end{array}$$

It is clear that (X, T, L) *is a left fuzzy near-ring. We define a mapping* σ *on* X *by* $\sigma(e) = e$ *and* $\sigma(a) = a$*. It is clear that* σ *is a fuzzy involution on* X*.*

We begin by proving the following result, which is the key essential to prove our subsequent theorems.

Theorem 3.1 *Let* (X, T) *be a fuzzy group. Then,*

$$(X, T) \text{ is abelian } \iff \forall x, y \in X, \ ((x \circ y) \circ (x^{-1} \circ y^{-1}))(e) > \theta.$$

In order to prove this theorem, we first show the following lemma:

Lemma 3.1 *Let (X,T) be a fuzzy group. Then,*

$$\forall x, y \in X,\ ((x \circ y) \circ (y^{-1} \circ x^{-1}))(e) > \theta.$$

Proof: Let $x, y \in X$ and taking $t \in X$ such that $T(x,y,t) > \theta$. Likewise, there is $d \in X$ satisfies $T(y^{-1}, x^{-1}, d) > \theta$. So that, because of Lemma 2.1 (3), we have $d = t^{-1}$, then $T(y^{-1}, x^{-1}, t^{-1}) > \theta$ and therefore

$$\big((x \circ y) \circ (y^{-1} \circ x^{-1})\big)(e) \geq T(x,y,t) \wedge T(y^{-1}, x^{-1}, t^{-1}) \wedge T(t, t^{-1}, e) > \theta.$$

□

Proof of Theorem 3.1:
First, assuming that (X,T) is abelian and prove that $\forall x, y \in X,\ \big((x \circ y) \circ (x^{-1} \circ y^{-1})\big)(e) > \theta$.
Let $x, y \in X$, we have

$$\big((x \circ y) \circ (x^{-1} \circ y^{-1})\big)(e) = \bigvee_{d_1, d_2 \in X} \Big(T(x,y,d_1) \wedge T(x^{-1}, y^{-1}, d_2) \wedge T(d_1, d_2, e)\Big).$$

Let $h_1, h_2 \in X$ such that $T(x,y,h_1) > \theta$ and $T(x^{-1}, y^{-1}, h_2) > \theta$. Since (X,T) is abelian, we conclude that $T(y,x,h_1) > \theta$. From Lemma 3.1, we have $\big((y \circ x) \circ (x^{-1} \circ y^{-1})\big)(e) > \theta$, which gives $T(h_1, h_2, e) > \theta$. Consequently,

$$\big((x \circ y) \circ (y^{-1} \circ x^{-1})\big)(e) \geq T(x,y,h_1) \wedge T(x^{-1}, y^{-1}, h_2) \wedge T(h_1, h_2, e) > \theta.$$

Now, assume that $\forall z, t \in T, ((t \circ z) \circ (t^{-1} \circ z^{-1}))(e) > \theta$, and check that (X,T) is abelian.
Let $x, y \in X$ and consider $h \in X$ such that $T(x,y,h) > \theta$. Our goal is to show that $T(y,x,h) > \theta$. Indeed, let $t, l \in X$ such that $T(y,x,t) > \theta$ and $T(x^{-1}, y^{-1}, l) > \theta$.
By our assumption, we have

$$((x \circ y) \circ (x^{-1} \circ y^{-1}))(e) > \theta$$

which implies that,

$$T(h, l, e) > \theta. \tag{3.1}$$

On the other hand, in view of Lemma 3.1, we have $((y \circ x) \circ (x^{-1} \circ y^{-1}))(e) > \theta$. Thereby,

$$T(t, l, e) > \theta. \tag{3.2}$$

From (3.1) and (3.2) together Lemma 2.1 (2), we find that $t = h$ and hence, $T(y,x,h) > \theta$.
Conversely, suppose that $T(y,x,h) > \theta$ and check that $T(x,y,h) > \theta$. For this purpose, let t, l be two elements of X satisfy $T(x,y,t) > \theta$ and $T(x^{-1}, y^{-1}, l) > \theta$ and using the same technics as used above, we arrive at $T(x,y,h) > \theta$ which means that (X,T) is abelian. This ends the proof.

Lemma 3.2 *i) Let (X,T,L) be a left fuzzy near-ring. Then,*

$$\forall\, k, x, y \in X,\ \Big(\big((x \circ y) * k\big) \circ \big((x \circ y) * k^{-1}\big)\Big)(e) > \theta.$$

ii) Let (X,T,L) be a right fuzzy near-ring. Then,

$$\forall\, k, x, y \in X,\ \Big(\big(k * (x \circ y)\big) \circ \big(k^{-1} * (x \circ y)\big)\Big)(e) > \theta.$$

Proof: *i)* Let $k, x, y \in X$, then there exist $t, t_1, t_2 \in X$ such that $T(x,y,t) > \theta$, $L(t,k,t_1) > \theta$ and $L(t, k^{-1}, t_2) > \theta$, which allowed us to deduce that $\big((x \circ y) * k\big)(t_1) > \theta$ and $\big((x \circ y) * k^{-1}\big)(t_2) > \theta$. On the other hand, from Lemma 2.2 (1), we have

$$\big((t * k) \circ (t * k^{-1})\big)(e) > \theta.$$

In view of $L(t,k,t_1)>\theta$ and $L(t,k^{-1},t_2)>\theta$, the last result gives $T(t_1,t_2,e)>\theta$ and thus

$$\begin{aligned}\Big(\big((x\circ y)*k\big)\circ\big((x\circ y)*k^{-1}\big)\Big)(e) &\geq \big((x\circ y)*k\big)(t_1)\wedge\big((x\circ y)*k^{-1}\big)(t_2)\wedge T(t_1,t_2,e)\\ &> \theta.\end{aligned}$$

ii) Let $k,x,y\in X$. Taking $t,t_1,t_2\in X$ such that $T(x,y,t)>\theta$, $L(k,t,t_1)>\theta$ and $L(k^{-1},t,t_2)>\theta$. It follows that $\big(k*(x\circ y)\big)(t_1)>\theta$ and $\big(k^{-1}*(x\circ y)\big)(t_2)>\theta$. Also, from Lemma 2.2 (2), we have

$$\Big((k*t)\circ(k^{-1}*t)\Big)(e)>\theta.$$

By taking into account $L(k,t,t_1)>\theta$ and $L(k^{-1},t,t_2)>\theta$ together the previous relation, we arrive at $T(t_1,t_2,e)>\theta$. Accordingly,

$$\begin{aligned}\Big(\big(k*(x\circ y)\big)\circ\big(k^{-1}*(x\circ y)\big)\Big)(e) &\geq \big(k*(x\circ y)\big)(t_1)\wedge\big(k^{-1}*(x\circ y)\big)(t_2)\wedge T(t_1,t_2,e)\\ &> \theta.\end{aligned}$$

□

Theorem 3.2 *Let (X,T,L) be a fuzzy near-ring. If (X,T,L) admits a fuzzy involution σ, then (X,T,L) is a fuzzy ring.*

Proof: • Firstly, suppose that (X,T,L) is a left fuzzy near-ring and show that (X,T,L) is a right fuzzy near-ring. For this, let $x,y,z,z_1,z_2\in X$ such that $\big((x\circ y)*z\big)(z_1)>\theta$ and $\big((x*z)\circ(y*z)\big)(z_2)>\theta$, our purpose is to verify that $z_1=z_2$.
Taking $t_1\in X$ satisfying $T(x,y,t_1)>\theta$ and using the fact that $\big((x\circ y)*z\big)(z_1)>\theta$, we obtain $L(t_1,z,z_1)>\theta$. In view of Definition 3.2(1.) and (2.), we infer that

$$T(\sigma(x),\sigma(y),\sigma(t_1))>\theta \text{ and } L(\sigma(z),\sigma(t_1),\sigma(z_1))>\theta. \tag{3.3}$$

Let $t_2,t_3\in X$ such that $L(x,z,t_2)>\theta$ and $L(y,z,t_3)>\theta$, the fact that $\big((x*z)\circ(y*z)\big)(z_2)>\theta$ gives $T(t_2,t_3,z_2)>\theta$. Once again invoking Definition 3.2 (1.) and (2.), we get

$$L(\sigma(z),\sigma(x),\sigma(t_2))>\theta\ ,\ \ L(\sigma(z),\sigma(y),\sigma(t_3))>\theta \text{ and } T(\sigma(t_2),\sigma(t_3),\sigma(z_2))>\theta. \tag{3.4}$$

Combining 3.3 and 3.4, we find that

$$\begin{aligned}\Big(\sigma(z)*\big(\sigma(x)\circ\sigma(y)\big)\Big)\big(\sigma(z_1)\big) &\geq T\Big(\sigma(x),\sigma(y),\sigma(t_1)\Big)\wedge L\Big(\sigma(z),\sigma(t_1),\sigma(z_1)\Big)\\ &> \theta,\end{aligned}$$

and

$$\begin{aligned}\Big(\big(\sigma(z)*\sigma(x)\big)\circ\big(\sigma(z)*\sigma(y)\big)\Big)\big(\sigma(z_2)\big) &\geq L\Big(\sigma(z),\sigma(x),\sigma(t_2)\Big)\wedge L\Big(\sigma(z),\sigma(y),\sigma(t_3)\Big)\\ &\quad \wedge T\Big(\sigma(t_2),\sigma(t_3),\sigma(z_2)\Big)\\ &> \theta.\end{aligned}$$

Taking account the condition (*iii*) in Definition 2.1, the last two relations give $\sigma(z_1)=\sigma(z_2)$. It follows that $\sigma(\sigma(z_1))=\sigma(\sigma(z_2))$, which means that $z_1=z_2$ by Definition 3.2 (3.).
Now, assume that (X,T,L) is a right fuzzy near-ring and prove that (X,T,L) is a left fuzzy near-ring. Let $x,y,z,z_1,z_2\in X$ satisfy $\big(z*(x\circ y)\big)(z_1)>\theta$ and $\big((z*x)\circ(z*y)\big)(z_2)>\theta$ and show that $z_1=z_2$. By using the similar approach as used in first case, we get the required result.
• Secondly, showing that (X,T) is commutative. It suffices to prove the result of Theorem 3.1, i.e,

$\forall x, y \in X$, $\big((x\circ y)\circ(x^{-1}\circ y^{-1})\big)(e) > \theta$. In fact, let $x, y \in X$ and taking $v_1, v_2, h \in X$ satisfy $T(x, y, v_1) > \theta$, $T(x^{-1}, y^{-1}, v_2) > \theta$ and $T(v_1, v_2, h) > \theta$. It follows that,

$$\big((x \circ y) \circ (x^{-1} \circ y^{-1})\big)(h) \geq T(x, y, v_1) \wedge T(x^{-1}, y^{-1}, v_2) \wedge T(v_1, v_2, h) > \theta. \tag{3.5}$$

Let $k \in X$, from Lemma 3.2 (ii), we have

$$\Big(\big(k * (x \circ y)\big) \circ \big(k^{-1} * (x \circ y)\big)\Big)(e) > \theta. \tag{3.6}$$

Now, let $m, n \in X$ such that $L(k, v_1, m) > \theta$ and $L(k^{-1}, v_1, n) > \theta$. As a consequence of this, we obtain

$$\big(k * (x \circ y)\big)(m) \geq T(x, y, v_1) \wedge L(k, v_1, m) > \theta \tag{3.7}$$

and

$$\big(k^{-1} * (x \circ y)\big)(n) \geq T(x, y, v_1) \wedge L(k^{-1}, v_1, n) > \theta. \tag{3.8}$$

In the light of (3.6), (3.7) and (3.8), we find that

$$T(m, n, e) > \theta. \tag{3.9}$$

Consider $t_1, t_2, h_1, h_2, l_1, l_2, r, p \in X$ such that

$$L(k, x, t_1) > \theta, \quad L(k, y, t_2) > \theta, \quad L(k^{-1}, x, h_1) > \theta, \quad L(k^{-1}, y, h_2) > \theta,$$
$$L(k, x^{-1}, l_1) > \theta, \quad L(k, y^{-1}, l_2) > \theta, \quad T(t_1, t_2, r) > \theta, \quad T(h_1, h_2, p) > \theta.$$

It follows that,

$$\big((k * x) \circ (k * y)\big)(r) \geq L(k, x, t_1) \wedge L(k, y, t_2) \wedge T(t_1, t_2, r) > \theta,$$

and

$$\big((k^{-1} * x) \circ (k^{-1} * y)\big)(p) \geq L(k^{-1}, x, h_1) \wedge L(k^{-1}, y, h_2) \wedge T(h_1, h_2, p) > \theta.$$

Combining (3.7) and (3.8) with the two last relations and using Definition 2.1 (iii), we arrive at $r = m$ and $p = n$, which show that $T(t_1, t_2, m) > \theta$ and $T(h_1, h_2, n) > \theta$.
On the other hand, by Lemma 2.2, we have $\big((k * x) \circ (k^{-1} * x)\big)(e) > \theta$ and $\big((k * x) \circ (k * x^{-1})\big)(e) > \theta$, which prove that $T(t_1, h_1, e) > \theta$ and $T(t_1, l_1, e) > \theta$; and therefore $l_1 = h_1$ by Lemma 2.1 (1-).
Similarly, we have $\big((k*y)\circ(k^{-1}*y)\big)(e) > \theta$ and $\big((k*y)\circ(k*y^{-1})\big)(e) > \theta$, which imply $T(t_2, h_2, e) > \theta$ and $T(t_2, l_2, e) > \theta$. Once again, in virtue of Lemma 2.1 (1-), we obtain $l_2 = h_2$ and therefore $L(k, x^{-1}, h_1) > \theta$ and $L(k, y^{-1}, h_2) > \theta$. So that,

$$\big((k * x^{-1}) \circ (k * y^{-1})\big)(n) \geq L(k, x^{-1}, h_1) \wedge L(k, y^{-1}, h_2) \wedge T(h_1, h_2, n) > \theta.$$

Next, choosing $w \in X$ for which $L(k, v_2, w) > \theta$, then

$$\big(k * (x^{-1} \circ y^{-1})\big)(w) \geq T(x^{-1}, y^{-1}, v_2) \wedge L(k, v_2, w) > \theta.$$

The Definition 2.1 (iii) shows that $w = n$, then

$$L(k, v_2, n) > \theta. \tag{3.10}$$

In view of $L(k, v_1, m) > \theta$ and from (3.9) and (3.10), we get

$$\big((k * v_1) \circ (k * v_2)\big)(e) \geq L(k, v_1, m) \wedge L(k, v_2, n) \wedge T(m, n, e) > \theta.$$

Taking $v_3 \in X$ such $L(k, h, v_3) > \theta$, then

$$\big(k * (v_1 \circ v_2)\big)(v_3) > \theta.$$

Once again by Definition 2.1 (iii), we can see that $v_3 = e$, then $L(k, h, e) > \theta$. Consequently, for all $z \in X$, we have $(z * (k * h))(e) \geq L(k, h, e) \wedge L(z, e, e) > \theta$. In virtue of the fuzzy primeness of (X, T, L), we get $h = e$ and thus $\big((x \circ y) \circ (x^{-1} \circ y^{-1})\big)(e) > \theta$. □

Theorem 3.3 *Let (X,T,L) be a prime fuzzy near-ring and I be a nontrivial fuzzy semigroup ideal of X such that $(\star)$ $T(x,y,z)>\theta \implies z\in I,\ \forall x,y\in I, \forall z\in X$. If I admits a fuzzy involution σ, then (X,T,L) is a fuzzy ring.*

Proof: • Case 1, suppose that (X,T,L) is a left fuzzy near-ring and I is a nontrivial fuzzy semigroup ideal of X. Our goal is to show that (X,T,L) is a right fuzzy near-ring, i.e, obtain the condition (iv) of Definition 2.1.

(i) By hypotheses given, and taking into account the first step of the proof of the previous theorem, we arrive at $\forall x,y,z,c_1,c_2\in I$, if $\big((x\circ y)*z\big)(c_1)>\theta$ and $\big((x*z)\circ(y*z)\big)(c_2)>\theta$, then $c_1=c_2$.

(ii) Now, showing that $\forall m,n,t_1,t_2\in X, \forall z\in I$, if $\big((m\circ n)*z\big)(t_1)>\theta$ and $\big((m*z)\circ(n*z)\big)(t_2)>\theta$, then $t_1=t_2$.
Let $m,n\in X$ and $z\in I$ and taking $t_1,t_2\in X$ such that $\big((m\circ n)*z\big)(t_1)>\theta$ and $\big((m*z)\circ(n*z)\big)(t_2)>\theta$.
Let $x\in I$ and choosing $t,h\in X$ such that $(x*m)(t)=L(x,m,t)>\theta$ and $(x*n)(h)=L(x,n,h)>\theta$.
Since I is a fuzzy semigroup ideal of X, then t and h must be two elements of I.
Taking $l,v\in X$ satisfy $T(t,h,l)>\theta$ and $L(l,z,v)>\theta$, then

$$\big((t\circ h)*z\big)(v)\geq T(t,h,l)\wedge L(l,z,v)>\theta. \tag{3.11}$$

In view of $L(l,z,v)>\theta$ and $z\in I$, the fact that I is a fuzzy semigroup guarantees $v\in I$. On the other hand, we have

$$((x*m)\circ(x*n))(l)\geq L(x,m,t)\wedge L(x,n,h)\wedge T(t,h,l)>\theta. \tag{3.12}$$

Taking $k_1,k\in X$ such that $T(m,n,k)>\theta$ and $L(x,k,k_1)>\theta$, then

$$(x*(m\circ n))(k_1)\geq T(m,n,k)\wedge L(x,k,k_1)>\theta. \tag{3.13}$$

In view of Definition 2.1 (iii) together (3.12) and (3.13), we get $k_1=l$ which means that $L(x,k,l)>\theta$. So that,

$$((x*k)*z)(v)\geq L(x,k,l)\wedge L(l,z,v)>\theta. \tag{3.14}$$

Since $T(m,n,k)>\theta$, the fact that $\big((m\circ n)*z\big)(t_1)>\theta$ implies that $L(k,z,t_1)>\theta$.
Letting $k_2\in X$ satisfies $L(x,t_1,k_2)>\theta$, then

$$(x*(k*z))(k_2)\geq L(k,z,t_1)\wedge L(x,t_1,k_2)>\theta. \tag{3.15}$$

Once again, by applying Definition 2.1 (ii) to relations (3.14) and (3.15), we arrive at $k_2=v$, and therefore

$$L(x,t_1,v)>\theta. \tag{3.16}$$

Similarly, let $s_1,s_2,s\in X$ such that $L(t,z,s_1)>\theta$, $L(h,z,s_2)>\theta$ and $T(s_1,s_2,s)>\theta$. We obtain,

$$\big((t*z)\circ(h*z)\big)(s)\geq L(t,z,s_1)\wedge L(h,z,s_2)\wedge T(s_1,s_2,s)>\theta. \tag{3.17}$$

Also, the fact that I is a fuzzy semigroup ideal and $t,h\in I$ together the assumed hypothesis $(\star)$, the expressions $L(t,z,s_1)>\theta$, $L(h,z,s_2)>\theta$ and $T(s_1,s_2,s)>\theta$ give $s_1,s_2,s\in I$.
Since $h,t,z,s,v\in I$, from the part (i) above, (3.17) and (3.11) give $s=v$ and therefore, $T(s_1,s_2,v)>\theta$.
Thus,
$((x*m)*z)(s_1)\geq L(x,m,t)\wedge L(t,z,s_1)>\theta$ and $((x*n)*z)(s_2)\geq L(x,n,h)\wedge L(h,z,s_2)>\theta$.
Now, choosing $v_1,v_2,v_3,v_4\in X$ such that $L(m,z,v_1)>\theta$, $L(n,z,v_2)>\theta$, $L(x,v_1,v_3)>\theta$ and $L(x,v_2,v_4)>\theta$, we get

$$(x*(m*z))(v_3)\geq L(m,z,v_1)\wedge L(x,v_1,v_3)>\theta \text{ and } (x*(n*z))(v_4)\geq L(n,z,v_2)\wedge L(x,v_2,v_4)>\theta.$$

Once again, because of Definition 2.1 (ii), we infer that $v_3=s_1$ and $v_4=s_2$; so that $L(x,v_1,s_1)>\theta$ and $L(x,v_2,s_2)>\theta$ and whence it follows that

$$((x*v_1)\circ(x*v_2)(v)\geq L(x,v_1,s_1)\wedge L(x,v_2,s_2)\wedge T(s_1,s_2,v)>\theta. \tag{3.18}$$

In view of $\big((m*z)\circ(n*z)\big)(t_2)>\theta$ together $L(m,z,v_1)>\theta$ and $L(n,z,v_2)>\theta$, we get $T(v_1,v_2,t_2)>\theta$. Consider $s_3\in X$ such that $L(x,t_2,s_3)>\theta$, which implies that

$$(x*(v_1\circ v_2))(s_3)\geq T(v_1,v_2,t_2)\wedge L(x,t_2,s_3)>\theta.$$

Applying Definition 2.1 (iii) to the last result and to (3.18), we conclude that $s_3=v$, and thus

$$L(x,t_2,v)>\theta.$$

Now, choose $t_3\in X$ such that $L(k,z^{-1},t_3)>\theta$. From Lemma 2.2(1.), we have

$$\big((k*z)\circ(k*z^{-1})\big)(e)>\theta.$$

In view of $L(k,z^{-1},t_3)$ and $L(k,z,t_1)$, the last result assures that $T(t_1,t_3,e)>\theta$.
On the other hand, we have

$$\big(x*(t_1\circ t_3)\big)(e)\geq T(t_1,t_3,e)\wedge L(x,e,e). \tag{3.19}$$

Let $l_1,d\in X$ satisfy $L(x,t_3,d)>\theta$ and $T(v,d,l_1)>\theta$. Since $L(x,t_1,v)>\theta$, then

$$\big((x*t_1)\circ(x*t_3)\big)(l_1)>\theta. \tag{3.20}$$

Let us apply Definition 2.1 (iii) to (3.19) and (3.20), we get $l_1=e$ and therefore $T(v,d,e)>\theta$. Also, using $L(x,t_2,v)>\theta$, we obtain

$$\big((x*t_2)\circ(x*t_3)\big)(e)\geq L(x,t_2,v)\wedge L(x,t_3,d)\wedge T(v,d,e)>\theta. \tag{3.21}$$

Consider $w_1,w\in X$ such that $T(t_2,t_3,w)>\theta$ and $L(x,w,w_1)>\theta$, then

$$\big(x*(t_2\circ t_3)\big)(w_1)>\theta. \tag{3.22}$$

From (3.21) and (3.22), we conclude that $w_1=e$; so that $L(x,w,e)>\theta$. Now, letting $z_1\in X$, then there exits $z_2\in X$ satisfies $(x*z_1)(z_2)=L(x,z_1,z_2)>\theta$. Since I is a fuzzy semigroup ideal of X, it follows that $z_2\in I$. In particular, putting z_2 instead of x, we get

$$L(z_2,w,e)=(z_2*w)(e)>\theta.$$

Consequently,

$$\big((x*z_1)*w\big)(e)\geq L(x,z_1,z_2)\wedge L(z_2,w,e)>\theta.$$

In the light of the fuzzy primeness of (X,T,L), the last result implies that either $x=e$ or $w=e$. Taking into account that I is nontrivial, we can consider $x\neq e$ and therefore $w=e$. Consequently, $T(t_2,t_3,e)>\theta$.
By applying Lemma 2.1 to $T(t_2,t_3,e)>\theta$ and $T(t_1,t_3,e)>\theta$, we conclude that $t_2=t_1$.

(iii) As the final step in Case 1, we verify that for all $\forall x,y,t,t_1,t_2\in X$, if $\big((x\circ y)*t\big)(t_1)>\theta$ and $\big((x*t)\circ(y*t)\big)(t_2)>\theta$, then $t_1=t_2$.
For this objective, let $x,y,t,t_1,t_2\in X$ such that $\big((x\circ y)*t\big)(t_1)>\theta$ and $\big((x*t)\circ(y*t)\big)(t_2)>\theta$.
Let $z\in I$ and choosing $s\in X$ which satisfies $(t*z)(s)>\theta$, then s must be an element of I by defining I.
Now, let $h_1,h_2,h,a,b\in X$ such that $T(x,y,a)>\theta$, $L(x,s,h_1)>\theta$, $L(y,s,h_2)>\theta$, $L(a,s,h)>\theta$ and $T(h_1,h_2,b)>\theta$. Then,

$$\big((x\circ y)*s\big)(h)\geq T(x,y,a)\wedge L(a,s,h)>\theta,$$

and

$$\big((x*s)\circ(y*s)\big)(b)\geq L(x,s,h_1)\wedge L(y,s,h_2)\wedge T(h_1,h_2,b)>\theta.$$

In virtue of $x,y,h,b\in X$ and $s\in I$, the step (ii) above assures that $h=b$; so that $T(h_1,h_2,h)>\theta$.
Also, we have

$$(a*(t*z))(h)\geq L(t,z,s)\wedge L(a,s,h)>\theta, \tag{3.23}$$

and from $T(x,y,a) > \theta$ together $\big((x \circ y) * t\big)(t_1) > \theta$, we get $L(a,t,t_1) > \theta$.
Now, taking $k \in X$ such that $L(t_1,z,k) > \theta$. It follows that,

$$\big((a * t) * z\big)(k) \geq L(a,t,t_1) \wedge L(t_1,z,k) > \theta. \tag{3.24}$$

By application of Definition 2.1 (ii) to relations (3.23) and (3.24), we deduce that $k = h$ and therefore,

$$L(t_1,z,h) > \theta. \tag{3.25}$$

On the other hand, we have

$$\big(x * (t * z)\big)(h_1) \geq L(t,z,s) \wedge L(x,s,h_1) > \theta \tag{3.26}$$

and

$$\big(y * (t * z)\big)(h_2) \geq L(t,z,s) \wedge L(y,s,h_2) > \theta. \tag{3.27}$$

Consider $m_1, m_2, m, n \in X$ which satisfy $L(x,t,m_1) > \theta$, $L(y,t,m_2) > \theta$, $L(m_1,z,m) > \theta$ and $L(m_2,z,n) > \theta$. It follows that

$$\big((x * t) * z\big)(m) > \theta \quad \text{and} \quad \big((y * t) * z\big)(n) > \theta.$$

Combining the last two results with (3.26) and (3.27) and applying Definition 2.1 (ii), we conclude $m = h_1$ and $n = h_2$ which give $L(m_1,z,h_1) > \theta$ and $L(m_2,z,h_2) > \theta$. Accordingly,

$$\big((m_1 * z) \circ (m_2 * z)\big)(h) \geq L(m_1,z,h_1) \wedge L(m_2,z,h_2) \wedge T(h_1,h_2,h) > \theta. \tag{3.28}$$

As well, from $L(x,t,m_1) > \theta$, $L(y,t,m_2) > \theta$, the relation $\big((x * t) \circ (y * t)\big)(t_2) > \theta$ implies that $T(m_1,m_2,t_2) > \theta$. Let $h^{'} \in X$ which satisfies $L(t_2,z,h^{'}) > \theta$, then

$$\big((m_1 \circ m_2) * z)\big)(h^{'}) \geq L(t_2,z,h^{'}) \wedge T(m_1,m_2,t_2) > \theta. \tag{3.29}$$

Since $z \in I$ and in view of the second step (ii), (3.28) and (3.29) assure that $h = h^{'}$ which implies that

$$L(t_2,z,h) > \theta. \tag{3.30}$$

Let $c, v \in X$ such that $L(e,z,c) > \theta$ and $T(c,c,v) > \theta$, we have

$$\big((e \circ e) * z\big)(c) \geq T(e,e,e) \wedge L(e,z,c) > \theta,$$

and

$$\big((e * z) \circ (e * z)\big)(v) \geq L(e,z,y) \wedge L(e,z,c) \wedge T(c,c,v) > \theta.$$

Using the conclusion of the preceding part, we get $v = c$ which implies that $T(c,c,c) > \theta$ and hence $c = e$ by Lemma 2.1 (4). Thus, $L(e,z,e) > \theta$.
Now, let $t_3 \in X$ such that $L(a,t^{-1},t_3) > \theta$ which implies that $\big((x \circ y) * t^{-1}\big)(t_3) > \theta$. From Lemma 2.2 (i), we have $\big((a * t) \circ \big(a * t^{-1}\big)\big)(e) > \theta$. Since $L(a,t,t_1) > \theta$ and $L(a,t^{-1},t_3) > \theta$, we find that

$$(t_1 \circ t_3)(e) = T(t_1,t_3,e) > \theta, \tag{3.31}$$

that is,

$$\big((t_1 \circ t_3) * z\big)(e) \geq T(t_1,t_3,e) \wedge L(e,z,e) > \theta.$$

Let $s_1, p \in X$ such that $L(t_3,z,s_1) > \theta$ and $T(h,s_1,p) > \theta$ and invoking (3.25), we obtain

$$\big((t_1 * z) \circ (t_3 * z)\big)(p) \geq L(t_1,z,h) \wedge L(t_3,z,s_1) \wedge T(h,s_1,p) > \theta.$$

In view of the preceding step, we conclude that $p = e$, and therefore $T(h,s_1,e) > \theta$.
By using (3.30), we get

$$\big((t_2 * z) \circ (t_3 * z)\big(e) \geq L(t_2,z,h) \wedge L(t_3,z,s_1) \wedge T(h,s_1,e) > \theta,$$

Now, choosing $v_1, v_2 \in X$ such that $T(t_2, t_3, v_1) > \theta$ and $L(v_1, z, v_2) > \theta$, then

$$\big((t_2 \circ t_3) * z\big)(v_2) > \theta.$$

Once again from the previous step, we get $v_2 = e$, so that $(v_1 * z)(e) > \theta$. Now, taking $z_1 \in X$, then there exits $z_2 \in X$ satisfying $(z_1 * z)(z_2) = L(z_1, z, z_2) > \theta$. By defining I, we obtain $z_2 \in I$. Now, put z_2 instead of z, we get

$$L(v_1, z_2, e) > \theta.$$

Consequently,

$$\big(v_1 * \big(z_1 * z\big)\big)(e) \geq L(z_1, z, z_2) \wedge L(v_1, z_2, e) > \theta.$$

In view of the fuzzy primeness of (X, T, L), the last result implies that either $z = e$ or $v_1 = e$. Taking into account that I is nontrivial, considering $z \neq e$, hence $v_1 = e$. And therefore,

$$(t_2 \circ t_3)(e) = T(t_2, t_3, e) > \theta. \tag{3.32}$$

Applying Lemma 2.1 (2) to (3.31) and (3.32), we obtain $t_1 = t_2$.

• Case 2, if (X, T, L) is a right fuzzy near-ring, then we can use the same arguments as used in the first case.

• To prove that (X, T) is commutative, it is enough to argue as in the proof of the previous theorem, precisely the second part, we obtain the desired result. This completes the proof of our Theorem. □

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