Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 1** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.77965

Relative Almost Uniform Convergent Sequence of Functions and its Topological Properties

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ABSTRACT: This paper introduces the concept of almost convergence for sequences of functions with respect to a scale function. We examine foundational properties such as linearity and completeness, and define the classes of relatively almost convergent, relatively null, and relatively bounded sequences of functions. In addition, we explore various topological aspects of the space, including convexity, strict convexity, separability, and symmetry.

Key Words: Almost convergent, relative convergent, scale function, completeness, symmetricity, convexity, separability.

Contents

1. Introduction

Throughout the article, σ denotes scale function on X.
In 1948, Lorentz [7] introduced the concept of almost convergent sequences of real numbers. The set of all almost convergent sequences of real numbers is denoted by $\hat{c}$ and is defined as follows:

$$\hat{c} = \left\{ x : \lim_{m \to \infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{i+n}, \text{exists uniformly in } n \right\}.$$

Let M be the set of bounded sequences of real numbers $x = (x_n)$ and it is a Banach space with respect to the linear norm

$$||x|| = \sup_{n \to \infty} x_n.$$

Nanda [10] studied the concept of strong methods of almost summability of real or complex sequences by infinite matrices. He introduced the class of strongly almost summable to zero, strongly almost summable and strongly almost bounded sequence spaces represented by $[\hat{A}, p]_0, [\hat{A}, p], [\hat{A}, p]_\infty$ respectively. Further, the completeness and topological properties of these three spaces were also discussed.

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2020 *Mathematics Subject Classification:* 46A03, 46A30, 46B99, 46E10.
Submitted July 21, 2025. Published January 19, 2026

Cunjalo [2] examined the definition of almost Cauchy in 2007 and showed that almost all double sequences of 0s and 1s are not almost convergent, while the set of almost convergent double sequences of 0s and 1s belong to the first category. Esi and Catalbas [5] extended the concept in triple sequence.
The notion of almost convergent was studied by different researchers from various aspects [3,4,6,8,9,11]. E. H. Moore introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Chittenden [1] defined the equivalent definition of relative uniform convergence for the case of sequence of functions of a real variable as follows:
A sequence (f_n) of single-valued, real-valued functions f_n of a variable x, ranging over a set X of real elements x, converges relatively uniformly on X in case there exist functions θ and σ, defined on X, and for every m an integer n_m (dependent on m), such that for every $n \geq n_m$ the inequality

$$m|\theta - f_n| \leq |\sigma|,$$

holds for every element x of X.
The following six propositions were established by Chittenden and are derived directly from the concept of relative uniform convergence.

1. Uniform convergence resembles the uniformity of convergence with respect to a constant scale function that is not zero.
2. For any scale function τ such that $|\tau| \geq |\sigma|$, uniform convergence of a sequence of functions with respect to a scale function σ implies uniform convergence of a sequence of functions with respect to any scale function τ.
3. For any scale function σ such that $A \leq |\sigma| \leq B$, where A and B are positive functions, then uniform convergence with respect to σ is equivalent to uniform convergence.
4. The scale function is not bounded if a sequence converges uniformly relative to the scale function but does not converge uniformly in the ordinary sense.

Several authors studied the notion of relative uniform convergence from the perspective of Korovkin approximation and double sequences.

2. Relative Almost Uniform Convergent Sequence of Functions

In this section, we introduce the following three classes and outline the concept of a relative almost uniform convergent sequence of functions.

1. The class of relative almost uniform convergent sequence of functions,
$\hat{c}(ru) : \sup_{i\geq 0} \lim_{k\to\infty} \frac{\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right|}{|\sigma(x)|} = f(x)$, uniformly in i.
2. The class of relative almost uniform null sequence of functions,
$\hat{c}_0(ru) : \sup_{i\geq 0} \lim_{k\to\infty} \frac{\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right|}{|\sigma(x)|} = 0$, uniformly in i.
3. The class of relative almost uniform bounded sequence of functions, $\hat{\ell}_\infty(ru) : \sup_k \frac{\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right|}{|\sigma(x)|} < M$, uniformly in i.

2.1. Preliminaries

Definition 2.1.1 *A sequence of functions $(f_n(x))$ of single-valued, real-valued defined on X is said to be relative almost uniform convergent to $f(x)$ on X with respect to the scale function $\sigma(x)$ defined on X if for every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, the inequality*

$$\sup_{i \geq 0} \left| \frac{1}{k} \sum_{n=i}^{i+k-1} f_n(x) - f(x) \right| < \varepsilon|\sigma(x)|, \textit{ for all } k > n_0 \textit{ and uniformly in } i.$$

i.e.,

$$\lim_{k\to\infty}\ \sup_{i\geq 0} \frac{\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x) - f(x)\right|}{|\sigma(x)|} = 0.$$

Definition 2.1.2 *A sequence of functions $(f_n(x))$ of single- valued, real-valued defined on X is said to be relative almost uniform null on X with respect to the scale function $\sigma(x)$ defined on X if for every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, the inequality*

$$\sup_{i\geq 0}\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right| < \varepsilon|\sigma(x)|, \text{ for all } k > n_0 \text{ and uniformly in } i.$$

Definition 2.1.3 *A sequence of functions $(f_n(x))$ of single- valued, real-valued defined on X is said to be relative almost uniformly bounded on X with respect to the scale function $\sigma(x)$ defined on X if for every $M > 0$ and $n_0 \in \mathbb{N}$, the inequality*

$$\sup_{k}\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right| < M|\sigma(x)|, \text{ for all } k > n_0 \text{ and uniformly in } i.$$

2.2. Characterization of Relative Almost Convergence

We formally characterize relative almost convergent sequences and examine how these sequences relate to null and bounded sequences.

Theorem 2.2.1 *Relative almost uniform convergence sequence of functions is equivalent to almost uniform convergence if the scale function is constant.*

Proof 1 *The theorem follows from the example below.*

Example 2.2.1 [**Ewert** [6], **Example 1.1**]
Let $\mathbb{R}$ be the set of real numbers. For each $n \geq 1$, we define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f_n = \begin{cases} 0, & \text{if } x \in (-\infty, n-1] \cup [n+1, \infty); \\ x^2 - n + 1, & \text{if } x \in [n-1, n]; \\ -x^2 + n + 1, & \text{if } x \in [n, n+1]. \end{cases}$$

The function f_n is almost uniformly convergent and also relatively almost uniformly convergent with respect to the scale function $\sigma(x) = 1$, for all $x \in \mathbb{R}$.

Theorem 2.2.2 *The scale function is unbounded if the sequence of functions does not converge almost uniformly, but is otherwise almost uniformly convergent with respect to the scale function.*

Proof 2 *The proof of the theorem follows from the example below.*

Example 2.2.2 [**Ewert** [6], **Example 1.1**] *Let R denote the set of real numbers with the natural topology. For each $n \geq 1$, we choose numbers a_n, b_n, c_n with $c_n \to 0$ and $0 < a_n < b_n < c_n$. Then, we define functions $f, f_n : \mathbb{R} \to \mathbb{R}$ assuming*

$$f_n = \begin{cases} 0, \ \text{if } x \in (-\infty, a_n] \cup [c_n, \infty); \\ \frac{x-a_n}{b_n-a_n}, \ \text{if } x \in [a_n, b_n]; \\ \frac{-x+c_n}{c_n-b_n}, \ \text{if } x \in [b_n, c_n]; \end{cases}$$

and $f(x) = 0, \forall x \in \mathbb{R}$.
We see that f_n is not almost uniform convergent. If we apply a scale function $\sigma(x) = \frac{1}{x}$ in the same domain of (f_n), we get $(f_n) \in \hat{c}(ru)$.

Theorem 2.2.3 *A sequence of functions that is relative almost uniform convergent is not neccessarily to be bounded.*

The theorem's proof can be found in Example 2.2.1. Although the sequence is not bounded, we may observe that it converges almost uniformly relative to a constant scale function.

3. Properties of Relative Almost Convergent Sequence of Functions

3.1. Linearity

Theorem 3.1.1 *The space $\hat{c}(ru), \hat{c}_0(ru), \hat{\ell}_\infty(ru)$ are linear.*

Proof 3 *Let $(f_n), (g_n) \in \hat{c}(ru)$.*
We define

$$h_n(x) = \alpha f_n(x) + \beta g_n(x).$$

Then,

$$\frac{1}{k}\sum_{n=i}^{i+k-1} h_n(x) = \alpha \cdot \frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x) + \beta \cdot \frac{1}{k}\sum_{n=i}^{i+k-1} g_n(x).$$

Taking the modulus and dividing by $|\sigma(x)|$, we get

$$\frac{1}{|\sigma(x)|}\left|\frac{1}{k}\sum_{n=i}^{i+k-1} h_n(x)\right| = \frac{1}{|\sigma(x)|}\left|\alpha \cdot \frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x) + \beta \cdot \frac{1}{k}\sum_{n=i}^{i+k-1} g_n(x)\right|.$$

Now as $k \to \infty$, we have f_n and g_n converge, so

$$\frac{1}{k}\sum_{n=i}^{i+k-1}\left|\frac{f_n(x)}{\sigma(x)}\right| \to f(x), \quad \frac{1}{k}\sum_{n=i}^{i+k-1}\left|\frac{g_n(x)}{\sigma(x)}\right| \to g(x),$$

and hence

$$\frac{1}{k}\sum_{n=i}^{i+k-1} h_n(x) \to \alpha f(x) + \beta g(x).$$

Therefore,

$$\lim_{k\to\infty}\left|\frac{1}{k}\sum_{n=i}^{i+k-1} h_n(x)\right| = |\alpha f(x) + \beta g(x)|,$$

which shows that $(h_n) \in \hat{c}(ru)$.
Linearity of $\hat{c}_0(ru)$ and $\hat{\ell}_\infty(ru)$ can be proved in the same manner.

3.2. Completeness

We consider the norm

$$||f|| = \begin{matrix}\sup \\ x \in X\end{matrix}\ \begin{matrix}\sup \\ i, k\end{matrix}\ \frac{1}{|\sigma(x)|}\left|\frac{1}{k}\sum_{n=i}^{i+k-1} f_n(x)\right|. \tag{3.1}$$

Theorem 3.2.1 *The space $\hat{c}(ru), \hat{c}_0(ru), \hat{\ell}_\infty(ru)$ are complete w. r. t. the norm given by Equation (1).*

We establish the proof of the theorem for $\hat{c}(ru)$.

Proof 4 *Consider a Cauchy sequence $f = (f_n^p)$ in $\hat{c}(ru)$. Then, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that*

$$\left\|\frac{f_n^p - f_n^q}{\sigma(x)}\right\| < \frac{\varepsilon}{2}, \forall p, q \geq K. \tag{3.2}$$

Also, (f_n^p) being a Cauchy sequence hence convergent to f_n (say).
i.e.,

$$\left\|\frac{f_n^p - f_n}{\sigma(x)}\right\| < \frac{\varepsilon}{2}, \forall p \geq K.$$

i.e.,

$$\begin{matrix}\lim \\ p \to \infty\end{matrix}\ \frac{\left|\frac{1}{k}\sum\limits_{p=i}^{i+k-1} f_n^p(x) - f_n(x)\right|}{|\sigma(x)|} = 0.$$

Taking limit $q \to \infty$ in Equation (1), we get

$$\left\| \frac{f_n^p - f_n}{\sigma(x)} \right\| < \frac{\varepsilon}{2}, \forall p \geq K.$$

This implies,

$$\left\| \frac{f_n^p - f}{\sigma(x)} \right\| < \frac{\varepsilon}{2}, \forall p \geq K, since\ f = f_n.$$

i.e.,

$$\lim_{p \to \infty} \frac{\left| \frac{1}{k} \sum_{p=i}^{i+k-1} f_n(x) - f(x) \right|}{|\sigma(x)|} = 0.$$

Then,

$$\|f_n - f\| = |f_n - f_n^i + f_n^i - f|| \leq ||f_n - f_n^i|| + ||f_n^i - f|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

This implies, $f \in \hat{c}(ru)$.

4. Topological Features of Function Spaces with Relative Almost Convergent Sequences

In this section we discuss the topological properties of the spaces of relative almost uniform convergent sequence of functions.

4.1. Convexity

A space X is said to be *convex* if, for any two functions $f, g \in X$ and any scalar $\lambda \in [0, 1]$, the convex combination $\lambda f + (1-\lambda)g$ is also in X. We study the conditions under which the spaces of relative almost uniform convergent sequence are convex.

Theorem 4.1.1 *The class of sequence of functions $\hat{c}(ru), \hat{c}_0(ru)$ and $\hat{\ell}_\infty(ru)$ are convex.*

Proof 5 *Let $\{f_n\}, \{g_n\} \in \hat{c}(ru)$ and let $\lambda \in [0, 1]$. We define $h_n = \lambda f_n + (1 - \lambda)g_n$. Then*

$$\frac{1}{k} \sum_{i=0}^{k-1} h_{n+i}(x) = \lambda \cdot \frac{1}{k} \sum_{i=0}^{k-1} f_{n+i}(x) + (1 - \lambda) \cdot \frac{1}{k} \sum_{i=0}^{k-1} g_{n+i}(x).$$

Using the triangle inequality and linearity:

$$\frac{1}{|\sigma(x)|} \left| \frac{1}{k} \sum_{i=0}^{k-1} h_{n+i}(x) \right| \leq \lambda \cdot \frac{1}{|\sigma(x)|} \left| \frac{1}{k} \sum_{i=0}^{k-1} f_{n+i}(x) \right| + (1 - \lambda) \cdot \frac{1}{|\sigma(x)|} \left| \frac{1}{k} \sum_{i=0}^{k-1} g_{n+i}(x) \right|.$$

Taking limits as $k \to \infty$, we obtain convergence of the convex combination to $\lambda f(x) + (1 - \lambda)g(x)$ in the relative almost uniform sense. Thus, $\{h_n\} \in \hat{c}(ru)$, proving convexity.

Example 4.1.1 *Consider the sequence of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R} \subset X$ defined by $f_n(x) = \left(\frac{1}{4}\right)^{nx} = g_n(x)$. We see that the sequences f_n and g_n are almost uniform convergent relative to a constant scale function $\sigma(x) = 1$, for all $x \in \mathbb{R}$. i.e., $f_n, g_n \in \hat{c}(ru)$.*
Further, $f_n + g_n = 2\left(\frac{1}{4}\right)^{nx} \in \hat{c}(ru)$, $\forall \lambda \in (0, 1)$. Hence $\hat{c}(ru)$ is convex.

4.2. Strict Convexity

A space X is *strictly convex* if the following holds:
For any two distinct functions f and g in X, the inequality

$$\varphi(\lambda f + (1 - \lambda)g) < \max\{\varphi(f), \varphi(g)\}, \quad \text{for all } \lambda \in (0, 1)$$

is satisfied. We examine the impact of strict convexity on relative almost convergent sequences.

Result 4.2.1 *The spaces $\hat{c}(ru), \hat{c}_0(ru), \hat{\ell}_\infty(ru)$ are not strictly convex.*

The result follows from the example below:

Example 4.2.1 *Consider the functions f_n and g_n such that*

$$f_n = g_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

We get $f_n \in \{\hat{c}(ru), \hat{c}_0(ru), \hat{\ell}_\infty(ru)\}$ with respect to $\sigma(x) = 1$, for all $x \in X$.
Also, $||f_n|| = 1 = ||g_n||$.
But, $||f_n + g_n|| \not< ||f_n|| + ||g_n||$.
Hence, $\hat{c}(ru), \hat{c}_0(ru), \hat{\ell}_\infty(ru)$ are not strictly convex.

4.3. Separability

A space is *separable* if it contains a countable dense subset. We investigate whether the space of relative almost convergent sequences is separable and explore the conditions for separability.

Theorem 4.3.1 *The spaces $\hat{c}(ru), \hat{c}_0(ru)$ are separable.*

Proof 6 *Let $f = (f_n) \in \hat{c}(ru)$. For every $\varepsilon > 0$ and $n > n_0(\varepsilon) \in \mathbb{N}$, we have*

$$\sup_{i \geq 0} \frac{\left| \frac{1}{k} \sum_{i=0}^{i+k-1} f_n(x) - f(x) \right|}{|\sigma(x)|} < \frac{\varepsilon}{2}, \tag{4.1}$$

uniformly in i.
Let (g_n) be a rational sequence of functions that is eventually constant.

$$g_n = \begin{cases} g_n, \forall n = 1 \text{ to } k \text{ and } n \leq n_0; \\ M, \text{if } n > n_0 \text{ and } M \in \mathbb{Q}; \end{cases}$$

such that

$$\left| \frac{f(x) - M}{\sigma(x)} \right| < \frac{\varepsilon}{2}. \tag{4.2}$$

Considering $g_n = M$ and Equations 4.1 and 4.2, we have

$$\left\| \frac{f_n - g_n}{\sigma(x)} \right\| < \varepsilon,$$

uniformly in i. Hence $\hat{c}(ru)$ is separable.
Separability for $\hat{c}_0(ru)$ can be established in the same manner.

Theorem 4.3.2 *The space $\hat{\ell}_\infty(ru)$ is not separable.*

Proof 7 *Consider the sequence of functions $(f_n) \in \hat{\ell}_\infty(ru)$ defined by*

$$f_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

We have, $||f_n|| = 1$. Consider

$$g_n = \begin{cases} f_n, & \text{if } n \text{ is odd;} \\ \frac{1}{3}, & \text{if } n \text{ is even.} \end{cases}$$

Let D be a dense subset of $\hat{\ell}_\infty$. D is uncountable since $||g_n|| < 1$ and hence, $\hat{\ell}_\infty$ is not separable.

4.4. Symmetric Spaces

A space X is symmetric if $(f_n) \in X$ then, $(f_n)_\alpha = (f_n)_{\alpha_{(i)}} \in X$, for all $\sigma \in \pi$. We study their relevance to the study of relative almost convergent sequences. We explore how these spaces interact with the convergence properties of function sequences.

Result 4.4.1 *The class of sequence of functions $\hat{c}(ru), \hat{c}_0(ru)$ and $\hat{\ell}_\infty(ru)$ are not symmetric.*

The result follows from the example below:

Example 4.4.1 *Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, defined by $f_n = \frac{n}{n+x}$, for all $x \in \mathbb{R}$. We see that the sequence f_n is relative almost uniform convergent w.r.t. the constant scale function $\sigma(x) = 1$, for all $x \in \mathbb{R}$. This implies that $(f_n) \in \hat{c}(ru)$ and $\hat{\ell}_\infty(ru)$.*
Let us consider the rearrangement sequence

$$f_n = \begin{cases} \frac{n}{n+x}, & \textit{if nis odd;} \\ 0, & \textit{if nis even.} \end{cases}$$

There does not exist any scale function for which the sequence g_n is relative almost uniform convergent.

5. Conclusion

We summarize the key findings of the paper, emphasizing the properties of relative almost convergent sequences, their completeness, linearity, and topological features such as convexity, separability, strict convexity, and symmetricity. The introduction of relative almost uniform convergent sequence of functions with respect to a scale function provides a rich framework for analyzing sequences of functions from both analytical and topological perspectives. By establishing the core properties and defining key subclasses such as relatively almost convergent, relatively null and relatively bounded sequences we lay the basic structure for further structural exploration. The investigation into the geometric and topological features of the space, including various convexity conditions, separability and symmetry, highlights the depth of the convergence concept, offering new approach for research in functional and sequence space theory.

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