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Nörlund Difference Sequence Spaces in 2-Normed Spaces Using Orlicz Function

Sunil K. Sharma, Ravi Kumar, Ayhan Esi and Ajay K. Sharma

ABSTRACT: In this paper, we introduce the concept of paranormed Nörlund-difference sequence spaces with Orlicz function over 2-normed spaces denoted by $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c_0}$, $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c}$ and $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{\ell_\infty}$ with Δ-transform are in c_0, c and ℓ_∞. Additionally, we investigate various topological characteristics of these spaces and analyze various inclusion relationships between these spaces. Furthermore, we establish and construct the $\alpha-,\beta-$ and $\gamma-$duals of these spaces, along with establishing the Schauder basis for $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c_0}$ and $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c}$.

Keywords: Difference sequence space, Nörlund matrix, paranormed sequence space, Orlicz function, $\alpha-,\beta-$ and $\gamma-$ duals.

Contents

1. Introduction and Preliminaries

In the paper, we will use ω to represent the class of all sequences formed by real or complex numbers, while ℓ_∞, c and c_0 will respectively denote the set of all bounded, convergent and null sequences. Additionally, we will use cs, bs, ℓ_1 and ℓ_p to represent the sequence spaces of all convergent, bounded, absolutely and p-absolutely convergent series, respectively. Gähler [4] introduced the concept of a 2-normed space as an innovative linear extension of a normed linear space. Subsequently, this concept has been further investigated by numerous researchers, as highlighted in the literature (refer to [3,5]).

Suppose Y be a real vector space having dimension d, where $2 \le d < \infty$. Consider the function $\|.,.\| : Y \times Y \to \mathbb{R}$, defined on Y, which acts as a 2-norm and satisfying the following properties:

1. $\|x,y\| = 0$ if and only if x and y are linearly dependent in Y;
2. $\|x,y\| = \|y,x\|$;
3. $\|\alpha x,y\| = |\alpha|\|x,y\|, \alpha \in \mathbb{R}$;
4. $\|x,y+z\| \le \|x,y\| + \|x,z\|$ for all $x.y,z \in Y$.

The pair $(Y,\|.,.\|)$ is known as 2-normed space, as in [5]. For instance, consider $Y = \mathbb{R}^2$ equipped with a 2-norm defined by $\|x,y\|$, where x and y are vectors in $\mathbb{R}^2$, representing the area of the parallelogram spanned by vectors x and y. This 2-norm is expressed as follows:

$$\|y_1,y_2\|_E = \text{abs}\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

Then, $(Y,\|.,.\|)$ is a 2-normed space. In a 2-normed space $(Y,\|.,.\|)$, a sequence (χ_k) is said to be converge to $L \in Y$ if

$$\lim_{k\to\infty} \|\chi_k - L, z\| = 0$$

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for every $z \in Y$.
In a 2-normed space $(Y, \|., .\|)$, a sequence (χ_k) is called Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \|\chi_k - \chi_p, z\| = 0$$

for every $z \in Y$. When every Cauchy sequence in Y has limit $L \in Y$, then Y is termed as complete with respect to the 2-norm. Moreover, any 2-normed space that is complete is termed as a 2-Banach space.

Definition 1.1 *Given that Y be a linear metric space, then $P : Y \to \mathbb{R}$ is a paranormed space if*

1. $P(y) \geq 0$
2. $P(-y) = P(y)$
3. $P(y + z) \leq P(y) + P(z)$ *for all* $y, z \in Y$
4. *if* α_k *be sequence of scalars such that* $\alpha_k \to \alpha$ *as* $k \to \infty$ *and* y_k *be the sequence of vectors such that* $P(y_k - y) \to 0$ *as* $k \to \infty$*, then* $P(\alpha_k y_k - \alpha y) \to 0$ *as* $k \to \infty$*.*

A total paranorm, denoted as P, is a paranorm such that $P(y) = 0$ implies $y = 0$. A total paranormed space is defined as a space Y equipped with a total paranorm P, i.e., (Y, P). The metric of every linear metric space is uniquely determined by some total paranorm see [25]. For additional details regarding sequence spaces, we refer to [1,15,16,17,20,21,22] and references therein.

Consider φ_1 and φ_2 as arbitrary sequence spaces. We define matrix $A = (a_{nk})$ as any infinite matrix, where $a_{nk} \in \mathbb{R}$ and $n, k \in \mathbb{N}$. Thus the matrix mapping from φ_1 into φ_2 defines by A and is represented by $A : \varphi_1 \to \varphi_2$, if for any $\xi = (\xi_n) \in \varphi_1$, the resulting sequence $A\xi = \{(A\xi)_n\} \in \varphi_2$, where

$$(A\xi)_n = \sum_k a_{nk}\xi_k, \text{ for } n \in \mathbb{N}. \tag{1.1}$$

By (φ_1, φ_2), we represent the collection of all matrices A, for each $n \in \mathbb{N}$ and every $\xi \in \varphi_1$ the series defined in equation (1.1) converges. We define that a sequence ξ is said to be A-summable to u if the sequence of matrix-vector products $A\xi$ converges to u. This convergence of Ax to u is termed as the A-limit of ξ.

Assuming (p_n) be a bounded sequence where each term is strictly positive, with $\sup p_n = Q$ and $M = \max\{1, Q\}$. Maddox in ([10,11,12]), Nakano [18] and Simons [23] defined the linear spaces $\ell_\infty, c(p)$ and $c_0(p)$ in the following manner:

$$\begin{aligned}
\ell_\infty &= \{\xi = (\xi_n) \in \omega : \sup_{n\in\mathbb{N}} |\xi_n|^{p_n} < \infty\}, \\
c(p) &= \{\xi = (\xi_n) \in \omega : \lim_{n\to\infty} |\xi_n - L|^{p_n} = 0\} \text{ for some } L > 0, \\
c_0(p) &= \{\xi = (\xi_n) \in \omega : \lim_{n\to\infty} |\xi_n|^{p_n} = 0\}.
\end{aligned}$$

In paranormed, they are complete sequence spaces

$$g(\xi) = \sup_{n\in\mathbb{N}} |\xi_n|^{\frac{p_n}{M}}, \text{ iff } \inf_{k\in\mathbb{N}} p_k > 0 \tag{1.2}$$

The set $S(u, v)$ is defined for the sequence space u and v as

$$S(u, v) = \{y \in \omega : \xi y \in v, \ \forall \xi \in u\} \tag{1.3}$$

The duals $\alpha-, \beta-, \gamma-$ of k, represented by $k^\alpha, k^\beta, k^\gamma$, respectively and defined as follows:
$k^\alpha = S(k, \ell_1), \ k^\beta = S(k, cs), \ k^\gamma = S(k, bs)$.

Definition 1.2 *The sequence x_n is known as Schauder basis of the paranormed sequence space (η, f), if $\xi \in \eta$, there exist a sequence (α_n) such that*

$$\lim_{k\to\infty} f(\xi - \sum_{n=0}^{k} \alpha_n x_n) = 0.$$

The Nörlund means, introduced by Peyerimhoff [19] and Mears [14], are a concept in mathematics that involve manipulating sequences. Given a sequence $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$, where $t_k \geq 0$ and $t_0 > 0$, the representation of Nörlund means for $t = (t_k)$ as a matrix is $\mathfrak{N}^t = (a^t_{nk})$, where

$$a^t_{nk} = \begin{cases} \frac{t_{n-k}}{T_n}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \text{for all } n \in \mathbb{N}.$$

Singh and MalikÂ [24] introduced paranormed Norlund $\mathfrak{N}^t$-difference sequence spaces. The Nörlund matrix $\mathfrak{N}^t$ transforms to the matrix $\mathcal{C}_1$ when $t = e = (1, 1, 1, \cdots)$, which corresponds to the matrix of arithmetic means. Now, let us consider $t_n = A^{r-1}_n$, where $r > -1$ and A^{r-1}_n denotes the n-th Abel mean of order $r - 1$. Then, the method $\mathfrak{N}^t$ gives the Cesàro method $\mathcal{C}_r$ for $r > -1$. In other words, the matrix $\mathfrak{N}^t$ for $t_n = A^{r-1}_n$ can be represented as follows:

$$A^r_n = \begin{cases} \frac{(r+1)(r+2)\cdots(r+n)}{n!}, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} \text{for } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, D_n denotes the determinant of square matrix, where $t_0 = D_0 = 1$ given, as

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \dots & 0 \\ t_2 & t_1 & 1 & 0 & \dots & 0 \\ t_3 & t_2 & t_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \dots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \dots & t_1 \end{vmatrix}$$

If $V^t = (r^t_{nk})$ is the inverse of $\mathfrak{N}^t = (a^t_{nk})$, [14], then

$$r^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases} \text{for all } n, k \in \mathbb{N}.$$

Also for all $k \in \mathbb{N}$, we have

$$D_k = \sum_{i=0}^{k-1} (-1)^{i-1} t_i D_{k-i} + (-1)^{k-1} t_k.$$

An Orlicz function $\mathfrak{M}$ is a function that is continuous, convex and nondecreasing with $\mathfrak{M}(0) = 0, \mathfrak{M}(u) > 0$ for $u > 0$ and $\mathfrak{M}(u) \to \infty$ as $u \to \infty$. This Orlicz function is used to define a sequence space, which was introduced by Lindenstrauss and Tzafriri [9]. The concept involves using the Orlicz function to characterize the properties of the sequence space. In other words, the sequence space $X_{\mathfrak{M}}$ associated with the Orlicz function $\mathfrak{M}$ consists of all sequences $\xi = (\xi_n)$ such that

$$\sum_{n=1}^{\infty} \mathfrak{M}(|\xi_n|) < \infty.$$

Alternatively, we can define $X_{\mathfrak{M}}$ as the set of all sequences such that the norm

$$\|\xi\|_{\mathfrak{M}} = \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \mathfrak{M}\left(\frac{|\xi_n|}{\lambda} \right) \right\}$$

is finite, where $\|.,.\|$ denotes the Orlicz norm induced by the Orlicz function $\mathfrak{M}$.
Furthermore, within every Orlicz sequence space $\ell_{\mathfrak{M}}$, there exists a subspace which is isomorphic to ℓ_p for $p \geq 1$ [9]. The Δ_2-condition states that $\mathfrak{M}(Ku) \leq K\mathfrak{M}(u)$ for all $u \geq 0$ and $0 < K < 1$. This condition verifies the boundedness of the Orlicz function.
Furthermore, the Orlicz function $\mathfrak{M}(u)$ is expressed in terms of an integral as:

$$\mathfrak{M}(u) = \int_0^u \eta(y)dy.$$

Here, $\eta(y)$ denotes the kernel of $\mathfrak{M}$. The kernel η exhibits right differentiability for $y \geq 0$ with $\eta(0) = 0, \eta(y) > 0, \eta$ is nondecreasing and $\eta(y) \to \infty$ as $y \to \infty$. This representation provides another perspective on the Orlicz function and its properties.
Kızmaz [7] introduced several distinct difference spaces, namely $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ which were extensively investigated by Et and Colak [2]. Let m (where $m \geq 0$) be an integer. Given any sequence space λ, we define $\lambda(\Delta^m) = \{t = (t_n) \in \omega : (\Delta^m \xi_n) \in \lambda\}$ for $\lambda = c_0, c$ and ℓ_∞, where $\Delta^m \xi = (\Delta^m \xi_n) = (\Delta^{m-1}\xi_n - \Delta^{m-1}\xi_{n+1})$. This can also be expressed as

$$\Delta^m \xi_n = \sum_{u=0}^{m} (-1)^u \binom{m}{u} \xi_{n+u}.$$

2. The Nörlund-Difference Sequence Spaces with Orlicz Function $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}$, $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$ Over 2-Normed Spaces

In this section, we introduce new paranormed Nörlund-difference sequence spaces equipped with Orlicz function defined over 2-normed spaces denoted as $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}$, $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$. We construct paranormed structures over these spaces and demonstrate their linear isomorphism.
The Nörlund sequence spaces $\mathfrak{N}^t(p)$ were introduced by Yeşilkayagil and Başar [26] as

$$\mathfrak{N}^t(p) = \{\xi = (\xi_n) \in \omega : \sum_n \| \frac{1}{T_k} \sum_{i=0}^{n} t_{k-i} x_i \|^{p_n} < \infty\}$$

where $0 < p_n \leq Q < \infty$.
Now, the Δ-Nörlund-difference Orlicz sequence spaces over 2-normed spaces are defined as follows

$$\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0} = \left\{\xi \in \omega : \lim_{n\to\infty} \left[\frac{1}{T_n} \sum_{i=0}^{n} \mathfrak{M}_k \left(\left\| \frac{t_{k-i}\Delta\xi_i}{\rho}, z \right\| \right)\right]^{p_n} = 0 \right\}$$

$$\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c = \left\{\xi \in \omega : \lim_{n\to\infty} \frac{1}{T_n} \sum_{i=0}^{n} \mathfrak{M}_k \left(\left\| \frac{t_{k-i}\Delta\xi_i - L}{\rho}, z \right\| \right)^{p_n} = 0 \right\}$$

$$\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty} = \left\{\xi \in \omega : \sup_n \left[\frac{1}{T_n} \sum_{i=0}^{n} \mathfrak{M}_k \left(\left\| \frac{t_{k-i}\Delta\xi_i}{\rho}, z \right\| \right)\right]^{p_n} < \infty \right\}.$$

where $\mathfrak{M} = (\mathfrak{M}_k)$ be the sequence of Orlicz functions and $p = p_n$ be the sequence of bounded positive real numbers.
Let $\chi = (\Delta\chi_n)$ be defined as the $\mathfrak{N}^t(\Delta)$-transform of the sequence $\xi = (\Delta\xi_n)$, then

$$\chi = (\chi_n) = \frac{1}{T_n} \sum_{i=0}^{n} t_{n-i} \Delta\xi_i \ \forall n \in \mathbb{N}.$$

Theorem 2.1 *For $0 \leq p_n \leq Q < \infty$ and $M = \max\{1, Q\}$, the spaces $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c, \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$ are the complete linear metric spaces paranormed by*

$$g(\xi) = \sup_n \left[\frac{1}{T_n} \sum_{i=0}^{n} \mathfrak{M}_k \left(\left\| \frac{t_{k-i}\Delta\xi_i}{\rho}, z \right\| \right)\right]^{\frac{p_n}{M}}.$$

Proof: The result has been proven for $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$. Since every bounded sequence has a finite supremum, the proof for the other spaces follows the similar method. Suppose that $\xi, \chi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, then

$$\sup_n \left[\frac{1}{T_n}\sum_{i=0}^n \mathfrak{M}_k\left(\left\|\frac{t_{k-i}\Delta(\xi_i+\chi_i)}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}}$$
$$\leq \sup_n \left[\frac{1}{T_n}\sum_{i=0}^n \mathfrak{M}_k\left(\left\|\frac{t_{k-i}\Delta\xi_i}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}} + \sup_n \left[\frac{1}{T_n}\sum_{i=0}^n \mathfrak{M}_k\left(\left\|\frac{t_{k-i}\Delta\chi_i}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}}$$

and for any $\alpha \in \mathbb{R}$, we have

$$\alpha^{p_k} \leq \{\max(1, \alpha^M)\}. \tag{2.1}$$

Given that $\mathfrak{M}(\theta) = 0$, it follows that $g(\theta) = 0$ and $g_1(\xi) = g_1(-\xi)\forall\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$. Therefore, inequalities (2.1) imply sub-additivity of the function g and $g(\alpha\xi) \leq \max(1, \alpha^M)g(\xi)$. Moreover, suppose $(\xi^{(n)}) \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, then $g(\xi^{(n)} - \xi) \to 0$ and let (α_n) is the sequence of scalars converging to α. Thus,

$$\begin{aligned} g(\alpha_n\xi^{(n)} - \alpha\xi) &= \sup_n \left[\frac{1}{T_n}\sum_{i=0}^n \mathfrak{M}_k\left(\left\|\frac{t_{k-i}\Delta(\alpha_n\xi_i^{(n)} - \alpha\xi_i)}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}} \\ &\leq \alpha_n - \alpha^{\frac{p_n}{M}} g(\xi^{(n)}) + \alpha^{\frac{p_n}{M}} g(\xi^{(n)} - \xi) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, g is paranorm.

Given a Cauchy sequence $\{\xi^i\} \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, where $\xi^i = \{\xi_0^{(i)}, \xi_1^{(i)}, \xi_2^{(i)}, \cdots\}$. For $\epsilon > 0$ be given, there exists $n_0(\epsilon)$ such that $g(\xi^i - \xi^j) < \epsilon$ for $i, j \geq n_0(\epsilon)$. Then, for any $k \in \mathbb{N}$, we get

$$\begin{aligned} &(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k - (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^j)_k^{\frac{p_n}{M}} \\ &\quad\leq \sup_n (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k - (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^j)_k^{\frac{p_n}{M}} \\ &\quad< \frac{\epsilon}{2}, \ \forall\ i, j > n_0(\epsilon) \end{aligned} \tag{2.2}$$

it follows that the sequence of real numbers forms a Cauchy sequence
$\{(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^0)_k, (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^1)_k, \ldots\}$, for $k \in \mathbb{N}$. Therefore,
$\{(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k \to (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_k$ as $i \to \infty$.
For the sequences $(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_0, (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_1, \ldots$ with infinitely many limits, there exist a sequence
$\{(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_0, (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_1, (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_2, \ldots\}$. By applying inequality (2.2) as $j \to \infty$, we get

$$(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k - \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_k < \frac{\epsilon}{2}, i \geq n_0(\epsilon).$$

Since $\xi^i = (\xi_n^i) \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ for each $i \in \mathbb{N}$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that

$$(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k^{\frac{p_n}{M}} < \frac{\epsilon}{2}, \text{ for every } i \geq n_0(\epsilon) \text{ and } k \in \mathbb{N}. \tag{2.3}$$

For fixed $i \geq n_0(\epsilon)$ and from (2.3), we get

$$\begin{aligned} (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_k^{\frac{p_n}{M}} &\leq (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k - (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_k^{\frac{p_n}{M}} \\ &+ (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi^i)_k^{\frac{p_n}{M}} < \epsilon \end{aligned}$$

for every $i \geq n_0(\epsilon)$. Hence, $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$. □

Remark 2.1 *The space $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ does not satisfy the absolute property, it follows that $\|g(\xi)\| \neq \|g(|\xi|)\|$. This indicates that $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ is non-absolute type.*

Theorem 2.2 *For $0 < p_n \leq Q < \infty$, the sequence spaces $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$, $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$, which are of non-absolute type, can be either paranorm or norm isomorphic to the spaces $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$, $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$ and $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ respectively.*

Proof: Consider the linear transformation $T : \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0} \to \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$ defined as $T\xi = (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0})\xi$. If $T\xi = 0$ only when $\xi = 0$, then T is injective.
Let $\chi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$ and $\xi = (\xi_n) = (\Delta\xi_n)$ be the sequence which is defined as
$\xi = (\xi_n) = \sum_{j=0}^{n}(-1)^{n-j} D_{n-j} T_j \Delta\chi_j$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
g(\xi) &= \sup_n \left[\frac{1}{T_n}\sum_{i=0}^{n} \mathfrak{M}_k\left(\left\|\frac{t_{n-i}\Delta\xi_i}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}} \\
&= \sup_n \left[\frac{1}{T_n}\sum_{i=0}^{n} \mathfrak{M}_k\left(\left\|\frac{t_{n-i}(\sum_{j=0}^{n}(-1)^{n-j}D_{n-j}T_j\Delta\chi_j)}{\rho}, z\right\|\right)\right]^{\frac{p_n}{M}} \\
&= \sup_n |\chi_n|^{\frac{p_n}{M}} < \infty.
\end{aligned}$$

Thus, $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, implying that T is surjective and its preservation under paranorm. Therefore $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ and $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$ are linearly isomorphic. In the same manner, we can establish that $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ are isomorphic to $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$ and $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ respectively. □

Theorem 2.3 *Consider the sequence $\Psi^{(n)}(t) = \{\Psi_k^{(n)}(t)\}$ which is defined as*

$$\Psi_k^{(n)}(t) = \begin{cases} (-1)^{(k-n)D_{k-n}T_n}, & 0 \leq n \leq k \\ 0, & n > k. \end{cases}$$

Then

1. *For $0 < p_n \leq Q < \infty$, the sequence $\{\Psi^{(n)}(t)\}_{n\in\mathbb{N}}$ forms a basis for $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ and $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ can be uniquely expressed as $\xi = \sum_n \alpha_n(t)\Psi^{(n)}(t)$, where $\alpha_n(t) = (\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_n$, for all $n \in \mathbb{N}$.*
2. *The set $\{e, \Psi^{(n)}(t)\}$ forms a basis of $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ and every $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ can be uniquely expressed as $\xi = \mu e + \sum_n[\alpha_n(t) - \mu]\Psi^{(n)}(t)$, where $\mu = \lim_{n\to\infty}(\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)\xi)_n$.*

Proof: (1) Clearly, $\{\Psi^{(n)}(t)\} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, also

$$\mathfrak{N}^t\Psi^{(n)}(t) = e^{(n)} \in \ell(c_0, \Delta), \forall\ n \in \mathbb{N}, 0 < p_n \leq Q < \infty. \tag{2.4}$$

For given $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ and for every non-negative integer m, we have

$$\xi^{[m]} = \sum_{n=0}^{m} \alpha_n(t)\Psi^{(n)}(t). \tag{2.5}$$

Thus by using $\mathfrak{N}^t$ and above equations, we have

$$\mathfrak{N}^t\xi^{[m]} = \sum_{n=0}^{m} \alpha_n(t)\mathfrak{N}^t\Psi^{(n)}(t) = \sum_{n=0}^{m}(\mathfrak{N}^t\xi)_n e^{(n)}.$$

Now, for all $i, m \in \mathbb{N}$, we have

$$\{\mathfrak{N}^t(\xi - \xi^{[m]})\}_i = \begin{cases} 0, & 0 \leq i \leq m \\ (\mathfrak{N}^t \xi)_i, & i > m \end{cases}.$$

For given $\epsilon > 0$, there exists a number $m_0 \in \mathbb{N}$ such that

$$\left[\sum_{i=m}^{\infty} (\mathfrak{N}^t \xi)_i^{p_n}\right]^{\frac{1}{M}} < \frac{\epsilon}{2}, \text{ for all } m \geq m_0.$$

Thus,

$$\begin{aligned} g[\mathfrak{N}^t(\xi - \xi^{[m]})] &= \left[\sum_{i=m}^{\infty} (\mathfrak{N}^t \xi)_i^{p_n}\right]^{\frac{1}{M}} \\ &\leq \left[\sum_{i=m_0}^{\infty} (\mathfrak{N}^t \xi)_i^{p_n}\right]^{\frac{1}{M}} \\ &< \epsilon, \text{ for all } m \geq m_0. \end{aligned}$$

Assume that $\xi = \sum_n \mu_n(t) \Psi^{(n)}(t)$. Then

$$\begin{aligned} (\mathfrak{N}^t \xi)_k &= \sum_n \mu_n(t) \{\mathfrak{N}^t \Psi^{(n)}(t)\}_k \\ &= \sum_n \mu_n(t) \epsilon_k^{(n)} = \mu_k(t), \text{ for all } k \in \mathbb{N}, \end{aligned}$$

which contradicts the assumption.
(2) Since $\{\Psi^{(n)}(t)\} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ and $e \in c$, the inclusion $\{e, \Psi^{(n)}(t)\} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ is trivial. For $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$, there exist unique μ satsifying equation (2.5). Thus, $\ell \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ when $\ell = \xi - \mu e$. Therefore, from part (1), the representation is unique. □

Theorem 2.4 *The inclusion* $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ *strictly holds.*

Proof: Let $\chi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$. Consequently $\mathfrak{N}^t \Delta \xi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$. As $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0} \subset \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$, it follows that $\mathfrak{N}^t \Delta \xi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$, thus implying that $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$. Thus the inclusion $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ is established.

Additionally, given that for every $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ we have $\mathfrak{N}^t \Delta \xi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$ and the inclusion $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0} \subset \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$ is strict for some $\mathfrak{N}^t \Delta \xi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c$, it follows that $\xi \notin \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$.

On the similar lines, it can be readily established that the inclusion
$\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ is strict.

□

Theorem 2.5 *For* $1 \leq p_n \leq p_{n+1}$*, the inclusion* $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_c \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$, $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ *and* $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{\ell_\infty} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ *holds for all* $n \in \mathbb{N}$.

Proof: For $p = e$, inclusions are obvious (see [13]). Now we examining the case for $\mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{\ell_\infty} \subset \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$. Given $\xi \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{\ell_\infty}$, we have $\Delta \xi^p \in \ell_\infty$, where $\xi^p = (\xi_k^{p_k})_{k=0}^{\infty}$. Choose a constant $m_0 \in \mathbb{N}$ such that $\Delta \xi_k^{p_k} < 1$ holds for all $k > m_0$. Thus for $n > m_0$, we have

$$\Delta \xi_k^{p_n} = (\Delta \xi_k^{p_k})^{\frac{p_n}{p_k}} \leq \Delta \xi_k^{p_k}, \ m_0 \leq k \leq n. \tag{2.6}$$

Given that $p=(p_n)$ is bounded and $p_k \leq p_n$ for $k \leq n$ and $n \in \mathbb{N}$ and , it follows that for $\mathcal{K}>0$, we get

$$\sup_n \left[\frac{1}{T_n}\sum_{i=0}^{m_0-1}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}\Delta\xi_i}{\rho},z\right\|\right)\right]^{p_n} \leq \mathcal{K}\sup_n\left(\frac{1}{T_n}\sum_{i=0}^{m_0-1}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)\right)^{p_n}|\Delta\xi_i|^{p_k} \tag{2.7}$$

Therefore, by applying Hölder inequality and using equation (2.6) and (2.7), we have
$|\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{\ell_\infty}|^{p_n}$

$$\begin{aligned}
&\leq \left[\sup_n\left(\frac{1}{T_n}\sum_{i=0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)\right)|\Delta\xi_i|\right]^{p_n}\\
&\leq \left[\sup_n\frac{1}{T_n}\sum_{i=0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)|\Delta\xi_i|^{p_n}\right]\left[\sup_n\left(\frac{1}{T_n}\sum_{i=0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)\right)\right]^{p_{n-1}}\\
&= \sup_n\frac{1}{T_n}\sum_{i=0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)|\Delta\xi_i|^{p_n}\\
&= \sup_n\frac{1}{T_n}\left[\sum_{i=0}^{n_0-1}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)|\Delta\xi_i|^{p_n}+\sum_{k=n_0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)|\Delta\xi_i|^{p_n}\right]\\
&\leq \frac{\mathcal{K}+1}{T_n}\sum_{k=0}^{n}\mathfrak{M}_k\left(\left\|\frac{t_{k-i}}{\rho},z\right\|\right)|\Delta\xi_i|^{p_n}\\
&= (\mathcal{K}+1)(\mathfrak{N}^t(\Delta\xi^p)).
\end{aligned}$$

Furthermore, given that $\Delta\xi^p \in \ell_\infty$, it follows that $\mathfrak{N}^t(\Delta\xi^p)\in\ell_\infty$,[13]. Using this, the previous inequality implies that $\mathfrak{N}^t\Delta\xi \in \mathfrak{N}^t(\mathfrak{M},p,\|.,.\|)_{\ell_\infty}$ and consequently $\xi \in \mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{\ell_\infty}$. Hence the inclusion $\mathfrak{N}^t(\mathfrak{M},p,\|.,.\|)_{\ell_\infty} \subset \mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{\ell_\infty}$ holds.

Similarly, by making certain modifications, we can prove that the inclusion $\mathfrak{N}^t(\mathfrak{M},p,\|.,.\|)_c \subset \mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_c$ and $\mathfrak{N}^t(\mathfrak{M},p,\|.,.\|)_{c_0} \subset \mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c_0}$ holds. □

3. The $\alpha-,\beta-$ and $\gamma-$ Duals of the Spaces $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{c_0}$, $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_c$ and $\mathfrak{N}^t(\Delta,\mathfrak{M},p,\|.,.\|)_{\ell_\infty}$

Lemma 3.1 *[6] For an infinite matrix (a_{nk}) defined over the field of complex number, then the following condition holds:*

1. *For $1<p_n\leq Q<\infty$, then $A=(a_{nk})\in(\ell(p),\ell_1)$ if and only if $R>1\in\mathbb{Z}$ such that*

$$\sup_{N\in\mathcal{F}}\sum_k\left|\sum_{n\in\mathbb{N}}\mathfrak{M}a_{nk}R^{-1}\right|^{p'_n}<\infty,\ \textit{for every}\ n\in\mathbb{N}.$$

2. *For $0<p_n\leq 1$, then $A=(a_{nk})\in(\ell(p),\ell_1)$ if and only if*

$$\sup_{N\in\mathcal{F}}\sum_{k\in\mathbb{N}}\left|\sum_{n\in\mathbb{N}}\mathfrak{M}a_{nk}\right|^{p_n}<\infty,\quad \textit{for every}\ n\in\mathbb{N}.$$

Lemma 3.2 *[8] For an infinite matrix (a_{nk}) defined over the field of complex number, then the following condition holds:*

1. *For $1<p_n\leq Q<\infty$, then $A=(a_{nk})\in(\ell(p),\ell_\infty)$ if and only if there exists $R>1\in\mathbb{Z}$ such that*

$$\sup_{n\in\mathbb{N}}\sum_k\mathfrak{M}|a_{nk}R^{-1}|^{p'_n}<\infty,\ \textit{for every}\ n\in\mathbb{N}. \tag{3.1}$$

2. For $0 < p_n \leq 1$, then $A = (a_{nk}) \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} \mathfrak{M}|a_{nk}|^{p_n} < \infty, \text{ for every } n \in \mathbb{N}. \tag{3.2}$$

Lemma 3.3 *[8] For $0 < p_n \leq Q < \infty$, then $A = (a_{nk}) \in (\ell(p), c)$ if and only if inequality (3.1) and (3.2) holds for $n \in \mathbb{N}$ and $\gamma_k \in \mathbb{C}$ such that $a_{nk} \to \gamma_k$, for each $k \in \mathbb{N}$.*

Theorem 3.1 *For $1 \leq p_n \leq Q < \infty$, the α-dual of the spaces $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}, \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c$, $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$ is*

$$\mathcal{D}_1 = \left\{ a \in \omega : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in\mathbb{N}} \mathfrak{M}(-1)^{n-k} \mathcal{D}_{n-k} T_n \Delta a_n R^{-1} \right|^{p'_n} < \infty \right\}, n \in \mathbb{N}$$

Proof: For $a = (a_n) \in \omega$, consider the following inequality

$$a_n \xi_n = \sum_{k=0}^{n} \mathfrak{M}(-1)^{n-k} \mathcal{D}_{n-k} T_n \Delta a_n \chi_k = (\mathcal{C}\chi)_n, \text{for all } n \in \mathbb{N} \tag{3.3}$$

where $\mathcal{C} = (\mathcal{C}_{nk})$ is defined as

$$\mathcal{C}_{nk} = \begin{cases} \mathfrak{M}(-1)^{n-k} \mathcal{D}_{n-k} T_n \Delta a_n, & 0 \leq k \leq n \\ 0, & k > n \end{cases},$$

for all $n, k \in \mathbb{N}$. Therefore, by combining equation (3.3) with part (a) of Lemma 3.1, we have $a\xi \in \ell_1$ when $\xi \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}, \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c$ or $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$ if and only if $\mathcal{C}\chi \in \ell_1$ when $\chi \in \mathfrak{N}^t(\mathfrak{M}, p, \|.,.\|)_{c_0}, \mathfrak{N}^t(\mathfrak{M}, p, \|.,.\|)_c$, $\mathfrak{N}^t(\mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$. Thus a is in α-dual of $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}, \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}$ if and only if $\mathcal{C} \in \mathfrak{N}^t(\mathfrak{M}, \ell_1, \|.,.\|)_{c_0} = \mathfrak{N}^t(\mathfrak{M}, \ell_1, \|.,.\|)_c = \mathfrak{N}^t(\mathfrak{M}, \ell_1, \|.,.\|)_{\ell_\infty}$. Therefore $a \in [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}]^\alpha = [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c]^\alpha = [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}]^\alpha$ if and only if $\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in\mathbb{N}} \mathfrak{M}(\mathcal{C}_{nk} R^{-1}) \right|^{p'_n} < \infty$ which leads to the consequence that

$$\mathcal{D}_1 = [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}]^\alpha = [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c]^\alpha = [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}]^\alpha. \tag{3.4}$$

□

Theorem 3.2 *Let $\mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ and $\mathcal{D}_5$ be sets defined as follow:*

$$\begin{aligned}
\mathcal{D}_2 &= \left\{ a \in \omega : \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \left| \sum_{i=k}^{n} \mathfrak{M}(-1)^{i-k} \mathcal{Y} R^{-1} \right|^{p'_n} < \infty \right\} \\
\mathcal{D}_3 &= cs \\
\mathcal{D}_4 &= \left\{ a \in \omega : \sup_{N\in\mathcal{F}} \sum_{k\in\mathbb{N}} \left| \sum_{i=k}^{n} \mathfrak{M}(-1)^{n-k} \mathcal{P} \Delta a_n \right|^{p'_n} < \infty \right\} \\
\text{and } \mathcal{D}_5 &= \left\{ a \in \omega : \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \left| \sum_{i=k}^{n} \mathfrak{M}(-1)^{i-k} \mathcal{P} \Delta a_i \right|^{p'_n} < \infty \right\}
\end{aligned}$$

where $\mathcal{Y} = \Delta a_i \mathcal{D}_{i-k} T_n, \mathcal{P} = \mathcal{D}_{i-k} T_n, 0 < p_n \leq 1, \forall n \in \mathbb{N}$. Then, $[\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{c_0}]^\beta = \mathcal{D}_2 \cap \mathcal{D}_3$, $[\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_c]^\beta = \mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathcal{D}_4$ and $[\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|.,.\|)_{\ell_\infty}]^\beta = \mathcal{D}_3 \cap \mathcal{D}_5$.

Proof: We now provide the β-dual for the space $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$. Consider the inequality

$$\sum_{k=0}^{n} a_k \xi_k = \sum_{k=0}^{n-1} \sum_{j=k}^{n} \mathfrak{M}(-1)^{j-k} \mathcal{P} \Delta a_j \chi_k + T_n \Delta a_n \chi_n = (E\chi)_n \tag{3.5}$$

where $\mathcal{P} = \mathcal{D}_{j-k} T_n$, for all $n \in \mathbb{N}$. We define the sequence $E = (e_{nk})$ as follow

$$e_{nk} = \begin{cases} \sum_{j=k}^{n} \mathfrak{M}(-1)^{j-k} \mathcal{D}_{j-k} T_n \Delta a_j, & 0 \leq k < n \\ T_n \Delta a_n, & k = n \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Then from equation (3.4), we have $a\xi = (a_n \xi_n) \in cs$ when $\xi = (\xi_n) \in \mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$ if and only if $E\chi \in c$ when $\chi = (\chi_n) \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$. Therefore, $a = (a_n) \in [\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}]^\beta$ if and only if $E \in \mathfrak{N}^t(\mathfrak{M}, p, \|., .\|)_{c_0}$. Thus by part (1) of Lemma 3.2, we have

$$\sup_{n \in \mathbb{N}} \sum_{k} \Big| \sum_{j=k}^{n} \mathfrak{M}(-1)^{j-k} \mathcal{D}_{j-k} T_n \Delta a_j R^{-1} \Big|^{p'_n} < \infty$$

and by Lemma 3.3, $\lim_n a_{nk}$ exists for all $k \in \mathbb{N}$. Therefore $[\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}]^\beta = \mathcal{D}_2 \cap \mathcal{D}_3$. Similarly, we can obtained the β-dual for the spaces $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$. □

Theorem 3.3 *The γ-dual of spaces $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{c_0}$, $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_c$ and $\mathfrak{N}^t(\Delta, \mathfrak{M}, p, \|., .\|)_{\ell_\infty}$ is $\mathcal{D}_2$.*

Proof: The proof of this theorem is directly follows from Theorem (3.2) with Lemma (3.3). □

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Sunil K. Sharma,
School of Mathematics,
Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J&K,
India.
E-mail address: sunilksharma42@gmail.com

and

Ravi Kumar,
Department of Mathematics,
Central University of Jammu, Rahya Suchani (Bagla), Samba-181143, J&K,
India.
E-mail address: ravikumar13849@gmail.com

and

Ayhan Esi,
Department of Basic Engi. Sci. (Math.Sect.),
Faculty of Engineering and Natural Sciences, Malatya Turgut Ozal University-44200, Malatya,
Türkiye.
E-mail address: aesi23@hotmail.com; ayhan.esi@ozal.edu.tr

and

Ajay K. Sharma,
Department of Mathematics,
Central University of Jammu, Rahya Suchani (Bagla), Samba-181143, J&K,
India.
E-mail address: aksju_76@yahoo.com