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Modules with 2-Pure Intersection Property

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ABSTRACT: In this work, new concept is introduced and studied, which 2-pure submodule. Many results concerns with these concept are given.

Key Words: pure submodules, 2-pure submodules.

Contents

1. Introduction

First note that in this paper, R denoted a commutative ring with unity, W is a left R-module. Moreover, all modules are unitary left R-modules. In what follow $H \leq W(H < W)$ refer to H is a submodule of W (H is a proper submodule of W)Cohen [8] defined a submodule H of W a pure submodule if the sequence $0 \to H \bigotimes D \to W \bigotimes D$ is exact for every R-module D. Anderson and Fuller [3] called a submodule H pure in W if $IH = IW \cap H$ for every $I \leq R$. The first statement implies the second, see , [[20],P.158], but the second statement does not implies the first one,[[20],P.158]. However, they are equivalent if W if flat. In this paper a pure a pure submodule means a pure submodule of Anderson and Fuller. A module W is regular if every submodule is pure. These consepts considered and studied by many authors, see [1,2,3,5,13]
Many generalizations of pure submodule presented see [19,17,10,9,7,18,21,22]. Ghaleb [15] introduce and studied 2-pure submodule, where a submodule H of W is 2-pure if $I^2W \cap H = I^2H, \forall I \leq R$. Farshadifar [7] named a submodule H is 2-pure if $IJW \cap IH \cap JH = IJH$ for every $I \leq R, J \leq R$. Also, she generalized this concept to n-pure submodule, $n \in Z_+$.
Note that a pure submodule implies these concept, but they are not equivalent, see [15] and [7]. Moreover, we see that 2-pure submodule(in sense [7]) does not implies 2-pure submodule(in sense of [15]) since $H =<\overline{4}>\leq Z_8$ as Z-module is 2-pure (in sense of [7](see Remarks and Examples 3.2 [7], but it is not 2-pure(in sense of [15] because $(2Z)^2Z_8 \cap <\overline{4}>=<\overline{4}>\neq (2Z)<\overline{4}>=<\overline{0}>$.
Bahar in [1] studied pure intersection property PIP and Galib in [15] studied and 2-pure(in sense of [15] intersection property.

In this work our definition of 2-pure submodule will be that of Farshadifar [7]. Also, we will denote it by 2-P.S.Also, we add some new result, see **Remarks and Examples**2.1 and **Propsition**2.2,2.3,2.4. In S.3, we consider and study modules which satisfy that the intersection of any two 2-pure submodules is 2-pure submodule (denoted by modules with 2-PIP). Many properties of these type of modules are presented, and some of them are analogous to that of modules with PIP.

2. 2-pure submodules

Definition 2.1 *[7] Let $H \leq W$,H is called 2-pure submodule(2-p.s) if $IJW \cap JH \cap JH = IJH\ \forall\ I, J \leq R$*
An ideal K of R is called 2-pure ideal (2-p.i), $IJ \cap IK \cap JK = IJK, \forall\ I, J \leq R$
The following proposition presents some results about 2-p.s, see [7].

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Proposition 2.1 *Let $H \leq W$.*

1. *If I and J are ideals of R with $IH \cap JH = IJH$, then H is a 2-p.s.*
2. *If for each $I \leq R$, IH is a pure in W, Then H is a 2-p.s. in W*
3. *If $H \leq K \leq W$ and H is a 2-p.s. in K and K a 2-p.s. in W, then H is a 2-p.s. in W.*
4. *If $H \leq K \leq W$ and H is a 2-p.s. in W, then K/H is a 2-p.s. in W/H*
5. *If K/H is a 2-p.s. in W/H and H is a 2-p.s. in W, then K is a 2-p.s. in W*

We added the following

Remarks and Examples 2.1 .

1. *If $H \leq K \leq W$. If H is a 2-p.s. in W, then H is a 2-p.s. in K.*
2. *The converse of **Proposition2.1**(1) may be not true, for example:*
   Consider $Z_{12} as Z_{12}$-module,$H =< \overline{3} >$, H is a 2-p.s., but for $I = J =< \overline{2} >$. $IH \cap JH =< \overline{6} > \neq IJH =< \overline{0} >$
3. *Let R be a P.I.R, $H \leq W$ where W is an R-modile H is 2-p.s. if and only if $abW \cap aH \cap bH = abH$.*
4. *Every submodule is a 2-p.s. and so every direct summand is a 2-P.S.*
5. *Let R be a P.I.R, $H \leq W$ and H is a faithful R-module.*
   If H is a weakly strongly 2-absorbing submodule(i.e whenever $a, b \in R, K \leq W$ and $abW \nsubseteq K$, then $aH \subseteq K$ or $bH \subseteq K$ or $ab \in annH$) [14], then H is a 2-P.S.

   Proof: *Let $a, b \in R$. Assume $ab \neq 0$, so $ab \notin annH = (0)$. Hence by [[7], **Proposition2.3**(c), $abH = abW \cap aH \cap bH$. Also, if $ab = 0$, then it is clear that $abH = abW \cap aH \cap bH$. Thus H is a 2-p.s.* □

6. *Consider the Z-module Z_{p^∞}. Any submodule H of Z_{p^∞} has the form $< \frac{1}{p^n} + Z >$. If $n = 2m, m \in Z$, then $P^2 Z_{p^\infty} \cap PH \cap PH = Z_{p^\infty} \cap < \frac{1}{p^{2m-1}} + Z >=< \frac{1}{p^{2m-1}} + Z >, P^2 H =< \frac{1}{p^{2m-2}} >$. Thus H is not a 2-p.s.*
   If $n = 2m + 1$, then $P^2 Z_{p^\infty} \cap PH \cap PH = PH < \frac{1}{p^{2m}} + Z >$ and $P^2 H =< \frac{1}{p^{2m-1}} + Z >$, so the H is a not 2-p.s. If $H =< \frac{1}{p} + Z >$, let $I = nZ, J = mZ$, when n, m are a relatively prime with P. We get, $nmZ_{p^\infty} \cap nH \cap mH = nH \cap mH = n < \frac{1}{p} + Z > \cap m < \frac{1}{p} + Z >=< \frac{1}{p} + Z >$. Also, nm is a relatively prime with P, so $nm < \frac{1}{p} + Z >=< \frac{1}{p} + Z >$. If $n = cp$ and m is a relatively prime with P, so $nmZ_{p^\infty} \cap n < \frac{1}{p} + Z > \cap m < \frac{1}{p} + Z >=< 0 >$ and $nm < \frac{1}{p} + Z >=< 0 >$ ...
   If n, m are multiple of P then $nmZ_{p^\infty} \cap nH \cap mH = nmH =< 0 >$. Thus $H =< \frac{1}{p} + Z >=< 0 >$.
   If n, m are multiple of P, Then $nmZ_{p^\infty} \cap nH \cap mH = nmH = 0$. Thus $H =< \frac{1}{p} + Z >$ is a 2-P.s.
   Note that H is not pure
7. *In the Z-module Q, every cyclic submodule $H \leq Q$ is not a 2-p.s. let $H =< \frac{m}{n} >= \frac{m}{n} Z$, so that $n.nQ \cap n(\frac{m}{n})Z \cap n(\frac{m}{n})Z = mZ$, but $n.n(\frac{m}{n})Z = nmZ$.*
8. *Let $H \leq W$. If $IJH = IH \cap JH$. Then H is a 2-p.s.*

   Proof: *$IJW \cap IH \cap JH = IJW \cap IJH = IJH$. Thus H is a 2-p.s.* □

Proposition 2.2 *If $W = \bigoplus_{i\in I} W_i$ be an R-module $W_i \leq W$, $\forall i \in I$and let $H_i \leq W_i$, $\forall i \in I$. Then $\bigoplus_{i\in I} H_i$ is a 2-p.s. in W if and only if H_i is a 2-p.s. in W_i $\forall i \in I$*

Proof: *$\Rightarrow$ Let $j \in I, H_j$ is a direct summand of $\bigoplus_{i\in I} H_i$, hence H_j is a 2-p.s. in $\bigoplus_{i\in I} H_i$. But $\bigoplus_{i\in I} H_i$ is a 2-p.s. in W, hence H_j is a 2-p.s. in W by* ***Proposition****2.1(3). Since $H_j \leq W_j$, so H_j is a 2-p.s. in W_j.*
$\Leftarrow$ Let I, J be ideals of R. Then, $IJ(\bigoplus_{i\in I} W_i) \cap I(\bigoplus_{i\in I} W_i) \cap J(\bigoplus_{i\in I} W_i) = \bigoplus_{i\in I}(IJW_i) \cap \bigoplus_{i\in I}(IH_i) \cap \bigoplus_{i\in I}(JH_i) = \bigoplus_{i\in I}(IJW_i) \cap IH_i \cap JH_i) = \bigoplus_{i\in I} IJH_i$(since H_i is a 2-p.s.)$= IJ(\bigoplus_{i\in I} H_i)$.Thus $\bigoplus_{i\in I} H_i$is a 2-p.s. in $\bigoplus_{i\in I} W_i$
Recall that an R-module W is named multiplication module if each $H \leq W, H = IW$ for some $I \leq R$. Equivalently, $\forall H \leq W, H = (H :_R W)W$ where $(H :_R W) = \{r \in R : rW \leq H\}$ [6]. ☐

Proposition 2.3 *Let W be a faithful multiplication R-module , $K \leq W$. If K is a 2-pure ideal in R, then KW is a 2-p.s. in W.*

Proof: *Let I and J be any two ideals of R. Since K is a 2-pure in R, $IJ \cap IK \cap JK = IJK$. Hence, $(IJ \cap IK \cap JK)W = (IJK)W$. Then by [[6],Theorem 2.1], $IJW \cap IKW \cap JKW = IJKW$ that is KW is a 2-p.s. in W.* ☐

Proposition 2.4 *Let W be a faithful finitely generated multiplication R-module. If $K \leq R$ with KW is a 2-p.s. in W, then K is a 2-pure ideal in R.*

Proof: *Let I and J be ideals of R. As KW is a 2-p.s. in W, then $IJW \cap IKW \cap JKW = IJKW$. Since W is a faithful multiplication, $(IJ \cap IK \cap JK)W = (IJK)W$ by [[6], Theorem 2.1]. Thus $IJ \cap IK \cap JK = IJK$ by [[6], Theorem 3.1] and so K is a 2-pure ideal in R* ☐

Corollary 2.1 *Let W be a faithful finitely generated multiplication R-module and $H \leq W$. Then the following statements are self same:*

1. *H is a 2-p.s. in W.*
2. *$H = JW$ for some 2-pure ideal in R.*
3. *$(H :_R W)$ is 2-pure ideal in R.*

Proof: *(1)$\Leftrightarrow$(2) It is a consequence of* ***Proposition****2.3,2.4*
(2) $\Leftrightarrow$ (3) It deduces directly , since $H = (H :_R W)W$. ☐

Proposition 2.5 *Suppose W be a distributive R-module, let $H_1$and H_2 be pure submodules in W. Then $H_1 + H_2$ is a 2-p.s. in W*

Proof: *Assume I and J be ideals in R. $IJW \cap I(H_1 + H_2) \cap J(H_1 + H_2) = [IJW \cap (IH_1 + IH_2)] \cap [IJW \cap (JH_1 + JH_2)] = [(IJW \cap IH_1) + (IJW \cap IH_2)] \cap [(IJW \cap JH_1) + (IJW \cap JH_2)]$(since W is distributive)*
$\subseteq [(IJW \cap H_1) + (IJW \cap H_2)] \cap [(IJW \cap H_1) + (IJW \cap H_2)] = (IJH_1 + IJH_2) \cap (IJH_1 + IJH_2)$, (Since H_1 and H_2 are pure in W)$IJ(H_1 + H_2) \cap IJ(H_1 + H_2) = IJ(H_1 + H_2)$. Thus $IJW \cap I(H_1 + H_2) \cap J(H_1 + H_2) \subseteq IJ(H_1 + H_2)$. The reverse inclusion is clear. Thus $IJW \cap I(H_1 + H_2) \cap J(H_1 + H_2) = IJ(H_1 + H_2)$. ☐

3. Modules with 2-PIP

Definition 3.1 *An R-module W satisfy 2-pure intersection property (shortly 2-PIP) if the intersection of 2-p.s. is a 2-p.s..*

Remarks and Examples 3.1 .

1. *Every fully 2-pure module W(every submodule of W is a 2-pure, see [7] satisfies 2-PIP)*

2. *Let W be an R-module, if $IJH = IH \cap JH, \forall\ I, J \leq R, H \leq W$, then W is a fully 2-pure by **Remarks and Examples**2.1(8). Hence by part (1) W satisfies 2-PIP. As example, Z_{12} as Z_{12}-module is fully 2-pure module.*

3. *Every semi-simple module has 2-PIP. Also, every regular module (every submodule is pure) has 2-PIP .*

4. *$Z_8$and Z_{16} as Z-modules has 2-PIP , since the 2-p.s. of Z_8 are Z_8, $<\overline{4}>, <\overline{0}>$ and the 2-p.s. of Z_{16},$<\overline{8}>, <\overline{0}>$*

5. *Every module over semi-simple ring(or regular ring)has 2-PIP*

6. *For a module hasn't 2-PIP, see (1) and (2)*

Definition 3.2 *An Rmodule W is termed 2-pure simple (shortly 2-p.simple)if (0), W are the only 2-p.s. in W*

Remark 3.1 *.*

1. *It is clear that every 2-p.simple has 2-PIP.*

2. *Every 2-p.simple module is pure simple but the contrary may be not valid, for example Z_8 as Z-module is pure simple but it is not 2-p.simple.*

3. *Z as Z-module is a 2-p.simple.*

Proposition 3.1 *Let W be an R-module such that $I(H \cap E) = IH \cap IE, \forall I \leq R, \forall H, E \leq W$. Then W has 2-PIP*

Proof: *Let $H, E \leq W, I, J \leq R$.*
$IJW \cap I(H \cap E) \cap J(H \cap E) = IJW \cap IH \cap IE \cap JE = \left[IJW \cap IH \cap JH\right] \cap \left[IJW \cap IE \cap JE\right] = IJH \cap IJE = IJ(H \cap E)$ □

An R-module W is called prime if $annW = annH\ \forall 0 \neq H \leq W$
Equivalently W is a prime module if whenever $a \in R, m \in W, am = 0$, then $m = 0$ or $a \in annW = (0 :_R W)$ [4]

Proposition 3.2 *Let W be a prime module over a P.I.R. Then W has 2-PIP.*

Proof: *Let $A, B \leq W$ such that each of them is 2-p.s. in W. LEt $I = (a)$ be any ideal of R. Claim that $I(A \cap B) = IA \cap IB$. Clearly $I(A \cap B) \leq IA \cap IB$. Let $x \in IA \cap IB = (a)A \cap (a)B$. Hence $x = ay = aw$ for some $y \in A, w \in B$ and $a(y-w) = 0$ if $y-w = 0$, then $y = w$ and so $x = ay \in I(A \cap B)$ If $y-w \neq 0$, then $a \in ann(y-w) = anny$; that is $x = ay = 0$ which is a contradiction. Thus, $x = ay \in a(A \cap B) = I(A \cap B)$. Then $I(A \cap B) = IA \cap IB$ and so by 3.1, W has been 2-PIP* □

Proposition 3.3 *Let W be a faithful cyclic R-module. Then W has 2-PIP.*

Proof: *Let $A, B \leq W$ such that each of them is a 2-p.s. and $W = Rx$ for some $x \in W$. Let I be an ideal of $R.I(A \cap B) \subseteq IA \cap IB$. Since $W = Rx$, W is a multiplication R-module hence $A = Kx$, $B = Jx$for some ideals K, J of R. Let $y \in IA \cap IB = IKx \cap IJx$ and hence $0 \neq y = rx = dx$ for some $r \in IK, d \in IJ$.*
*Then $rx - dx = 0$, i.e$(r-d)x = 0$ and so $r - d \in ann(x) = 0$. It follows that $r = d \in IK \cap IJ$, which implies that $y = rx \in IKx \cap IJx = IA \cap IB$. Then $I(A \cap B) = IA \cap IB$. so that by **Proposition**3.1, W has 2-PIP.* □

Proposition 3.4 *If W is an R-module such that W has 2-PIP and A is a 2-p.s. in W, then A has 2-PIP.*

Proof: *Let $B, C \leq A$ such that each of them is a 2-p.s. in A. Since A is a 2-p.s. in W, hence by **Proposition**2.1(3), B and C are 2-pure submodule in W. So that $B \cap C$ is a 2-p.s. in W. But $B \cap C \subseteq A$ implies $B \cap C$ is a 2-p.s. in A by 2.1(8)* □

Corollary 3.1 *If W has a 2-PIP and A is a direct summand of W, then A has a 2-PIP*

Proposition 3.5 *If W has a 2-PIP and H is a 2-p.s. in W, then W/H has a 2-PIP*

Proof: *Assume each of A/H and B/H are 2-p.s. in W/H. Since H is a 2-p.s. in W, then each of A, B is a 2-p.s. in W by **Proposition**2.1(5). As W has PIP, $A \cap B$ is a 2-p.s. in W. Accordingly $A \cap B/H$ is a 2-p.s. in W/H by 2.1(4). Thus $A/H \cap B/H$ is a 2-p.s. in W/H.* □

Theorem 3.1 *If $W = A \bigoplus B$ is an R-module satisfying 2-PIP, let $f : A \to B$ be an R-homomorphism such that each of A and $A + imf$ is a 2-p.s. in W.*
Then $Kerf$ is a 2-p.s. in W.

Proof: Since $f : A \to B$, then $A \cap Imf \subseteq A \cap B = (0)$. Put $E = \{x + f(x) : x \in A\}$, hence $E \subseteq A+Imf$. Claim E is a 2-p.s. in W. Assume $y \in IJW \cap IE \cap JE$ where $I, J \leq R$. As $E \subseteq A+Imf$, so $y \in IJW \cap I(A + Imf) \cap J(A + Imf)$. But $A + Imf$ is a 2-p.s. in W, so $y \in IJ(A + Imf)$. Hence $y = \sum_{i=1}^{n} r_i(a_i + y_i)$ for some $n \in Z_+, a_i \in A, y_i \in Imf \ \forall \ i = 1, ...n$.
On other hand $y \in E$, implies $y = x + f(x)$ for some $x \in A$. Thus $x + f(x) = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} r_i y_i$. It follows that $x - \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} r_i y_i - f(x) \in A \cap Imf = (0)$; that is $x = \sum_{i=1}^{n} r_i a_i \in IJA$, Which implies that $y = x + f(x) = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} r_i f(a_i) = \sum_{i=1}^{n} r_i(a_i + f(a_i)) \in IJE$. Thus $IJE \cap IE \cap JE \subseteq IJE$ and since the reverse inclusion is valid always, then $IJE \cap IE \cap JE = IJE$, i.e. E is a 2-p.s. in W. Claim that $Kerf = A \cap E$, let $X \in A \cap E$ and $x \in A, x \in E$ and $x = a + f(a)$ for some $a \in A$. Ist follows that $x = a = f(a) \in A \cap Imf = 0$; that $x - a = 0 = f(a)$, thus $x = a$ and $f(a) = 0$, then $f(x) = 0$ and $x \in Kerf$. Thus $A \cap E \subseteq Kerf$. Now let $x \in Kerf \subseteq A$, so $f(x) = 0$ which implies $x = x + f(x) \in E$, Thus $Kerf \subseteq A \cap E$. Therefor $Kerf = A \cap E$. As W has a 2-PIP, $A \cap E$ is a 2-p.s. , that is $Kerf$ is a 2-p.s. □

Remark 3.2 *By applying 3.1, the next examples of modules don't have 2-PIP, also they explain that the direct sum of modules with 2-PIP need not satisfy 2-PIP.*

Example 1 *Let $W = Z_{p^\infty} \bigoplus Z_{p^\infty}$ as Z-module. As Z_{p^∞} has only $< 0 >, < \frac{1}{p} + Z >, Z_{p^\infty}$ are 2-p. submodule in W, hence Z_{p^∞} has 2-PIP.*
Define $f : Z_{p^\infty} \to Z_{p^\infty}$ defined by $f\left(\frac{n}{p^m} + Z\right) = P^2\left(\frac{n}{p^m} + Z\right) = \left(\frac{n}{p^{m-2}} + Z\right)$.
$Kerf = \left(\frac{1}{p^2} + Z\right)$ which doesn't satisfy 2-PIP. Hence by 3.1, W has not 2-PIP. But, Z_{p^∞} as Z-module has 2-PIP.

Example 2 *Consider the Z-module Z_8. This module has 2-PIP. But $W = Z_8 \bigoplus Z_8$ as Z-module since $\exists f : Z_8 \to Z_8$ defined by $f(\overline{x}) = 4\overline{x} \ \forall \ \overline{x} \in Z_8$*

$Kerf =< \overline{2} >$ which is not a 2-p.s. in Z_8 since if $I = 2Z = J$, then $IJZ_8 \cap I < \overline{2} > \cap J < \overline{2} > = < \overline{4} >$ but $IJ < \overline{2} > = < \overline{0} >$. Hence $W = Z_8 \bigoplus Z_8$ hasn't 2-PIP.

Proposition 3.6 *Let each of the R-module W_1 and W_2 have 2-PIP and $annW_1 + annW_2 = R$. Then $W = W_1 \bigoplus W_2$ has 2-PIP.*

Proof: *Let each of C, D be a 2-P.s. in W. Since $annW_1 + annW_2 = R$, then $C = C_1 + C_2$, $D = D_1 + D_2$ for some $C_1, D_1 \leq W_1$, $C_2, D_2 \leq W_2$. Since C, D are 2-pure in W, then each C_1 and D_1 is a 2-P.s. in W_1 and each C_2 and $D_2$is a 2-P.s. in W_2 by 2.2. Now $C \cap D = (C_1 \cap D_1) \bigoplus (C_2 \cap D_2)$.As W_1 and W_2 have 2-PIP, $C_1 \cap D_1$ is a 2-p.s. in W_1,$C_2 \cap D_2$ is a 2-p.s. in W_2 and hence, $(C_1 \cap D_1) \bigoplus (C_2 \cap D_2)$ is a 2-p.s. in $W = W_1 \bigoplus W_2$.Thus $C \cap D$ is a 2-p.s.* □

Recall that submodule A of an R-module W is called fully invariant if $\forall f \in End(W), f(A) \subseteq A$, [16,12]

Theorem 3.2 *Let $W = \bigoplus_{i\in I} W_i, W_i \leq W, \forall i \in I$. If W has a 2-PIP,then W_i has 2-PIP, $\forall i \in I$. The converse hold if each 2-p.s. in W is a fully invariant.*

Proof: $\Rightarrow$ *suppose W has 2-PIP. Since W_i is a direct summand of $W, \forall i \in I$, then W_i has 2-PIP by 2.1*
$\Leftarrow$ *Let $A, B \leq W$ such that each of them is a 2-p.s. in W. Since A, B are fully invariant submodules of W, $A = \bigoplus_{i\in I}(A \cap W_i)$, $B = \bigoplus_{i\in I}(B \cap W_i)$ [16].*

But $A \cap W_i \leq W_i$ and $B \cap W_i \leq W_i$, $\forall i \in I$. Hence $A \cap W_i, B \cap W_i$ are 2-pure submodules in W_i, $\forall i \in I$, so that $(A\cap W_i)\cap(B\cap W_i)$ is a 2-p.s. in $W_i, \forall i \in I$. But $A\cap B = \bigoplus_{i\in I}(A\cap W_i)\cap\bigoplus_{i\in I}(B\cap W_i) = \bigoplus_{i\in I}[(A \cap W_i) \cap (B \cap W_i)]$. Then by 2.2, $A \cap B$ is a 2-p.s. in $W = \bigoplus_{i\in I} W_i$. Thus W has 2-PIP. □

Recall that an R-module W is named quasi-Dedekind if $Hom\left(\frac{W}{N}, W\right) = 0, \forall N \leq W, N \neq 0$ Equivalently, $\forall f \in End(W), f \neq 0$ implies $Kerf = 0$ [11]

Theorem 3.3 *Let W_1 and W_2 be R-modules, W_1 is a 2-pure simple and $W_1 \bigoplus W_2$ has 2-PIP. Then either $Hom(W_1, W_2) = 0$ or $W_1 \cong$a submodule of W_2 and every nonzero homeomorphism from W_1 to W_2 is monomorphism and W_1 is quasi-Dedekind.*

Proof: *Assume $f : W_1 \to W_2$ be an R-homomorohism. Since$W_1 \bigoplus W_2$ has a 2-PIP, then by Theorem 3.9, $Kerf$ is a 2-p.s. in W_1. But W_1 is a 2-pure simple, so that $Kerf = 0$. Thus f is a monomorphism and hence $W_1 \cong$ a submodule of W_2 To prove W_1 is a q-Dedekind, let $g : W_1 \to W_1, g \neq 0$. Then $f \circ g : W_1 \to W_2$. It follows that $Ker(f \circ g) = Kerg$. As $W_1 \bigoplus W_2$ has 2-PIP , so $Kerg$ a 2-p.s. in W_1 and so $Kerg = 0$, since W_1 is a 2-pure simple. Thus W_1 is a q-Dedekind.* □

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