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Intutionistic Fuzzy $\mathfrak{FG}$-Metric Space

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ABSTRACT: In order to extend the concept of an Intutionistic Fuzzy metric space ($\widehat{IFMS}$), the prospect of an Intutionistic fuzzy $\mathfrak{FG}$- metric space is presented in this article. Some new concepts in an Intutionistic fuzzy $\mathfrak{FG}$- metric space are discussed along with some characteristics and various features. A few related results are proved in the proposed framework.

Key Words: t-norm, fuzzy metric, intutionistic fuzzy $\mathfrak{FG}$-metric space.

Contents

1. Introduction and Elementary Interpretations

After Zadeh [19] presented the idea of fuzzy sets, numerous authors explored fuzzy set notions in various fields, including Kramosil and Michalek's [13] fuzzy metric space.

Definition 1.1 *[19] Let $\bigotimes \neq \emptyset$. A function $\varphi : \bigotimes \times \bigotimes \to \mathbb{R}$ is said to be metric if it satisfies the following axioms $\forall\ \rho, \varrho, \zeta \in \bigotimes$:*

1. ***Non-negativity:*** $\varphi(\rho, \varrho) \geq 0$;
2. ***Symmetry:*** $\varphi(\rho, \varrho) = \varphi(\varrho, \rho)$;
3. ***Triangle Inequality:*** $\varphi(\rho, \varrho) \leq \varphi(\rho, \zeta) + \varphi(\zeta, \varrho)$;
4. ***Identity of Indiscernibles:*** $\varphi(\rho, \varrho) = 0$ *if and only if* $\rho = \varrho$.

Definition 1.2 *[16] A binary operation $\circledast : [0,1] \times [0,1] \to [0,1]$ is a continuous t- norms $(\widehat{CTN})$ if it meets the requirements listed below:*

1. $\circledast$ *is associative and commutative;*
2. $\circledast$ *is continuous;*
3. $\rho \circledast 1 = \rho\ \forall\ \rho \in [0,1]$;
4. $\rho \circledast \varrho \leq \zeta \circledast \ddot{\mathrm{d}}$ *whenever* $\rho \leq \zeta$ *and* $\varrho \leq \ddot{\mathrm{d}}\ \forall\ \rho, \varrho, \zeta, \ddot{\mathrm{d}} \in [0,1]$.

The concept of Fuzzy Metric Space $(\widehat{FMS})$ was examined by Kramosil and Michalek [13] in 1975 and was defined as follows:

Definition 1.3 *[13] Let $\bigotimes \neq \emptyset$. A triplet $(\bigotimes, \xi, \circledast)$ is a $\widehat{FMS}$ if $\circledast$ is a $\widehat{CTN}$ and ξ is a fuzzy set $(\widehat{FS})$ on $\bigotimes \times \bigotimes \times [0, \infty) \to [0,1]$ that satisfying the following conditions:*

1. $\xi(\rho, \varrho, \check{t}) = 0$, $\forall\ \rho, \varrho \in \bigotimes$ *and* $\check{t} > 0$;
2. $\xi(\rho, \varrho, \check{t}) = 1\ \forall\ \check{t} > 0$ *iff* $\rho = \varrho$;

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3. $\xi(\rho,\varrho,\check{t}) = \xi(\varrho,\rho,\check{t})\ \forall\ \rho,\varrho \in \bigotimes$ *and* $\check{t} > 0$;

4. $\xi(\rho,\zeta,\check{t}+\check{s}) \geq \xi(\rho,\varrho,\check{t}) \circledast \xi(\varrho,\zeta,\check{s})\ \forall\ \rho,\varrho,\zeta \in \bigotimes\ \&\ \check{t},\check{s} > 0$;

5. $\xi(\rho,\varrho,\cdot) : (0,\infty) \to [0,1]$ *is left-continuous* $\forall\ \rho,\varrho \in \bigotimes$.

The $\widehat{FMS}$ concept of Kramosil and Michalek [13] was altered by George and Veeramani [7] in 1994. They acquired a Hausdorff and first countable topology on the modified $\widehat{FMS}$.

Definition 1.4 *[7] A* $\widehat{FMS}$ *is an ordered triplet* $(\bigotimes,\xi,\circledast)$ *such that:* $\bigotimes \neq \emptyset$, $\circledast$ *is a* $\widehat{CTN}$, *and* ξ *is a* $\widehat{FS}$ *on* $\bigotimes \times \bigotimes \times (0,\infty) \to [0,1]$ *satisfying the following conditions:*

1. $\xi(\rho,\varrho,\check{t}) > 0$;

2. $\xi(\rho,\varrho,\check{t}) = 1$ *iff* $\rho = \varrho$;

3. $\xi(\rho,\varrho,\check{t}) = \xi(\varrho,\rho,t)$;

4. $\xi(\rho,\zeta,\check{t}+\check{s}) \geq \xi(\rho,\varrho,\check{t}) \circledast \xi(\varrho,\zeta,\check{s})$;

5. $\xi(\rho,\varrho,\cdot) : (0,\infty) \to [0,1]$ *is continuous for all* $\rho,\varrho,\zeta \in \bigotimes$ *and* $\check{s},\check{t} > 0$.

Then ξ *is called a fuzzy metric* $(\widehat{FM})$ *on* $\bigotimes$.

Remark 1.1 *[7] In* $\widehat{FMS}$ $(\bigotimes,\xi,\circledast)$, *George and Veeramani considered the following:*

1. *If* $\xi(\rho,\varrho,\check{t}) > 1-\bar{q}\ \forall\ \rho,\varrho \in \bigotimes$, $\check{t} > 0$, $0 < \bar{q} < 1$, *We can locate a* $\check{t}_0$, $0 < \check{t}_0 < \check{t}$, *such that* $\xi(\rho,\varrho,\check{t}_0) > 1-\bar{q}$.

2. *For any* $\bar{q}_1 > \bar{q}_2$ *in* $(0,1)$, *we can locate* $\bar{q}_3 \in (0,1)$, *such that* $\bar{q}_1 \circledast \bar{q}_3 \geq \bar{q}_2$, *and for any* $\bar{q}_4 \in (0,1)$, *we can locate* $\bar{q}_5 \in (0,1)$, *such that* $\bar{q}_5 \circledast \bar{q}_5 \geq \bar{q}_4$.

The outcomes in this field have been examined by numerous researchers ([8], [9], [18]). Using this notion, Atanassov [1] recommended the notion of an intuitionistic fuzzy sets $(\widehat{IFS})$ in 2004 as follows:

Definition 1.5 *[1] Let* $\acute{\mho}$ *be a universe. An* $\widehat{IFS}$ $\acute{\wp}$ *on* $\acute{\mho}$ *can be defined as follows:*

$$\acute{\wp} = \{\langle \mu_{\acute{\wp}}(\rho), \gamma_{\acute{\wp}}(\rho)\rangle : \rho \in \acute{\mho}\}$$

where

$$\mu_{\acute{\wp}} : \acute{\mho} \to [0,1] \quad \text{and} \quad \gamma_{\acute{\wp}} : \acute{\mho} \to [0,1]$$

such that

$$0 \leq \mu_{\acute{\wp}}(\rho) + \gamma_{\acute{\wp}}(\rho) \leq 1 \quad \text{for any } \rho \in \acute{\mho}.$$

Here, $\mu_{\acute{\wp}}(\rho)$ *and* $\gamma_{\acute{\wp}}(\rho)$ *represent the degree of membership and degree of non-membership of the element* ρ, *respectively.*

Next, Park [14] expanded the notion of $\widehat{FMS}$ to $\widehat{IFMS}$. Park used $\widehat{CTN}$ [16] and continuous t-conorms $(\widehat{CTCN})$ [16] to describe this idea.

Definition 1.6 *[16] A binary operation* $\blacktriangle : [0,1] \times [0,1] \to [0,1]$ *is called a* $\widehat{CTCN}$ *if* $\blacktriangle$ *meets the requirements listed below:*

1. $\rho \blacktriangle 0 = \rho,\ \forall\ \rho \in [0,1]$;

2. $\rho \blacktriangle \varrho = \varrho \blacktriangle \rho$ *and* $\rho \blacktriangle (\varrho \blacktriangle \zeta) = (\rho \blacktriangle \varrho) \blacktriangle \zeta$, $\forall\ \rho, \varrho, \zeta \in [0,1]$;

3. *If* $\rho \leq \zeta$ *and* $\varrho \leq \ddot{\mathrm{d}}$, *then* $\rho \blacktriangle \varrho \leq \zeta \blacktriangle \ddot{\mathrm{d}}$, $\forall\ \rho, \varrho, \zeta, \ddot{\mathrm{d}} \in [0,1]$;

4. $\blacktriangle$ *is continuous.*

Remark 1.2 *[14]*

1. *For any* $\bar{q}_1,\ \bar{q}_2 \in (0,1)$ *with* $\bar{q}_1 > \bar{q}_2$, $\exists\ \bar{q}_3,\ \bar{q}_4 \in (0,1)$ *such that* $\bar{q}_1 \circledast \bar{q}_3 \geq \bar{q}_2$ & $\bar{q}_1 \geq \bar{q}_4 \blacktriangle \bar{q}_2$.

2. *For any* $\bar{q}_5 \in (0,1)$, $\exists\ \bar{q}_6,\ \bar{q}_7 \in (0,1)$ *such that* $\bar{q}_6 \circledast \bar{q}_6 \geq \bar{q}_5$ & $\bar{q}_7 \blacktriangle \bar{q}_7 \leq \bar{q}_5$.

Definition 1.7 *[14] Let* ξ *and* N *be fuzzy sets on* $\bigotimes^2 \times (0,\infty)$, *and let* $\circledast$ *be a* $\widehat{CTN}$ *and* $\blacktriangle$ *be a* $\widehat{CTCN}$. *If* ξ *and* N *satisfy the following conditions for all* $\rho, \varrho, \zeta \in \bigotimes$, $\check{s}, \check{t} > 0$, *we say that* (ξ, N) *is an intuitionistic fuzzy metric* ($\widehat{IFM}$) *on* $\bigotimes$:
(IFM1) $\xi(\rho,\varrho,\check{t}) + N(\rho,\varrho,\check{t}) \leq 1$;
(IFM2) $\xi(\rho,\varrho,\check{t}) > 0$;
(IFM3) $\xi(\rho,\varrho,\check{t}) = 1$ *if and only if* $\rho = \varrho$;
(IFM4) $\xi(\rho,\varrho,\check{t}) = \xi(\varrho,\rho,\check{t})$;
(IFM5) $\xi(\rho,\varrho,\check{t}) \circledast \xi(\varrho,\zeta,\check{s}) \leq \xi(\rho,\zeta,\check{t}+\check{s})$;
(IFM6) $\xi(\rho,\varrho,\cdot) : (0,\infty) \to (0,1]$ *is continuous;*
(IFM7) $N(\rho,\varrho,\check{t}) < 1$;
(IFM8) $N(\rho,\varrho,\check{t}) = 0$ *if and only if* $\rho = \varrho$;
(IFM9) $N(\rho,\varrho,\check{t}) = N(\varrho,\rho,\check{t})$;
(IFM10) $N(\rho,\varrho,\check{t}) \blacktriangle N(\varrho,\zeta,\check{s}) \geq N(\rho,\zeta,\check{t}+\check{s})$;
(IFM11) $N(\rho,\varrho,\cdot) : (0,\infty) \to (0,1]$ *is continuous.*

The five- tuple $(\bigotimes, \xi, N, \circledast, \blacktriangle)$ *is called* $\widehat{IFMS}$. *The functions* $\xi(\rho,\varrho,\check{t})$ *and* $N(\rho,\varrho,\check{t})$ *express the degree of nearness and the degree of nonnearness between* ρ *and* ϱ *with respect to* $\check{t}$, *respectively.*

Over the past few decades, numerous intriguing extensions of metric spaces including fuzzy metric spaces have been created and researched.(*e.g.* b-metric by Czerwik [2], S-metric by Sedghi *et al.* [17], $\mathfrak{F}$-metric by Jleli and Samet [11], Modified intuitionistic fuzzy metric spaces by Saadati *et al.* [15], Neutrosophic metric spaces by M.Kirišci and N. Šimšek [12]). Many other authors have investigated numerous results in various extended fuzzy metric space (eg. [4], [6], [5], [10]). The authors employed a family $\mathfrak{F}$ of real-valued functions in $\mathfrak{F}$-metric space (Jleli and Samet [11]) that had the following properties:

Given $g : (0,\infty) \to \mathbb{R}$ that satisfies the following conditions:

($\mathfrak{F}1$) g is non-decreasing on $(0,\infty)$;
($\mathfrak{F}2$) for every sequence$\{\rho_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \rho_n = 0 \iff \lim_{n\to\infty} g(\rho_n) = -\infty$.

Using the above family, Das *et al.* introduced fuzzy $\mathfrak{F}$-metric space [3], as a generalization of the George and Veeramani-type fuzzy metric space [7] by involving some special kind of functions.

Let $\mathfrak{F}$ denote the set of all functions $f : [0,1] \to [0,1]$ that satisfy the following conditions:

($\mathfrak{F}1$) f is strictly increasing on $[0,1)$;
($\mathfrak{F}2$) for every sequence $\{\mathtt{t}_n\} \subset [0,1]$, $\lim_{n\to\infty} \mathtt{t}_n = 1 \iff \lim_{n\to\infty} f(\mathtt{t}_n) = 1$.

In this work, a new generalization known as Intutionistic fuzzy $\mathfrak{FG}$- metric space is introduced.

2. Results and Discussion

Definition 2.1 *Let* $\mathfrak{G}$ *denote the set of all functions* $g : [0,1] \to [0,1]$ *that satisfy the following conditions:*

($\mathfrak{G}1$) *g is strictly decreasing on* $(0,1]$;
($\mathfrak{G}2$) *For every sequence* $\{\mathtt{t}_n\}$ *in* $[0,1]$, $\lim_{n\to\infty} \mathtt{t}_n = 1 \implies \lim_{n\to\infty} g(\mathtt{t}_n) = 0$.

Example 2.1 *Some examples of $\mathfrak{G}$ are:*
(i) $g(\rho) = \rho^{-n}$, $\rho \in (0,1]$, $n \in \mathbb{N}$
(ii) $g(\rho) = 1 - \sqrt{\rho}$, $\rho \in [0,1]$.

Now, using the functions $f \in \mathfrak{F}$ and $g \in \mathfrak{G}$, we define Intutionistic fuzzy $\mathfrak{FG}$- metric space by relaxing the axiom (IFM5) and (IFM10) of $\widehat{IFMS}$.

Definition 2.2 *Let* $\bigotimes \neq \emptyset$, $\xi : \bigotimes \times \bigotimes \times (0,\infty) \to [0,1]$ *and* $N : \bigotimes \times \bigotimes \times (0,\infty) \to [0,1]$ *be two mappings,* $\circledast$ *and* $\blacktriangle$ *be* $\widehat{CTN}$ *and* $\widehat{CTCN}$ *respectively. If there exist* $(f,\alpha) \in \mathfrak{F}\times(0,1]$ *and* $(g,\beta) \in \mathfrak{G}\times(0,1]$ *such that* ξ *and* N *satisfying the following conditions:*

- ($\mathfrak{FG}_{\xi N}1$) $\xi(\rho,\varrho,\check{t}) + N(\rho,\varrho,\check{t}) \leq 1$, *for all* $\rho,\varrho \in \bigotimes$ *and* $\check{t} > 0$;
- ($\mathfrak{FG}_{\xi N}2$) $\xi(\rho,\varrho,\check{t}) > 0$, *for all* $\rho,\varrho \in \bigotimes$ *and* $\check{t} > 0$;
- ($\mathfrak{FG}_{\xi N}3$) $\xi(\rho,\varrho,\check{t}) = 1$, *for all* $\check{t} > 0$ *if and only if* $\rho = \varrho$;
- ($\mathfrak{FG}_{\xi N}4$) $\xi(\rho,\varrho,\check{t}) = \xi(\varrho,\rho,\check{t})$, *for all* $\rho,\varrho \in \bigotimes$ *and* $\check{t} > 0$;
- ($\mathfrak{FG}_{\xi N}5$) *For every* $(\rho,\varrho) \in \bigotimes \times \bigotimes$, *for every* $N \in \mathbb{N}$, $N \geq 2$, *and for every* $u_i \in \bigotimes$ *with* $u_1 = \rho$ & $u_N = \varrho$, *we have*

  $$\xi(\rho,\varrho,\check{t}) < 1 \implies (f(\xi(\rho,\varrho,\check{t}))^{\alpha} \geq f(\xi(u_1,u_2,\check{t}_1) \circledast \xi(u_2,u_3,\check{t}_2) \circledast \cdots \circledast \xi(u_{N-1},u_N,\check{t}_{N-1}));$$

- ($\mathfrak{FG}_{\xi N}6$) $N(\rho,\varrho,\check{t}) < 1$, $\forall\ \rho,\varrho \in \bigotimes$ & $\check{t} > 0$;
- ($\mathfrak{FG}_{\xi N}7$) $N(\rho,\varrho,\check{t}) = 0$, $\forall\ \check{t} > 0$ *iff* $\rho = \varrho$;
- ($\mathfrak{FG}_{\xi N}8$) $N(\rho,\varrho,\check{t}) = N(\varrho,\rho,\check{t})$, $\forall\ \rho,\varrho \in \bigotimes$ *and* $\check{t} > 0$;
- ($\mathfrak{FG}_{\xi N}9$) *For every* $(\rho,\varrho) \in \bigotimes \times \bigotimes$, *for every* $N \in \mathbb{N}$, $N \geq 2$, & *for every* $u_i \in \bigotimes$ *with* $u_1 = \rho$ & $u_N = \varrho$, *we have*

  $$N(\rho,\varrho,\check{t}) > 0 \implies (g(N(\rho,\varrho,\check{t}))^{\beta} \geq g(N(u_1,u_2,\check{t}_1) \blacktriangle N(u_2,u_3,\check{t}_2) \blacktriangle \cdots \blacktriangle N(u_{N-1},u_N,\check{t}_{N-1})),$$

  where $\check{t} = \check{t}_1 + \check{t}_2 + \cdots + \check{t}_{N-1}$; $\check{t}_i > 0$ *for* $i = 1,2,\ldots,N-1$.

Then, (ξ, N) *is called an Intutionistic fuzzy* $\mathfrak{FG}$*-metric on* $\bigotimes$*, and the 9-tuple* $(\bigotimes,\xi,N,f,g,\alpha,\beta,\circledast,\blacktriangle)$ *is called an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space.*

Remark 2.1 *It has been observed that every* $\widehat{IFMS}$ *is an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space, since if* $(\bigotimes,\xi,N,\circledast,\blacktriangle)$ *is an* $\widehat{IFMS}$*, then clearly* ξ *and* N *satisfy* ($\mathfrak{FG}_{\xi N}1$)-($\mathfrak{FG}_{\xi N}4$) *and* ($\mathfrak{FG}_{\xi N}6$)-($\mathfrak{FG}_{\xi N}8$)*. We need to verify the condition* ($\mathfrak{FG}_{\xi N}5$) *and* ($\mathfrak{FG}_{\xi N}9$)*. For* $\rho,\varrho,\zeta \in \bigotimes$ *with* $\rho \neq \varrho$ *and* $\check{s},\check{t} > 0$*, we have, from (IFM5),*

$$\xi(\rho,\varrho,\check{s}+\check{t}) \geq \xi(\rho,\zeta,\check{s}) \circledast \xi(\zeta,\varrho,\check{t})$$

implies

$$f(\xi(\rho,\varrho,\check{s}+\check{t})) \geq f(\xi(\rho,\zeta,\check{s}) \circledast \xi(\zeta,\varrho,\check{t})) \quad using\,(\mathfrak{F}1)$$

$$\Rightarrow f(\xi(\rho,\varrho,\check{s}+\check{t}))^{\alpha} \geq f(\xi(\rho,\zeta,\check{s}) \circledast \xi(\zeta,\varrho,\check{t})) \quad since\ \alpha \in (0,1].$$

If we write $\check{s}+\check{t} = T$ *and* $f(\check{t}) = \check{t}\ \forall\ \check{t} \in [0,1]$ *and* $\alpha = 1$*, we get*

$$f(\xi(\rho,\varrho,T))^{\alpha} \geq f(\xi(\rho,\zeta,\check{s}) \circledast \xi(\zeta,\varrho,\check{t})).$$

and from (IFM10),

$$N(\rho,\varrho,\check{s}+\check{t}) \leq N(\rho,\zeta,\check{s}) \blacktriangle N(\zeta,\varrho,\check{t})$$

implies

$$g(N(\rho,\varrho,\check{s}+\check{t})) \geq g(N(\rho,\zeta,\check{s}) \blacktriangle N(\zeta,\varrho,\check{t})) \quad using\,(\mathfrak{G}1)$$

$$\Rightarrow g(N(\rho,\varrho,\check{s}+\check{t}))^{\beta} \geq g(N(\rho,\zeta,\check{s}) \blacktriangle N(\zeta,\varrho,\check{t})) \quad since\,\beta \in (0,1].$$

If $\check{s}+\check{t}=T$ & $g(\check{t})=-\check{t}$ *for all* $\check{t}\in[0,1]$ *and* $\beta=1$*, then*

$$g(N(\rho,\varrho,T))^{\beta} \geq g(N(\rho,\zeta,\check{s}) \blacktriangle N(\zeta,\varrho,\check{t})).$$

Thus, (ξ,N) *is an Intutionistic fuzzy* $\mathfrak{FG}$*-metric on* $\bigotimes$ *with* $f(\check{t})=\check{t}$ *and* $g(\check{t})=-\check{t}$*,* $\forall\ \check{t}\in[0,1]$ *and* $\alpha=1$ & $\beta=1$.

The Intutionistic fuzzy $\mathfrak{FG}$-metric is the broader version of the $\widehat{IFM}$. The following claim demonstrates that, in specific circumstances, an Intutionistic fuzzy $\mathfrak{FG}$-metric induces an $\widehat{IFM}$.

Proposition 2.1 *Let* $(\bigotimes,\xi,N,f,g,\alpha,\beta,\circledast,\blacktriangle)$ *be an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space. Define a function* $m:\bigotimes\times\bigotimes\times(0,\infty)\to[0,1]$ *and* $n:\bigotimes\times\bigotimes\times(0,\infty)$
$\to[0,1]$ *by*

$$m(\rho,\varrho,\check{t})=\sup\left\{\xi(u_1,u_2,\check{t}_1)\ \circledast\ \dots\ \circledast\ \xi(u_{N-1},u_N,\check{t}_{N-1}):N\in\mathbb{N},N\geq 2\right\} \tag{1}$$

$$n(\rho,\varrho,\check{t})=\inf\left\{N(u_1,u_2,\check{t}_1)\blacktriangle\ \dots\ \blacktriangle N(u_{N-1},u_N,\check{t}_{N-1}):N\in\mathbb{N},N\geq 2\right\} \tag{2}$$

with $(u_1,u_N)=(\rho,\varrho)\ \forall\ \rho,\varrho\in\bigotimes$ *and* $\check{t}>0$*, where* $\check{t}=\check{t}_1+\check{t}_2+\cdots+\check{t}_{N-1}$.

If $\xi(\rho,\varrho,\cdot)$ *is a continuous, non-decreasing function of* $\check{t}\ \forall\ \rho,\varrho\in\bigotimes$ & *If* $N(\rho,\varrho,\cdot)$ *is a continuous, non-increasing function of* $\check{t}\ \forall\ \rho,\varrho\in\bigotimes$*, then* $(\bigotimes,m,n,\circledast,\blacktriangle)$ *is an* $\widehat{IFMS}$.

Proof: As $(\bigotimes,\xi,N,f,g,\alpha,\beta,\circledast,\blacktriangle)$ is an Intutionistic fuzzy $\mathfrak{FG}$-metric space, $\exists$ a pair $(f,\alpha)\in\mathfrak{F}\times(0,1]$ and $(g,\beta)\in\mathfrak{G}\times(0,1]$ with respect to which ξ and N satisfying the condition $(\mathfrak{FG}_{\xi N}5)$ and $(\mathfrak{FG}_{\xi N}9)$ respectively.

As m satisfies $(IFM1)$, $(IFM2)$, $(IFM4)$, $(IFM6)$ trivially and n satisfies $(IFM1)$, $(IFM7)$, $(IFM9)$, $(IFM11)$ trivially and we need to verify $(IFM3)$, $(IFM5)$, $(IFM8)$ and $(IFM10)$. Now,

(i) If $\rho=\varrho$, then $\xi(\rho,\varrho,\check{t})=1\ \forall\ \check{t}>0$ and, hence, $m(\rho,\varrho,\check{t})=1\ \forall\ \check{t}>0$.
Conversely, if possible, suppose that $\exists\ \rho,\varrho\ (\rho\neq\varrho)\in\bigotimes$, such that $m(\rho,\varrho,\check{t})=1\ \forall\ \check{t}>0$. Then, $\exists$ a $\check{t}_0>0$ such that

$$\xi(\rho,\varrho,\check{t}_0)<1. \tag{3}$$

Let $0<\epsilon<1$. Then, definition of m implies, $\exists\ N\in\mathbb{N},N\geq 2$ & $\{u_i\}_i^N\subset X$ with $(u_1,u_N)=(\rho,\varrho)$, such that

$$1-\epsilon<\xi(u_1,u_2,\check{t}_1)\ \circledast\ \dots\ \circledast\ \xi(u_{N-1},u_N,\check{t}_{N-1}),$$

where $\check{t}_0=\check{t}_1+\check{t}_2+\cdots+\check{t}_{N-1}$, implying

$$\begin{aligned} f(1-\epsilon) &< f(\xi(u_1,u_2,\check{t}_1)\ \circledast\ \dots\ \circledast\ \xi(u_{N-1},u_N,\check{t}_{N-1})) \\ &\leq (f(\xi(\rho,\varrho,\check{t}_0)))^{\alpha} \quad (by\ \ (\mathfrak{F}1)\ and\ (\mathfrak{FG}_{\xi N}5)). \end{aligned}$$

Since $0<\epsilon<1$ is chosen arbitrarily, we have

$$f(\xi(\rho,\varrho,\check{t}_0))^{\alpha}>f(1)=1.$$

Thus, we get

$$f(\xi(\rho,\varrho,\check{t}_0))=1 \quad \text{implying} \quad \xi(\rho,\varrho,\check{t}_0)=1,$$

which contradicts the relation(3). Hence, $m(\rho,\varrho,\check{t})=1\ \forall\ \check{t}>0$ implies $\rho=\varrho$.

(ii) Let $\rho,\varrho,\zeta\in\bigotimes$ and $0<\epsilon<1$. Then, according to the definition of m, $\exists$ 2 chains of points $\rho=u_1,u_2,\dots,u_n=\varrho$ and $\varrho=u_n,u_{n+1},\dots,u_N=\zeta$, such that

$$m(\rho,\varrho,\check{s})-\epsilon<\xi(u_1,u_2,\check{t}_1)\ \circledast\ \dots\ \circledast\ \xi(u_{n-1},u_n,\check{t}_{n-1})$$

and

$$m(\varrho,\zeta,\check{t})-\epsilon<\xi(u_n,u_{n+1},\check{t}_n)\ \circledast\ \ldots\ \circledast\ \xi(u_{N-1},u_N,\check{t}_{N-1}),$$

where $s=\check{t}_1+\cdots+\check{t}_{n-1}$ and $\check{t}=\check{t}_n+\check{t}_{n+1}+\cdots+\check{t}_{N-1}$, with $t_i>0$, $i=1,2,\ldots,N-1$.

Therefore,

$$\begin{aligned}m(\rho,\zeta,T)&\geq\xi(u_1,u_2,\check{t}_1)\ \circledast\ \cdots\circledast\xi(u_{n-1},u_n,\check{t}_{n-1})\ \circledast\ \xi(u_n,u_{n+1},\check{t}_n)\ \circledast\ \cdots\circledast\xi(u_{N-1},u_N,\check{t}_{N-1})\\&>(m(\rho,\varrho,\check{s})-\epsilon)\ \circledast\ (m(\varrho,\zeta,\check{t})-\epsilon),\end{aligned}$$

where $T=\check{s}+\check{t}$. Since $0<\epsilon<1$ is arbitrary, we obtain as $\epsilon\to 0^+$:

$$m(\rho,\zeta,T)\geq m(\rho,\varrho,\check{s})\ \circledast\ m(\varrho,\zeta,\check{t}).$$

Thus, m meets the inequality $(IFM5)$. Furthermore, because ξ is a non-decreasing function of $\check{t}$, the definition of m implies that $m(\rho,\varrho,\check{t})$ is likewise non-decreasing with respect to $\check{t}$, for any $\rho,\varrho\in\bigotimes$.

(iii) If $\rho=\varrho$, then $N(\rho,\varrho,\check{t})=0$ for all $\check{t}>0$ and, hence, $n(\rho,\varrho,\check{t})=0\ \forall\ \check{t}>0$. Conversely, if possible, suppose that there exist $\rho,\varrho\ (\rho\neq\varrho)\in\bigotimes$, such that $n(\rho,\varrho,\check{t})=0\ \forall\ \check{t}>0$. Then, $\exists$ a $\check{t}_0>0$ such that

$$N(\rho,\varrho,\check{t}_0)>0. \tag{4}$$

Let $0<\epsilon<1$. Then, definition of n implies, $\exists\ N\in\mathbb{N}$, $N\geq 2$ & $\{u_i\}_i^N\in X$ with $(u_1,u_N)=(\rho,\varrho)$, such that

$$0+\epsilon>N(u_1,u_2,\check{t}_1)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{N-1},u_N,\check{t}_{N-1}),$$

where $\check{t}_0=\check{t}_1+\check{t}_2+\cdots+\check{t}_{N-1}$, implying

$$g(\epsilon)<g(N(u_1,u_2,\check{t}_1)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{N-1},u_N,\check{t}_{N-1}))\leq(g(N(\rho,\varrho,\check{t}_0)))^{\beta}\quad(\text{by}\quad(\mathfrak{G}1)\text{ and }(\mathfrak{FG}_{MN}9)).$$

Since $0<\epsilon<1$ is chosen arbitrarily, we have

$$g(N(\rho,\varrho,\check{t}_0))^{\beta}\geq g(0)=1.$$

Thus, we get

$$g(N(\rho,\varrho,\check{t}_0))=1\quad\text{implying}\quad N(\rho,\varrho,\check{t}_0)=0,$$

which contradicts the relation (4). Hence, $n(\rho,\varrho,\check{t})=0$ for all $\check{t}>0$ implies $\rho=\varrho$.

(iv) Let $\rho,\varrho,\zeta\in\bigotimes$ and $0<\epsilon<1$. Then, by the definition of n, there exist two chains of points $\rho=u_1,u_2,\ldots,u_n=\varrho$ and $\varrho=u_n,u_{n+1},\ldots,u_N=\zeta$, such that

$$n(\rho,\varrho,\check{s})+\epsilon>N(u_1,u_2,\check{t}_1)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{n-1},u_n,\check{t}_{n-1})$$

and

$$n(\varrho,\zeta,\check{t})+\epsilon>N(u_n,u_{n+1},\check{t}_n)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{N-1},u_N,\check{t}_{N-1}),$$

where $\check{s}=\check{t}_1+\cdots+\check{t}_{n-1}$ and $\check{t}=\check{t}_n+\check{t}_{n+1}+\cdots+\check{t}_{N-1}$, with $\check{t}_i>0$, $i=1,2,\ldots,N-1$. Therefore,

$$\begin{aligned}n(\rho,\zeta,T)&\leq N(u_1,u_2,t_1)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{n-1},u_n,\check{t}_{n-1})\ \blacktriangle\ N(u_n,u_{n+1},\check{t}_n)\ \blacktriangle\ \ldots\ \blacktriangle\ N(u_{N-1},u_N,\check{t}_{N-1})\\&<(n(\rho,\varrho,\check{s})+\epsilon)\ \blacktriangle\ (n(\varrho,\zeta,\check{t})+\epsilon),\end{aligned}$$

where $T=\check{s}+\check{t}$. Since $0<\epsilon<1$ is arbitrary, we obtain as $\epsilon\to 0^+$:

$$n(\rho,\zeta,T)\leq n(\rho,\varrho,\check{s})\ \blacktriangle\ n(\varrho,\zeta,\check{t}).$$

Thus, n meets the inequality $(IFM10)$. Furthermore, because N is a non-increasing function of $\check{t}$, the definition of n implies that $n(\rho,\varrho,\check{t})$ is similarly non-increasing with respect to $\check{t}$, for any $\rho,\varrho\in\bigotimes$. Thus, $(\bigotimes,m,n,\circledast,\blacktriangle)$ is a $\widehat{IFMS}$. □

Theorem 2.1 *Let $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ be an Intutionistic fuzzy $\mathfrak{F}\mathfrak{G}$-metric space, such that f & g are continuous from the left and right respectively, & suppose that $\xi(\rho, \varrho, \check{t})$ is a continuous and non-decreasing function of t, $N(\rho, \varrho, \check{t})$ is a continuous and non-increasing function of $\check{t}$ $\forall$ $\rho, \varrho \in \bigotimes$. If $(\bigotimes, m, n, \circledast, \blacktriangle)$ represents the induced $\widehat{IFMS}$ on $\bigotimes$, then the following holds:*

$$\rho, \varrho \in \bigotimes \ \textit{with}\ \xi(\rho, \varrho, \check{t}) < 1 \,\forall\ \check{t} > 0, \ \Rightarrow \ f(\xi(\rho, \varrho, \check{t})) \leq f(m(\rho, \varrho, \check{t})) \leq f(\xi(\rho, \varrho, \check{t}))^{\alpha}, \tag{5}$$

and

$$\rho, \varrho \in \bigotimes \ \textit{with}\ N(\rho, \varrho, \check{t}) > 0 \,\forall\ \check{t} > 0, \ \Rightarrow \ g(N(\rho, \varrho, \check{t})) \leq g(n(\rho, \varrho, \check{t})) \leq g(N(\rho, \varrho, \check{t}))^{\beta}. \tag{6}$$

Proof: (i) Let $(\rho, \varrho) \in \bigotimes \times \bigotimes$ be such that $\xi(\rho, \varrho, \check{t}) < 1$ $\forall$ $\check{t} > 0$. Then definition of m states that $m(\rho, \varrho, \check{t}) \geq \xi(\rho, \varrho, \check{t})$ which implies

$$f(m(\rho, \varrho, \check{t})) \geq f(\xi(\rho, \varrho, \check{t})) \quad (\text{by } (\mathfrak{F}1)). \tag{7}$$

Let $0 < \epsilon < 1$ be arbitrary. Definition of m implies that $\exists$ $N \in \mathbb{N}$, $N \geq 2$, and a sequence of points $u_1, u_2, \ldots, u_N \in \bigotimes$ with $u_1 = \rho$, $u_N = \varrho$, such that for $\check{t} = \check{t}_1 + \check{t}_2 + \cdots + \check{t}_{N-1}$, we have

$$\begin{aligned} m(\rho, \varrho, \check{t}) - \epsilon &< \xi(u_1, u_2, \check{t}_1) \ \circledast \ \cdots \ \circledast \ \xi(u_{N-1}, u_N, \check{t}_{N-1}), \\ \Rightarrow f(m(\rho, \varrho, \check{t}) - \epsilon) &< f(\xi(u_1, u_2, \check{t}_1) \ \circledast \ \cdots \ \circledast \ \xi(u_{N-1}, u_N, \check{t}_{N-1})), \\ \Rightarrow (f(\xi(\rho, \varrho, \check{t})))^{\alpha} &> f(m(\rho, \varrho, \check{t}) - \epsilon) \qquad (\textit{by } (\mathfrak{F}\mathfrak{G}_{\xi N}5)). \end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrary, Allowing $\epsilon \to 0^{+}$ yields

$$f(\xi(\rho, \varrho, \check{t}))^{\alpha} \geq f(m(\rho, \varrho, \check{t})). \tag{8}$$

The relations (7) and (8) together give

$$f(\xi(\rho, \varrho, \check{t})) \leq f(m(\rho, \varrho, \check{t})) \leq f(\xi(\rho, \varrho, \check{t}))^{\alpha}.$$

(ii) Let$(\rho, \varrho) \in \bigotimes \times \bigotimes$ be such that $N(\rho, \varrho, \check{t}) > 0$ for every $\check{t} > 0$. The definition of n indicates that $n(\rho, \varrho, \check{t}) \leq N(\rho, \varrho, \check{t})$, implying

$$g(n(\rho, \varrho, \check{t})) \geq g(N(\rho, \varrho, \check{t})) \quad (\text{by } \ (\mathfrak{G}1)). \tag{9}$$

Let $0 < \epsilon < 1$ be arbitrary. Then, the definition of n implies, $\exists$ $N \in \mathbb{N}$, $N \geq 2$, a sequence of points $u_1, u_2, \ldots, u_N \in \bigotimes$ with $u_1 = \rho$, $u_N = \varrho$, such that for $\check{t} = \check{t}_1 + \check{t}_2 + \cdots + \check{t}_{N-1}$, we have

$$\begin{aligned} n(\rho, \varrho, \check{t}) + \epsilon &> N(u_1, u_2, \check{t}_1) \ \blacktriangle \ \cdots \ \blacktriangle \ N(u_{N-1}, u_N, \check{t}_{N-1}), \\ \Rightarrow g(n(\rho, \varrho, \check{t}) + \epsilon) &< g(N(u_1, u_2, \check{t}_1) \ \blacktriangle \ \cdots \ \blacktriangle \ N(u_{N-1}, u_N, \check{t}_{N-1}) \\ &\leq g(N(u_1, u_N, \check{t}_1 + \cdot + \check{t}_{N-1}))^{\beta} \qquad (\textit{by } (\mathfrak{F}\mathfrak{G}_{MN}9)), \\ \Rightarrow \ g(n(\rho, \varrho, \check{t}) + \epsilon) &< g(N(\rho, \varrho, \check{t}))^{\beta}. \end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrary, by letting $\epsilon \to 0^{+}$, we acquire

$$(g(n(\rho, \varrho, \check{t}))) \leq g(N(\rho, \varrho, \check{t}))^{\beta}. \tag{10}$$

The relations (9)and (10) together give

$$g(N(\rho, \varrho, \check{t})) \leq g(n(\rho, \varrho, \check{t})) \leq g(N(\rho, \varrho, \check{t}))^{\beta}.$$

□

Now we define characteristics of an Intutionistic fuzzy $\mathfrak{F}\mathfrak{G}$-metric space (eg. topology, convergence, cauchyness, completeness etc.)

Definition 2.3 *Assume* $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ *is an intutionistic fuzzy* $\mathfrak{FG}$*-metric space. Define an open ball as follows:* $\rho \in \bigotimes$ *and* $\bar{q} > 0$, $\check{t} > 0$,

$$B(\rho, \bar{q}, \check{t}) = \left\{ \varrho \in \bigotimes : \xi(\rho, \varrho, \check{t}) > 1 - \bar{q},\ N(\rho, \varrho, \check{t}) < \bar{q} \right\}.$$

Proposition 2.2 *Let* $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ *be an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space. Then*

$$\tau(\xi, N) = \left\{ \wp \subseteq \bigotimes : \text{for each } \rho \in \wp,\ \exists\ \bar{q} \in (0,1),\ \check{t} > 0 \text{ such that } B(\rho, \bar{q}, \check{t}) \subseteq \wp \right\}$$

is called topology on $\bigotimes$.

Definition 2.4 *Assume* $\{\rho_n\}$ *is a sequence in an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space* $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$. *Then*

(i) $\{\rho_n\}$ *is said to be* ***convergent*** *if* $\exists\ \rho \in \bigotimes$, *such that for any* $0 < \bar{q} < 1$, $\exists$ *a natural number* $N \in \mathbb{N}$ *such that* $\forall\ \check{t} > 0$,

$$\xi(\rho_n, \rho, \check{t}) > 1 - \bar{q} \quad \text{and} \quad N(\rho_n, \rho, \check{t}) < \bar{q} \quad \forall n \geq N.$$

(ii) $\{\rho_n\}$ *is said to be* ***Cauchy sequence*** *if for each* $\check{t} > 0$ *and* $0 < \bar{q} < 1$, $\exists$ *a natural number* $N \in \mathbb{N}$ *such that*

$$\xi(\rho_n, \rho_m, \check{t}) > 1 - \bar{q} \quad \text{and} \quad N(\rho_n, \rho_m, \check{t}) < \bar{q} \quad \forall m, n \geq N.$$

(iii) $\bigotimes$ *is considered to be* ***complete*** *if every Cauchy sequence in* $\bigotimes$ *converges to some point in set* $\bigotimes$.

The following results are simply proven.

Proposition 2.3 *Consider* $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ *is an Intutionistic fuzzy* $\mathfrak{FG}$*-metric space, let* $\{\rho_n\} \subset \bigotimes$ *is a sequence, &* $\rho \in \bigotimes$. *Then*

(i) $\{\rho_n\}$ *is convergent to* ρ *iff*

$$\lim_{n \to \infty} \xi(\rho_n, \rho, \check{t}) = 1 \quad \& \quad N(\rho_n, \rho, \check{t}) = 0 \quad \forall \quad \check{t} > 0.$$

(ii) $\{\rho_n\}$ *is Cauchy iff*

$$\lim_{m,n \to \infty} \xi(\rho_n, \rho_m, \check{t}) = 1 \quad \& \quad N(\rho_n, \rho_m, \check{t}) = 0 \quad \forall \quad \check{t} > 0.$$

Example 2.2 *Assume* $\bigotimes = \mathbb{R}$ *and define*

$$\xi(\rho, \varrho, \check{t}) = \frac{\check{t}}{\check{t} + |\rho - \varrho|} \quad \& \quad N(\rho, \varrho, \check{t}) = \frac{|\rho - \varrho|}{\check{t} + |\rho - \varrho|} \quad \forall \quad \rho, \varrho \in \mathbb{R},\ \check{t} > 0.$$

Take the sequence $\rho_n = \frac{1}{n}$ *in* $\mathbb{R}$.

(i) For $\xi(\rho_n, \rho_m, \check{t})$, *we have*

$$\xi(\rho_n, \rho_m, \check{t}) = \frac{\check{t}}{\check{t} + \left|\frac{1}{n} - \frac{1}{m}\right|}$$

As $n, m \to \infty$, *we have* $|\rho_n - \rho_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \to 0$, *so*

$$\xi(\rho_n, \rho_m, \check{t}) \to 1.$$

This demonstrates that sequence is Cauchy in terms of Nearness.

Similarly, take $N(\rho_n, \rho_m, \check{t})$*:*

$$N(\rho_n, \rho_m, \check{t}) = \frac{|\rho_n - \rho_m|}{\check{t} + |\rho_n - \rho_m|}.$$

As $n, m \to \infty$*, we again have* $|\rho_n - \rho_m| \to 0$*, which implies*

$$N(\rho_n, \rho_m, \check{t}) \to 0.$$

This demonstrates that the sequence is Cauchy in terms of Non- Nearness.

Proposition 2.4 *Limit of a convergent sequence in* $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ *is unique.*

Proof: Let $\rho, \varrho \in \bigotimes$ be such that a sequence $\{\rho_n\}$ converges to both ρ & ϱ. Then

$$\lim_{n\to\infty} \xi(\rho_n, \rho, \check{t}) = \lim_{n\to\infty} \xi(\rho_n, \varrho, \check{t}) = 1 \quad \forall \quad \check{t} \to 0.$$

$$\lim_{n\to\infty} N(\rho_n, \rho, \check{t}) = \lim_{n\to\infty} N(\rho_n, \varrho, \check{t}) = 0 \quad \forall \quad \check{t} \to 0.$$

Since $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ is an Intutionistic fuzzy $\mathfrak{FG}$-metric space, $\exists$ $(f, \alpha) \in \mathfrak{F} \times (0, 1]$ and $(g, \beta) \in \mathfrak{G} \times (0, 1]$ and satisfying $(\mathfrak{FG}_{\xi N}5)$ and $(\mathfrak{FG}_{\xi N}9)$.

Now suppose $\rho \neq \varrho$. Then $\exists$ $\check{t}_0 > 0$ such that $\xi(\rho, \varrho, \check{t}_0) < 1$ and $N(\rho, \varrho, \check{t}_0) > 0$, and hence, by $(\mathfrak{FG}_{\xi N}5)$ and $(\mathfrak{FG}_{\xi N}9)$:

$$\begin{aligned} &(f(\xi(\rho, \varrho, \check{t}_0)))^\alpha \geq f(\xi(\rho, \rho_n, \check{t}_1) \ \circledast \ \xi(\rho_n, \varrho, \check{t}_2)) \ \forall \ n \in \mathbb{N}, \ \check{t}_0 = \check{t}_1 + \check{t}_2, \\ \Rightarrow &(f(\xi(\rho, \varrho, \check{t}_0)))^\alpha \geq \lim_{n\to\infty} f(\xi(\rho, \rho_n, \check{t}_1) * \xi(\rho_n, \varrho, \check{t}_2)). \end{aligned}$$

On the other hand, by $(\mathfrak{F}2)$, we have

$$\begin{aligned} &\lim_{n\to\infty} f(\xi(\rho, \rho_n, \check{t}_1) \ \circledast \ \xi(\rho_n, \varrho, \check{t}_2)) = 1, \\ &\Rightarrow (f(\xi(\rho, \varrho, \check{t}_0)))^\alpha \geq 1 \quad \text{so} \quad f(\xi(\rho, \varrho, \check{t}_0))^\alpha = 1, \\ &\Rightarrow f(\xi(\rho, \varrho, \check{t}_0)) = 1 \Rightarrow \xi(\rho, \varrho, \check{t}_0) = 1 \ \ (by\ (\mathfrak{F}2)). \end{aligned}$$

and

$$\begin{aligned} &(g(N(\rho, \varrho, \check{t}_0)))^\beta \leq g(N(\rho, \rho_n, \check{t}_1)) \blacktriangle N(\rho_n, \varrho, \check{t}_2)) \ \ \forall \ \ n \in \mathbb{N}, \ \check{t}_0 = \check{t}_1 + \check{t}_2, \\ \Rightarrow &(g(N(\rho, \varrho, \check{t}_0)))^\beta \leq \lim_{n\to\infty} g(N(\rho, \rho_n, \check{t}_1)) \blacktriangle N(\rho_n, \varrho, \check{t}_2)). \end{aligned}$$

Now using $(\mathfrak{G}2)$, we have

$$\begin{aligned} &\lim_{n\to\infty} g(N(\rho, \rho_n, \check{t}_1) \blacktriangle N(\varrho, \rho_n, \check{t}_2)) = 1, \\ &\Rightarrow (g(N(\rho, \varrho, \check{t}_0)))^\beta \leq 1 \quad \text{so} \quad (g(N(\rho, \varrho, \check{t}_0)))^\beta = 1, \\ &\Rightarrow g(N(\rho, \varrho, \check{t}_0)) = 1 \Rightarrow N(\rho, \varrho, \check{t}_0) = 0 \ \ (by\ (\mathfrak{G}2)) \end{aligned}$$

which contradicts our assumption that $\xi(\rho, \varrho, \check{t}_0) < 1$ and $N(\rho, \varrho, \check{t}) > 0$. □

Proposition 2.5 *In* $\widehat{IFMS}$ $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$*, every convergent sequence is a Cauchy sequence.*

Proof: Consider $\{\rho_n\}$ is a convergent sequence in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ converging to $\rho \in \bigotimes$. Then

$$\lim_{n\to\infty} \xi(\rho_n, \rho, \check{t}) = 1 \ \ and \ \ N(\rho_n, \rho, \check{t}) = 0 \ \ \forall \quad \check{t} > 0. \tag{11}$$

Let $(f, \alpha) \in \mathfrak{F} \times (0, 1]$ *and* $(g, \beta) \in \mathfrak{G} \times (0, 1]$ be such that $(\mathfrak{F}\mathfrak{G}_{\xi N}5)$ and $(\mathfrak{F}\mathfrak{G}_{\xi N}9)$ hold respectively. Let $0 < \epsilon < 1$. By $(\mathfrak{F}2)$, $\exists\ 0 < \delta < 1$ such that

$$1 - \delta < \check{t} < 1 \implies 1 - \epsilon < f(\check{t}) < 1. \tag{12}$$

and by $(\mathfrak{G}2)$, $\exists\ 0 < \delta < 1$ such that

$$1 - \delta < \check{t} < 1 \implies g(\epsilon) > g(\check{t}) > 0. \tag{13}$$

By Remark (1.2), for $\delta \in (0, 1)$, we can choose $\beta \in (0, 1)$ such that

$$(1 - \beta) \ \circledast \ (1 - \beta) \geq 1 - \delta. \tag{14}$$

$$\beta \blacktriangle \beta \leq 1 - \delta \tag{15}$$

Again, (11) implies that for $\check{t}_1 > 0$ and $\check{t}_2 > 0$, $\exists\ N_1(\check{t}_1)$ and $N_2(\check{t}_2) \in \mathbb{N}$, such that

$$\xi(\rho_n, \rho, \check{t}_1) > 1 - \beta \quad \text{for all } n \geq N_1(\check{t}_1),$$

$$\xi(\rho_m, \rho, \check{t}_2) > 1 - \beta \quad \text{for all } m \geq N_2(\check{t}_2).$$

and

$$N(\rho_n, \rho, \check{t}_1) < \beta \quad \text{for all } n \geq N_1(\check{t}_1),$$
$$N(\rho_m, \rho, \check{t}_2) < \beta \quad \text{for all } m \geq N_2(\check{t}_2).$$

Let $\check{t} = \check{t}_1 + \check{t}_2$ and $N(\check{t}) = \max\{N_1(\check{t}_1), N_2(\check{t}_2)\}$. Then, using (12), (14) and $(\mathfrak{F}\mathfrak{G}_{\xi N}5)$, we have

$$\begin{aligned}
&\xi(\rho_n, \rho, \check{t}_1) \ \circledast \ \xi(\rho_m, \rho, \check{t}_2) > (1 - \beta) \ \circledast \ (1 - \beta) \geq (1 - \delta) \ \forall\ m, n \geq N(\check{t}),\\
\implies &f(\xi(\rho_n, \rho_m, \check{t}))^{\alpha} > 1 - \epsilon \ \forall\ m, n \geq N(\check{t}), \ \ \forall\ \check{t} > 0,\\
\implies &\lim_{m,n\to\infty} f(\xi(\rho_n, \rho_m, \check{t})) = 1 \ \forall\ \check{t} > 0,\\
\implies &\lim_{m,n\to\infty} \xi(\rho_n, \rho_m, \check{t}) = 1 \ \forall\ \check{t} > 0.
\end{aligned}$$

using (13), (15) and $(\mathfrak{F}\mathfrak{G}_{\xi N}9)$, we have

$$\begin{aligned}
&N(\rho_n, \rho, \check{t}_1) \blacktriangle N(\rho_m, \rho, \check{t}_2) < \beta \blacktriangle \beta \leq (1 - \delta) \ \forall\ m, n \geq N(\check{t}),\\
\implies &g(N(\rho_n, \rho, \check{t}_1) \blacktriangle N(\rho_m, \rho, \check{t}_2)) \geq g(\epsilon)\\
\implies &g(N(\rho_n, \rho_m, \check{t}))^{\beta} \geq g(\epsilon)\\
\implies &g(N(\rho_n, \rho_m, \check{t}) \geq g(\epsilon)\\
\implies &N(\rho_n, \rho_m, \check{t}) \leq \epsilon\\
\implies &\lim_{m,n\to\infty} N(\rho_n, \rho_m, \check{t}) = 0 \ \forall\ \check{t} > 0.
\end{aligned}$$

This demonstrates that in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$, $\{\rho_n\}$ is a Cauchy sequence.

□

The following finding demonstrates that the convergence characteristics of a sequence's cauchyness and convergent point remain invariant in an intuitionistic fuzzy $\mathfrak{F}\mathfrak{G}$-metric space $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ and an Intutionistic fuzzy metric space $(\bigotimes, \xi, N, \circledast, \blacktriangle)$, where ξ and N induces m and n respectively as in relation (1) and (2).

Theorem 2.2 *Let $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ be an Intutionistic fuzzy $\mathfrak{FG}$-metric space, such that $\xi(\rho, \varrho, \cdot)$ is continuous and non-decreasing and $N(\rho, \varrho, \cdot)$ is continuous and non-increasing with respect to $\check{t}$ for all $\rho, \varrho \in \bigotimes$ and $\check{t} > 0$, & m and n be the induced metric in the relation (1) and (2). Let $(f, \alpha) \in \mathfrak{F} \times (0, 1]$ and $(g, \beta) \in \mathfrak{G} \times (0, 1]$ with respect to which $(\mathfrak{FG}_{\xi N}5)$ and $(\mathfrak{FG}_{\xi N}9)$ hold. Then:*

(i) If $\{\rho_n\}$ is convergent to $\rho \in \bigotimes$ in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$, then $\{\rho_n\}$ converges to ρ in $(\bigotimes, m, n, \circledast, \blacktriangle)$.

(ii) If $\{\rho_n\}$ is a Cauchy sequence in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$, then $\{\rho_n\}$ is a Cauchy sequence in $(\bigotimes, m, n, \circledast, \blacktriangle)$.

(iii) $\bigotimes$ is complete $\Leftrightarrow$ $\bigotimes$ is complete with respect to the fuzzy metric m and n.

Proof: *(i)* First, suppose that $\{\rho_n\}$ converges to $\rho \in \bigotimes$ in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$. This means that for every ϵ, $0 < \epsilon < 1$, for each $\check{t} > 0$, there exists an $N(\check{t}) \in \mathbb{N}$ such that

$$\xi(\rho_n, \rho, \check{t}) > 1 - \epsilon \qquad \textit{and} \qquad N(\rho_n, \rho, \check{t}) < \epsilon \quad \forall\, n \geq N(\check{t}).$$

From the definition of m and n, we get

$$m(\rho_n, \rho, \check{t}) \geq \xi(\rho_n, \rho, \check{t}) \quad \forall\, n \quad \text{or} \quad m(\rho_n, \rho, \check{t}) > 1 - \epsilon \quad \forall\, n \geq N(\check{t}).$$

$$n(\rho_n, \rho, \check{t}) \leq N(\rho_n, \rho, \check{t}) \quad \forall\, n \quad \text{or} \quad n(\rho_n, \rho, \check{t}) < \epsilon \quad \forall\, n \geq N(\check{t}).$$

As a result, $\{\rho_n\}$ converges to $\rho \in \bigotimes$ using the fuzzy metric m and n.

Conversely, Assume $\{\rho_n\}$ converges to ρ in the intutionistic fuzzy metric space $(\bigotimes, m, n, \circledast, \blacktriangle)$ with $0 < \epsilon < 1$. Then, by condition $(\mathfrak{F}2)$ and $(\mathfrak{G}2)$, if $0 < f(1-\epsilon) < 1$ and $0 < g(\epsilon) < 1$, then $\exists\ \delta > 0$ such that

$$1 - \frac{\delta}{2} < \check{t} < 1 \quad \Rightarrow \quad f(1-\epsilon) < f(\check{t}) < 1.$$

$$0 < \check{t} < \frac{\delta}{2} \quad \Rightarrow \quad 0 > g(\check{t}) > g(\epsilon).$$

Again, for any $\check{t} > 0$, $\exists\ N(\check{t}) \in \mathbb{N}$ such that

$$m(\rho_n, \rho, \check{t}) > 1 - \frac{\delta}{4} \qquad \textit{and} \qquad n(\rho_n, \rho, \check{t}) < \frac{\delta}{4} \quad \forall\, n \geq N(\check{t}).$$

From m, we have

$$m(\rho_n, \rho, \check{t}) - \frac{\delta}{4} < \xi(\rho_n, \rho, \check{t}) \quad \Rightarrow \quad 1 - \frac{\delta}{2} < \xi(\rho_n, \rho, \check{t}) < 1 \quad \text{for } n \geq N(\check{t}).$$

and

$$n(\rho_n, \rho, \check{t}) + \frac{\delta}{4} > N(\rho_n, \rho, \check{t}) \quad \Rightarrow \quad \frac{\delta}{2} > N(\rho_n, \rho, \check{t}) > 0 \quad \forall\, n \geq N(\check{t}).$$

which implies

$$f(1-\epsilon) < f(\xi(\rho_n, \rho, \check{t})) < 1 \quad \forall\, n \geq N(\check{t}),$$

or equivalently,

$$\xi(\rho_n, \rho, \check{t}) > 1 - \epsilon \quad \forall\, n > N(\check{t}).$$

and

$$g(\epsilon) < g(N(\rho_n, \rho, \check{t})) < 1,$$

or equivalently,

$$N(\rho_n, \rho, \check{t}) < \epsilon.$$

This proves that $\{\rho_n\}$ converges to ρ in $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ as well.

Similar steps can be taken to prove (ii), and (iii) is straightforward. □

Next, boundedness is defined as:

Definition 2.5 *Assume $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ is an intutionistic fuzzy $\mathfrak{FG}$ - metric space. A subset A of $\bigotimes$ is called bounded iff $\exists\ \check{t} > 0\ \&\ 0 < \bar{q} < 1$, such that $\xi(\rho, \varrho, \check{t}) > 1 - \bar{q} \quad \& \ N(\rho, \varrho, \check{t}) < \bar{q}\ \forall\ \rho, \varrho \in A$.*

Theorem 2.3 *In an Intutionistic fuzzy $\mathfrak{FG}$- metric space, every convergent sequence is bounded.*

Proof: Assume $(\bigotimes, \xi, N, f, g, \alpha, \beta, \circledast, \blacktriangle)$ is an Intutionistic fuzzy $\mathfrak{FG}$-metric space and $\{\rho_n\}$ be a sequence in $\bigotimes$ such that $\rho_n \to \rho$ as $n \to \infty$.

Let $0 < \epsilon < 1$. Then, by $(\mathfrak{F}2)$, $\exists\ \delta \in (0, 1)$ implies

$$1 - \delta < \check{t} \leq 1 \quad \Rightarrow \quad 1 - \epsilon < f(\check{t}) \leq 1 \ and \tag{16}$$

$$0 < \check{t} < \delta \quad \Rightarrow \quad 1 > g(\check{t}) > \epsilon. \tag{17}$$

Since $\rho_n \to \rho$ as $n \to \infty$, for $\delta \in (0, 1)$, $\exists\ N \in \mathbb{N}$ such that $\forall\ \check{t} > 0$,

$$\xi(\rho_n, \rho, \check{t}_0) > 1 - \delta \quad \forall\ n \geq N \ and$$

$$N(\rho_n, \rho, \check{t}_0) < \delta \quad \forall\ n \geq N.$$

Particularly, for given $\check{t}_0 > 0$, we have

$$\xi(\rho_N, \rho, \check{t}_0) > 1 - \delta \ and$$

$$N(\rho_N, \rho, \check{t}_0) < \delta.$$

Now, by (1.2), $\exists\ \bar{\mathrm{q}} \in (0, 1)$ which implies

$$\xi(\rho_N, \rho, \check{t}_0) \ \circledast \ (1 - \bar{\mathrm{q}}) \geq 1 - \delta \ and \tag{18}$$

$$\bar{\mathrm{q}} \blacktriangle N(\rho_N, \rho, \check{t}_0) \leq \delta. \tag{19}$$

Consider a sequence $\{\alpha_n\}$ in $(0, 1)$ such that $\alpha_n \to 1 \quad as \quad n \to \infty$ & $\{\beta_n\}$ in $(0, 1)$ such that $\beta_n \to 0 \quad as \quad n \to \infty$. Assume $\{\rho_n\}$ is not bounded. Then, for a given $\check{t}_0$, for each α_k and β_k,, $\exists\ \ \rho_{n_k} \in \{\rho_n\}$ such that

$$\xi(\rho_{n_k}, \rho_N, 2\check{t}_0) \leq 1 - \alpha_k. \tag{20}$$

and

$$N(\rho_{n_k}, \rho_N, 2\check{t}_0) \geq \beta_k. \tag{21}$$

Since $\rho_n \to \rho$ as $n \to \infty$, we have $\rho_{n_k} \to x \quad as \quad k \to \infty$. Thus, for $\check{t}_0$, $\exists \quad m(t_0) \in \mathbb{N}$ which implies

$$\xi(\rho_{n_k}, \rho, \check{t}_0) > 1 - \bar{\mathrm{q}} \quad \forall\ k \geq m(\check{t}_0)$$

and

$$N(\rho_{n_k}, \rho, \check{t}_0) < \bar{\mathrm{q}} \quad \text{for all } k \geq m(\check{t}_0).$$

Hence, we can write

$$\xi(\rho_N, \rho, \check{t}_0) \ \circledast \ \xi(\rho_{n_k}, \rho, \check{t}_0) > \xi(\rho_N, \rho, \check{t}_0) \ \circledast \ (1 - \bar{\mathrm{q}}) \quad \text{for all } k \geq m(\check{t}_0)$$

$$\Rightarrow \ \xi(\rho_N, \rho, \check{t}_0) \ \circledast \ \xi(\rho_{n_k}, \rho, \check{t}_0) \geq (1 - \delta) \quad \text{for all } k \geq m(\check{t}_0) \quad (using (18))$$

which implies

$$f(\xi(\rho_N,\rho,\check{t}_0)) \circledast (\xi(\rho_{n_k},\rho,\check{t}_0)) > 1-\epsilon \quad \text{for all } k \geq m(\check{t}_0). \ (using \ (16))$$

Thus,

$$(f(\xi(\rho_N,\rho_{n_k},2\check{t}_0)))^{\alpha} > 1-\epsilon \quad \forall\ k \geq m(\check{t}_0) \ \ (using \ \ (\mathfrak{FG}_{\xi N}5)).$$

or equivalently

$$(f(1-\alpha_k))^{\alpha} > 1-\epsilon \quad \text{for all } k \geq m(\check{t}_0) \ \ (using \ \ (20)).$$

This leads to

$$\lim_{k\to\infty}(f(1-\alpha_k))^{\alpha} \geq 1-\epsilon.$$

Since $0<\epsilon<1$ is arbitrary, letting $\epsilon \to 0^+$ yields

$$\lim_{k\to\infty}(f(1-\alpha_k))^{\alpha} = 1 \Rightarrow \lim_{k\to\infty}(f(1-\alpha_k)) = 1$$

But from $(\mathfrak{F}2)$, we know that

$$\lim_{k\to\infty}(1-\alpha_k) = 1,$$

implies

$$\lim_{k\to\infty}\alpha_k = 0.$$

Similarly

$$\begin{aligned}
&N(\rho_N,\rho,\check{t}_0) \blacktriangle N(\rho_{n_k},\rho,\check{t}_0) < N(\rho_N,\rho,\check{t}_0) \blacktriangle \bar{\mathrm{q}} \leq \delta \quad \forall\ k \geq m(\check{t}_0) \ \ (using \ \ (19))\\
\Longrightarrow\ & g(N(\rho_N,\rho,\check{t}_0) \blacktriangle N(\rho_{n_k},\rho,\check{t}_0)) > \epsilon \quad \forall\ k \geq m(\check{t}_0) \ \ (using \ \ (17))\\
\Longrightarrow\ & (g(N(\rho_N,\rho_{n_k},2\check{t}_0)))^{\beta} > \epsilon \quad \forall\ k \geq m(\check{t}_0) \ \ (using \ \ (\mathfrak{FG}_{\xi N}9))\\
.\ \Longrightarrow\ & (g(\beta_k))^{\beta} > \epsilon \quad \forall\ k \geq m(\check{t}_0) \ \ (using \ \ (21))\\
\Longrightarrow\ & \lim_{k\to\infty} g(\beta_k) \geq \epsilon.
\end{aligned}$$

Since $0<\epsilon<1$ is arbitrary, letting $\epsilon \to 0^+$ yields

$$\lim_{k\to\infty}(g(\beta_k))^{\beta} = 0 \Rightarrow \lim_{k\to\infty} g(\beta_k) = 0.$$

But from $(\mathfrak{G}2)$, we know that

$$\lim_{k\to\infty}\beta_k = 1.$$

which contradicts the assumption that $\alpha_k \to 1$ and $\beta_k \to 0$. Therefore, the assumption that $\{\rho_n\}$ is unbounded leads to a contradiction. Hence the result. □

Conclusion

A novel concept for a generalised $\widehat{FMS}$ called an Intutionistic fuzzy $\mathfrak{FG}$-metric space by taking into account a family of functions. This leads to a further generalization of fuzzy metric spaces, extending its applicability and contributing to a deeper understanding of fuzzy metrics and their properties in mathematical analysis.

Conflicts of Interest

The authors declare no conflicts of interest.

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