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Entropy solution for double phase elliptic (p, q)-Laplacien problem with Right-Hand Side Measure

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ABSTRACT: In this work, we establish the existence of entropy solutions for a double phase elliptic (p,q)-Laplacian problem, subject to Dirichlet boundary conditions and measure data on the right-hand side. Our main strategy combines the variational method with the framework of Sobolev spaces, by verifying the conditions of the Minty–Browder theorem.

Key Words: double phase elliptic, entropy solution, existence, weighted Sobolev space, measure data..

Contents

1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$ $(N \geq 2)$ with a connected Lipschitz boundary $\partial\Omega$, $p(x) \in (1, \infty)$ for all $x \in (0, \infty)$ and μ be a Radon measure on Ω. In the present work, we study the existence of entropy solutions for double phase elliptic (p,q)-Laplacien problem with variable exponent as follows:

$$\begin{cases} -div(\Phi_{p,q}(x, \nabla u, \theta(u))) + |u|^{p-2}u = \mu & in \ \ \Omega, \\ u = 0 & on \ \ \partial\Omega, \end{cases} \tag{1.1}$$

such as $\Phi_{p,q}(., ., .)$ is defined precisement in the next section and μ is a diffuse measure on Ω, and θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$ that satisfies suitable assumptions(see assumption (H_3) below) and $a(x) \geq 0$ for all $x \in \overline{\Omega}$..

The study of partial differential equations (PDEs) and variational problems is a very rich field that has attracted the attention of several researchers. In this paper, we focus on an important type of PDEs, namely nonlinear partial differential equations with variable exponents. These types of PDEs model several physical phenomena, including elastic mechanics, electrorheological fluid dynamics, and image processing, among others. The manifestation of degenerate phenomena has arisen in diverse fields such as oceanography, turbulent fluid flows, induction heating, and electrochemical problems, as evidenced in sources such as [6], [7].

The first result on the existence and uniqueness of solutions for the Dirichlet problem with measure data dates back to 1965. In this paper, G. Stampacchia used a duality method (see [15]), and a few years later, H. Brezis studied some semilinear elliptic equations such as:

$$\begin{cases} -\Delta u + g(u) = \mu & in \ \ \Omega, \\ u = 0 & on \ \ \partial\Omega, \end{cases}$$

where $g \in C(\mathbb{R}, \mathbb{R})$ is a nondecreasing function. The case $\mu \in L^1(\Omega)$ is solved in the well-known papers by Brezis-Strauss [16], for the case where Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary. A well-known result of Bénilan-Brezis is devoted to the case of the Thomas-Fermi equation where μ is a

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measure on Ω (see [17] and the recent paper [18]). For nonlinear elliptic equations $A(u) = \mu$ with $u = 0$ on $\partial\Omega$, L. Boccardo, T. Gallouet, and L. Orsina proved the existence and uniqueness of entropy solutions in [3].

In this paper, we will prove the existence of solutions for the nonlinear degenerate weighted elliptic $p(.)$-Laplacian problem with measure data. We achieve this by using the regularization approach: we regularize problem (1) and investigate the existence of weak solutions for approximate problems. Secondly, we establish a set of a priori estimates crucial for demonstrating the existence of an entropy solution to problem (1). This type of problem has been dealt with by several authors. For example, for $\theta = 0$, $\omega(x) = 1$, and $f \in L^\infty$ data, M. Chipot and H. B. de Oliveira used the Schauder fixed-point theorem to prove the existence of weak solutions for some $p(.)$-Laplacian problems. With the same conditions and $f \in L^1$ data, C. Zhang and X. Zhang in [11] proved the existence of entropy solutions to problem (1.1). The case when $p(x)$ is constant and $f \in L^\infty$ is already treated by A. Sabri and A. Jamea in [1], where they prove the existence and uniqueness of weak solutions.

This paper is structured as follows. Section 2 introduces the necessary preliminary results and notations for our existence proof. In the following section, we prove the existence of entropy solutions for an approximate problem using variational methods, relying on a fundamental theorem of Minty-Browder type. Finally, we establish the existence of entropy solutions for problem (1) via Fatou's lemma, supported by additional technical lemmas to ensure the required conditions.

2. Preliminaries and notations

In the present section we give some definitions, notations and results which well be used in this work. Let φ function from $\Omega \times \mathbb{R}^+$ to $\mathbb{R}^+$ defined by

$$\varphi(x, y) = y^p + a(x)\ y^q$$

where a and p, q verify condition (H_1).

(H_1) $a : \Omega \to \mathbb{R}$ is a Lipschitz continuous and $p > \frac{Nq}{N+q-1}$ i.e $(\frac{q}{p} < 1 + \frac{q-1}{N})$

The function φ is a generalized N-function and

$$\varphi(x, 2y) = 2^p \varphi(x, y)$$

Now we define the Musielak-Orlicz space $L^\varphi(\Omega)$ by

$$L^\varphi(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{g is measurable and} \int_\Omega \varphi(x, |v|)dx < \infty \right\}$$

endowed with the Luxemburg norm

$$\|v\|_\varphi = inf \left\{ \lambda > 0, \int_\Omega \varphi(x, \frac{|v|}{\lambda})dx \leq 1 \right\}.$$

The Sobolev space corresponds to the L^φ space is

$$W^{1,\varphi}(\Omega) = \{v \in L^\varphi(\Omega) \ \text{ such that } \nabla v \in L^\varphi(\Omega)\}$$

With the norm

$$\|v\|_{1,\varphi} = \|v\|_\varphi + \|\nabla v\|_\varphi$$

Theorem 2.1 *i) If $q \neq N$ for all $r \in [1, q*]$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$ with* $q* = \begin{cases} \frac{Nq}{N-q} & if\, q¡N; \\ +\infty & if\, q \geq N. \end{cases}$

ii) If $q = N$ for all $r \in [1, +\infty[$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$.

iii) If $q \leq N$ for all $r \in [1, q*]$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$ compactly.

iv) *If* $q > N$ $W^{1,\varphi}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ *compactly.*

v) $W^{1,\varphi}(\Omega) \hookrightarrow L^{q}(\Omega)$.

We define now the weighted Lebesgue space $L^q_a(\Omega)$ by

$$L^q_a(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{ is mesurable } \middle| \begin{array}{l} \|v\|_{q,a} < \infty \\ \|v\|_{q,a} = \int_\Omega a(x)|v|^q dx \end{array} \right\}$$

On the space $L^{\varphi}(\Omega)$ we consider the function $\varrho_\varphi : L^{\varphi}(\Omega) \to \mathbb{R}^+$ defined by

$$\varrho_\varphi(v) = \int_\Omega [|v|^p + a(x)|v|^q] dx$$

The connection between ϱ_φ and $\|.\|_\varphi$ is established by the next result.

Proposition 2.1 **([4])** *Let u be an element of $L^{\varphi}(\Omega)$. The following assertions hold:*

i) $\|u\|_\varphi < 1$(*respectively* $>, = 1$) $\Leftrightarrow \varrho_\varphi(u) < 1$(*respectively* $>, = 1$),

ii) *If* $\|u\|_\varphi < 1$ *then* $\|u\|^p_\varphi \le \varrho_\varphi(u) \le \|u\|^q_\varphi$,

iii) *If* $\|u\|_\varphi > 1$ *then* $\|u\|^q_\varphi \le \varrho_\varphi(u) \le \|u\|^p_\varphi$,

iv) $\|u\|_\varphi \to 0 \Leftrightarrow \varrho_\varphi(u) \to 0$ *and* $\|u\|_\varphi \to \infty \Leftrightarrow \varrho_\varphi(u) \to \infty$.

Definition 2.1 Let $\varphi : [0, \infty) \to [0, \infty]$. We denote by φ^* the conjugate function of φ which is defined, for $u \ge 0$, by

$$\varphi^*(u) := \sup_{t \ge 0}(tu - \varphi(t))$$

Proposition 2.2 **([4])** *For any functions $u \in L^{\varphi}(\Omega)$, $v \in L^{\varphi^*}(\Omega)$, and under the assumption that hypothesis (H_1) be satisfied, we have:*

$$\int_\Omega |uv| dx \le 2\|u\|_\varphi \|v\|_{\varphi^*}.$$

In the following of this paper, the space $W^{1,\varphi}_0(\Omega)$ denote the closure of C^∞_0 in $W^{1,\varphi}(\Omega)$ with respect the norm $\|.\|_{1,\varphi}$ (see [10]).

Proposition 2.3 **([4])** *The spaces $(L^{\varphi}(\Omega), \|.\|_\varphi)$ and $(W^{1,\varphi}(\Omega), \|.\|_{1,\varphi})$ are separable and uniformly convex (hence reflexive) Banach space.*

We have

$$L^p(\Omega) \hookrightarrow L^{\varphi}(\Omega) \hookrightarrow L^p(\Omega) \bigcap L^q_a(\Omega)$$

Proposition 2.4 *([8]) Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded set, and let $u \in W^{1,\varphi}(\Omega)$*

$$\|u\|_\varphi \le C_0 \|\nabla v\|_\varphi$$

C_0 is a strictly positive constant depends on the exponent $diam(\Omega)$ and the dimension N.

Proposition 2.5 **([7])** *Let $1 < p < +\infty$ There exist two positive constants μ_p and ρ_p such that for every $x, y \in \mathbb{R}^N$, it holds that*

$$\begin{array}{rcl} \mu_p(|x| + |y|)^{p-2}|x - y|^2 & \le & \\ \langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle & \le & \rho_p(|x| + |y|)^{p-2}|x - y|^2 \end{array}$$

Proposition 2.6 ([6]) *Suppose that $\Omega \subset \mathbb{R}^N$ be a bounded set. Then, for all $u \in W_0^{1,\varphi}(\Omega, \|\cdot\|_{1,\varphi})$, the inequality*

$$\|u\|_\infty \leq C'\|\nabla u\|_\varphi,$$

is satisfied where the constant C' depends on the exponent $p(.), meas(\Omega)$ and the dimension N.

Definition 2.2 Given a constant $k > 0$; we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k(u)$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
By [8] we define also the space

$$\mathcal{T}_0^{1,\varphi}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} \text{u is a measurable function} \\ T_k(u) \in W_0^{1,\varphi}(\Omega) \text{ for all } \; k > 0 \end{array} \right\}.$$

Proposition 2.7 *For every $u \in \mathcal{T}_0^{1,\varphi}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{|u|<k}$, a.e. in Ω for every $k > 0$, where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $W_0^{1,\varphi}(\Omega)$, then v coincides with the standard distributional gradient of u, and we will denote it by $v = \nabla u$.*

Lemma 2.1 *Let us Ω a open bounded subset of $\mathbb{R}^N$. Then $u \in W_0^{1,\varphi}(\Omega)$ if and only if $u \in \mathcal{T}_0^{1,\varphi}(\Omega)$ and $\nabla u \in L_0^\varphi(\Omega)$.*

Definition 2.3 ([8]) The φ-capacity of any compact set $K \subseteq \Omega$ is first defined as

$$cap_\varphi(K, \Omega) = \{\varrho_\varphi(u) < \infty \,;\; u \in C_C^\infty\,(\Omega);\;\; u \geq \chi_K\}\,,$$

The φ-capacity of any open subset $U \subset \Omega$ is then defined by

$$cap_\varphi(U, \Omega) = \sup\{cap_\varphi(K, \Omega)\,; \text{K compact set of } \; U\}\,.$$

Finally, the φ-capacity of any subset $B \subset \Omega$ is then defined by

$$cap_\varphi(U, \Omega) = \inf\{cap_\varphi(B, \Omega)\,; \text{U open subset of } \; \Omega;\;\; U \subset B\}\,.$$

Definition 2.4 ([8]) A function u defined on Ω is said to be cap_φ-quasi continuous if for every $\varepsilon > 0$ there exists $B \subseteq \Omega$ with $cap_\varphi(B, \Omega) < \varepsilon$ such that the restriction of u to $\Omega \setminus B$ is continuous.

Remark 2.1 *For every $u \in W_0^{1,\varphi}(\Omega)$ has a cap_φ-quasi continuous representative. With this convention for every subset B of Ω we have*

$$cap_\varphi(B, \Omega) = \inf\left\{\varrho_\varphi(u)\,;\; \textit{for all} \;\; v \in W_0^{1,\varphi}(\Omega)\right\}.$$

Definition 2.5 ([8]) We define the spaces $M_b(\Omega)$ and $M_b^\varphi(\Omega))$ respectively as follows:

$$M_b(\Omega) = \{\mu : \Omega \to \mathbb{R} \;\text{ measure of Radon on } \Omega \text{ such that } \mu(\Omega) < \infty\}.$$

$$M_b^\varphi(\Omega) = \left\{ \mu \in M_b(\Omega)\,; \mu(E) = 0 \;\middle|\; \begin{array}{l} E \subseteq \Omega \\ cap_\varphi(E, \Omega) = 0 \end{array} \right\}.$$

Theorem 2.2 ([8]) *μ be an element of $M_b(\Omega)$. Then there exist an element $f \in L^1(\Omega)$ and $F \in (L^{\varphi*}(\Omega))^N$, such that $\mu = f - div(F)$, if and only if μ be an element of $M_b^\varphi(\Omega)$.*

Lemma 2.2 ([3]) *For $\xi,\eta\in\mathbb{R}^N$ and $1<p<\infty$, we have:*

$$\frac{1}{p}|\xi|^p-\frac{1}{p}|\eta|^p\leq|\xi|^{p-2}\xi(\eta-\xi).$$

where a dot denote the Euclidean scalar product in $\mathbb{R}^N$.

Lemma 2.3 ([3]) *For $a>0,b>0$ and $1\leq p<\infty$ we have*

$$(a+b)^p\leq 2^{p-1}(a^p+b^p).$$

Lemma 2.4 ([5]) *Let p et p' be two real numbers such that $p>1$, $p>1$, and $\frac{1}{p}+\frac{1}{p'}=1$. There existed a positive constant m such that*

$$|(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)|^{p'}\leq m\{(\xi-\eta)(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\}^{\frac{\beta}{2}}\{\xi^p+\eta^{p'}\}^{1-\frac{\beta}{2}},$$

For all $\xi,\eta\in\mathbb{R}^N$, $\beta=2$ if $1<p\leq 2$, and $\beta=p'$ if $p>2$.

Definition 2.6 ([13]) Let Y be a reflexive Banach space and let A be an operator from Y to its dual $Y^{'}$. We say that A is *monotone* if

$$\langle Au-Av,u-v\rangle\geq 0\qquad\forall\, u,v\in Y.$$

Theorem 2.3 ([13]) *Let Y be a reflexive real Banach space and $A:Y\to Y^{'}$ be a bounded operator, hemi-continuous, coercive and monotone on space Y. Then the equation $Au=v$ has at least one solution $u\in Y$ for each $v\in Y^{'}$.*

3. Assumptions and proof of the main claim

In this section, we first recall the notion of entropy solutions for problem (1.1), discussing their theoretical basis and existence. Under assumptions (H_1) and (H_2) we introduce further hypotheses to advance our analysis. The proof of the main theorem unfolds in three stages:

1. **Approximation:** Using a regularization method, we construct approximate solutions to problem (3.2) and show their existence as entropy solutions.
2. **Uniform bounds:** We derive key a priori estimates, ensuring the necessary compactness for the limiting process.
3. **Convergence:** Passing to the limit, we prove the existence of an entropy solution to the original problem (1.1).

(H_3) θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0)=0$ and for all real numbers x,y we have $|\theta(x)-\theta(y)|<\lambda_0|x-y|$ where λ_0 is a real constant such that $0<\lambda_0<\frac{1}{2C_0}$

(H_4) $\mu\in M_b^{\varphi}(\Omega)$.

Definition 3.1 A function $u\in\mathcal{T}_0^{1,\varphi}(\Omega)$ is an entropy solution of degenerate elliptic problem (1.1) if and only if

$$\begin{aligned}\int_\Omega\Phi_{p,q}(x,\nabla u,\theta(u))\nabla T_k(u-\psi)dx &+ \int_\Omega|u|^{p-2}uT_k(u-\psi)dx\\ &\leq \int_\Omega fT_k(u-\varphi)dx\\ &+ \int_\Omega F\nabla T_k(u-\psi)dx,\end{aligned}\tag{3.1}$$

for all $k>0$, $\psi\in W_0^{1,\varphi}(\Omega)\cap L^\infty(\Omega)$, where $\mu=f-div(F)$, such that $f\in L^1(\Omega)$ and $F\in(L^{\varphi*}(\Omega))^N$.

Theorem 3.1 *Let hypothesis $(H_1)-(H_4)$ be satisfied, then the problem (1.1) has a entropy solution(in the sense of Definition (3.1)).*

Proof: We consider the following approximate problem:

$$\begin{cases} -div(\Phi_{p,q}(x,\nabla u_n,\theta(u_n))+|u_n|^{p-2}u_n = g_n & in\ \ \Omega, \\ u_n = 0 & on\ \ \partial\Omega, \end{cases} \tag{3.2}$$

where $g_n = f_n - div(F)$ and $(f_n)_{n\in\mathbb{N}}$ is a sequence of smooth functions such that

- $f_n \to f$ in $L^1(\Omega)$.
- $|f_n| \le |f|\ \forall n \in \mathbb{N}^*$.

Let us the operator $B_n : W_0^{1,\varphi}(\Omega) \to (W_0^{1,\varphi}(\Omega))'$, where $(W_0^{1,\varphi}(\Omega))'$ is the dual space of $W_0^{1,\varphi}(\Omega)$ and let

$$B_n = A_n - L_n + K \quad and \quad A_n = B_n^1 + B_n^2$$

where for $u_n, v \in W_0^{1,\varphi}(\Omega)$

$$\begin{aligned} \langle B_n^1 u_n, v\rangle &= \int_\Omega \Phi_{p,q}(x,\nabla u_n,\theta(u_n))\nabla v dx, \\ \langle B_n^2 u_n, v\rangle &= \int_\Omega T_n(\omega|u_n|^{p-2}u_n) v dx, \\ \langle L_n, v\rangle &= \int_\Omega f_n v dx, \\ \langle K, v\rangle &= \int_\Omega F\nabla v dx. \end{aligned}$$

Firstly we will prove that B_n is bounded and is of type(M). For that, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\varphi}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u\ \ weakly\ \ in\ \ W_0^{1,\varphi}(\Omega), \\ B_n u_k \rightharpoonup \chi\ \ weakly\ \ in\ \ (W_0^{1,\varphi}(\Omega))', \\ \limsup_{k\to+\infty} \langle A_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{3.3}$$

Now, using ((3.2)), (H_3) and (H_4), then by Lebesgue dominated convergence theorem $B_n^2 u_k \rightharpoonup B_n^2 u - L_n - K \in (W^{1,\varphi}(\Omega))'$ and $B_n^2 u_k \rightharpoonup B_n^1 u - L_n - K \in (W^{1,\varphi}(\Omega))'$. Consequently $B_n^1 u = \chi - B_n^2 u + L_n + K$. This implies that $B_n u = \chi$. Therefore B_n is of type (M).
By using proposition (2.1), lemma (2.3) and hypothesis (H_3) , then we have for $u_n \in W_0^{1,\varphi}(\Omega)$, $v \in W_0^{\varphi *}(\Omega)$
Step 1. The operator B_n^1 is bounded.

We use Holder inequality, Lemma (2.4) and hypothesis (H_3), for any $u, v \in W_0^{1,\varphi}(\Omega)$ we have

$$\begin{aligned}
|\langle B_n^1 u_n, v\rangle| &\leq \int_\Omega |\nabla u_n - \theta(u_n)|^{p-1}|\nabla v|dx + \int_\Omega a(x)|\nabla u_n - \theta(u_n)|^{q-1}|\nabla v|dx,\\
&\leq \int_\Omega 2^{p-2}(|\nabla u_n|^{p-1} + |\theta(u_n)|^{p-1})|\nabla v|dx\\
&+ \int_\Omega a(x)2^{q-2}(|\nabla u_n|^{q-1} + |\theta(u_n)|^{q-1})|\nabla v|dx,\\
&\leq 2^{p-2}\int_\Omega (|\nabla u_n|^{p-1} + |\theta(u_n)|^{p-1})|\nabla v|dx\\
&+ 2^{q-2}\int_\Omega a(x)(|\nabla u_n|^{q-1} + |\theta(u_n)|^{q-1})|\nabla v|dx,\\
&\leq 2^{p-2}\left(\int_\Omega (|\nabla u_n|^{p-1}|\nabla v|dx + \int_\Omega \lambda_0^{p-1}|u_n|^{p-1}|\nabla v|dx\right)\\
&+ 2^{q-2}\left(\int_\Omega a(x)(|\nabla u_n|^{q-1}|\nabla v|dx + \int_\Omega a(x)\lambda_0^{q-1}|u_n|^{q-1}|\nabla v|dx\right),\\
&\leq 2^{p-2}\left(2\|\nabla u_n\|_p^{p-1}\|\nabla v\|_{p'} + 2\lambda_0^{p-1}\|u_n\|_p^{p-1}\|\nabla v\|_{p'}\right)\\
&+ 2^{q-2}\left(2\|\nabla u_n\|_{q,a}^{q-1}\|\nabla v\|_{q',a} + 2\lambda_0^{q-1}\|u_n\|_{q,a}^{q-1}\|\nabla v\|_{q',a}\right)\\
&\leq 2^{p-1}(\|\nabla u_n\|_p^{p-1}\|\nabla v\|_{p'} + \lambda_0^{p-1}{C_0}^p\|\nabla u_n\|_p^{p-1}\|\nabla v\|_{p'})\\
&+ 2^{q-1}(\|\nabla u_n\|_{q,a}^{q-1}\|\nabla v\|_{q,a} + \lambda_0^{p-1}{C_0}^q\|\nabla u_n\|_{q,a}^{q-1}\|\nabla v\|_{q',a})\\
&\leq 2^{p-1}(1+{C_0}^p)\|\nabla u_n\|_p^{p-1}\|\nabla v\|_{p'} + 2^{q-1}(1+{C_0}^q)\|\nabla u_n\|_{q,a}^{q-1}\|\nabla v\|_{q',a}\\
&\leq C\|\nabla u\|_\varphi^{p-1}\|\nabla v\|_{\varphi^*}\\
&\leq C\|u_n\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi^*}
\end{aligned}$$

where

$$C = 2^p(1+{C_0}^p),$$

and we have

$$\begin{aligned}
|\langle B_n^2 u, v\rangle| &\leq \int_\Omega |u|^{p-1}|v|dx\\
&\leq \|u_n\|_\varphi^{p-1}\|v\|_{\varphi^*}\\
&\leq \|u_n\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi^*}
\end{aligned}$$

We get immediately the boundedness of A_n, K and L_n. Hence, B_n is bounded.

Step 2. The operator T is coercive.
For any $u \in W_0^{1,\varphi}(\Omega)$ remark that by application hypothesis (H_3), there exists a positive constant C_3 such that

$$\int_\Omega f u dx \leq \|f\|_\infty \|u\|_{W_0^{1,\varphi}}.$$

On other hand for u large enough and by Lemma (2.3), we get

$$\begin{aligned}\langle B_n^1 u, u\rangle &= \int_\Omega |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u))\nabla u dx\\ &+ \int_\Omega a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))\nabla u dx\\ &\geq \int_\Omega \frac{1}{p}|\nabla u - \theta(u)|^p dx - \int_\Omega \frac{1}{p}|\theta(u)|^p dx\\ &+ \int_\Omega a(x)\frac{1}{q}|\nabla u - \theta(u)|^q dx - \int_\Omega a(x)\frac{1}{q}|\theta(u)|^q dx\end{aligned}$$

and by Lemma (2.4) we find

$$\tfrac{1}{2^{p-1}}|\nabla u|^p - |\theta(u)|^p \leq |\nabla u - \theta(u)|^p$$

$$\begin{aligned}\langle B_n^1 u, u\rangle &\geq \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}\int_\Omega |\theta(u)|^p dx\\ &+ \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx - \frac{2}{q}\int_\Omega a(x)|\theta(u)|^q dx\end{aligned}$$

By (H_3) we have $|\theta(u)| \leq \lambda_0|u|$

$$\begin{aligned}&\geq \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}\int_\Omega \lambda_0^p|u|^p dx\\ &+ \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx - \frac{2}{q}\int_\Omega a(x)\lambda_0^q|u|^q dx\end{aligned}$$

Then by Proposition (2.4) we have

$$\begin{aligned}&\geq \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}{C_0}^p\lambda_0^p\int_\Omega |\nabla u|^p dx + \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx\\ &- \frac{2}{q}{C_0}^q\lambda_0^q\int_\Omega a(x)|\nabla u|^q dx\\ &\geq \left(\frac{1}{p}\frac{1}{2^{p-1}} - \frac{2}{p}{C_0}^p\lambda_0^p\right)\int_\Omega |\nabla u|^p dx + \left(\frac{1}{q}\frac{1}{2^{q-1}} - \frac{2}{q}{C_0}^q\lambda_0^q\right)\int_\Omega a(x)|\nabla u|^q dx\\ &\geq M\int_\Omega |\nabla u|^p + a(x)|\nabla u|^q dx\\ &\geq M\|u\|_{1,\varphi}^p\end{aligned}$$

where

$$M = \sup\left(\frac{1}{p}\frac{1}{2^{p-1}} - \frac{2}{p}{C_0}^p\lambda_0^p, \frac{1}{q}\frac{1}{2^{q-1}} - \frac{2}{q}{C_0}^q\lambda_0^q\right).$$

Then

$$\frac{\langle B_n^1 u, u\rangle}{\|u\|_{1,\varphi}} \longrightarrow +\infty \text{ as } \|u\|_{1,\varphi} \longrightarrow +\infty$$

$$\begin{aligned}|\langle B_n^2 u, u\rangle| &\leq \int_\Omega |u|^p dx\\ &\leq \|u\|_{1,\varphi}^p\end{aligned}$$

Then B_n^1, $L_n - F$ are coercive.
Finally the operator B_n is coercive
Step 3. The operator B_n^1 is hemi-continous.
To prove that the operator B_n is hemi-continous, it suffices to show that B_n^1 is hemi-continuous. For that, let us $(u_k)_{k\in\mathbb{N}}$ a sequence of $W_0^{1,\varphi}(\Omega)$ and $u \in W_0^{1,\varphi}(\Omega)$ such that $u_k \longrightarrow u$ in $W_0^{1,\varphi}(\Omega)$. This is implies that $u_k \longrightarrow u$ is a.e and $(u_k - u)_{k\in\mathbb{N}}$ is bounded. In addition, we have $\nabla u_k \longrightarrow \nabla u$ in $(L^{\varphi}(\Omega))^N$. This is implies that $\nabla u_k \longrightarrow \nabla u$ is a.e and $(\nabla u_k - \nabla u)_{k\in\mathbb{N}}$ is bounded. Using (H_3), we have $\theta(u_k) \longrightarrow \theta(u)$ is a.e and $(\theta(u_k)-\theta(u))_{k\in\mathbb{N}}$ is bounded. Then $\Phi_{p,q}(x,\nabla u_k,\theta(u_k)) \longrightarrow \Phi_{p,q}(x,\nabla u,\theta(u))$ is a.e and $(\Phi_{p,q}(x,\nabla u_k,\nabla(u_k)))_{k\in\mathbb{N}}$ is bounded. This is implies by using Lebesgue's dominated convergence theorem that

$$\lim_{k\to+\infty}\int_{\Omega}(\Phi_{p,q}(x,\nabla u_k,\theta(u_k)-\Phi_{p,q}(x,\nabla u,\theta(u))\varphi dx = 0,$$

for any $\varphi \in W_0^{1,\varphi}(\Omega)$.
From where $B_n^1 u_k \rightharpoonup B_n^1 u$ in $(W_0^{1,\varphi}(\Omega))'$. This implies that B_n^1 is hemi-continuous. Finally we deduce that A_n is hemi-continuous. Hence, using theorem (2.3), there exists $u_n \in W_0^{1,\varphi}(\Omega)$ such that

$$\begin{aligned}\int_{\Omega}\Phi_{p,q}(x,\nabla u_n,\theta(u_n))\nabla v dx + \int_{\Omega}T_n(|u_n|^{p-2}u_n)v dx &= \int_{\Omega}T_n(f)v dx \\ &+ \int_{\Omega}F\nabla v dx,\end{aligned}$$

for all $v \in W_0^{1,\varphi}(\Omega)$. □

A priori estimates

The a priori estimates presented here are derived using Boccardo and Gallouët's approach, building upon the next lemmas, each proved in detail.

Lemma 3.1 *Assume that conditions $(H_1)-(H_4)$ hold. Then $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{\varphi}(\Omega))^N$.*

Proof: we take the test function $v = T_k(u_n)$ in ((3.4)), we obtain that

$$\begin{aligned}\int_{\Omega_k(n)}\Phi_{p,q}(x,\nabla T_k(u_n),\theta(u_n))\nabla T_k(u_n)dx &\leq \langle B_n u_n, u_n\rangle \\ &\leq \int_{\Omega}T_n(f)T_k(u_n)dx \\ &+ \int_{\Omega}F\nabla T_k(u_n)dx, \\ &\leq k\left(\|f\|_1 + \sum_{i=1}^{N}\left\|\frac{\partial F}{\partial x_i}\right\|_{\varphi*}\right),\end{aligned} \tag{3.4}$$

where $\Omega_k(n) = \{|u_n| \leq k\}$.So, by using the same arguments used to prove the coercivity of B_n, we obtain that

$$C_5\|T_k(u_n)\|_{1,\varphi}^p \leq k\left(\|f\|_1 + \sum_{i=1}^{N}\left\|\frac{\partial F}{\partial x_i}\right\|_{\varphi*}\right), \tag{3.5}$$

then

$$\|T_k(u_n)\|_{1,\varphi} \leq C_6^{\frac{1}{p}}, \tag{3.6}$$

where

$$C_6 = \frac{k}{C_5}\left(\|f\|_1 + \sum_{i=1}^{N}\left\|\frac{\partial F}{\partial x_i}\right\|_{\varphi*}\right).$$

This implies that, for any $k > 0$, $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,\varphi}(\Omega)$. □

Lemma 3.2 *Suppose that hypotheses $(H_1) - (H_4)$ are satisfied, the sequence $(u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function u.*

Proof: The proof proceeds by demonstrating that $(u_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in measure. For an appropriately chosen $k>0$, we note the equivalence $\{|T_k(u_n)|\geq k\} = \{|u_n|\geq k\}$. Applying inequality (3.6) together with Markov's inequality gives

$$\begin{aligned} meas(|u_n|\geq k|) &\leq \frac{1}{k^p}\int_\Omega |T_k(u_n)|^p dx, \\ &\leq \frac{1}{k^p}\|T_k(u_n)\|_p^p \\ &\leq \frac{1}{k^p}\|T_k(u_n)\|_{1,\varphi}^p \\ &\leq \frac{C_6}{k^p} \end{aligned} \tag{3.7}$$

this implies that

$$meas(|u_n|\geq k) \leq \frac{C_6}{k^p},$$

Therefore

$$meas(|u_n|>k)\to 0 \text{ as } k\to+\infty, \text{ uniformly with respect to n.} \tag{3.8}$$

Moreover, for every fixed $t>0$, every real positive k, let use $n,m\in\mathbb{N}$ and $\varepsilon>0$ we know that

$$\{|u_n-u_m|>t\}\subset\{|u_n|>t\}\cup\{|u_m|>t\}\cup\{|T(u_n)-T(u_m)|>t\}. \tag{3.9}$$

The strong convergence of $T_k(u_n)$ in $L^\varphi(\Omega)$ ensures that it is a Cauchy sequence in this space. An application of Markov's inequality then yields.

$$\begin{aligned} meas(|T_k(u_n)-T_k(u_m)|>t) &\leq meas(|T_k(u_n)-T_k(u_m)|^p>t^p) \\ &\leq \int_\Omega \frac{|T_k(u_n)-T_k(u_m)|^p}{t^p}dx \\ &\leq \frac{1}{t^p}\int_\Omega |T_k(u_n)-T_k(u_m)|^p dx \\ &\leq \frac{\varepsilon}{3}. \end{aligned} \tag{3.10}$$

Then, using (3.7) - (3.10), we get that

$$\begin{aligned} meas\{|u_n-u_m|>t\} &\leq meas\{|u_n|>t\}+meas\{|u_m|>t\} \\ &+ meas\{|T(u_n)-T(u_m)|>t\} \\ &\leq \varepsilon, \end{aligned} \tag{3.11}$$

for all $n,m\geq n_0(t,\varepsilon)$. This proves that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and it converges almost everywhere to some measurable function u.
Therefore

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \text{ in } W_0^{1,\varphi}(\Omega), \\ T_k(u_n) &\to T_k(u) \text{ in } L^\varphi(\Omega), \text{ and a.e. in } \Omega. \end{aligned} \tag{3.12}$$

□

Lemma 3.3 *Given that conditions (H_1) - (H_4) are fulfilled, we have convergence in measure: $\nabla u_n \xrightarrow{meas} \nabla u$ where ∇u is the weak gradient.*

Proof: Indeed, let ε, t, k, l are positive real numbers and let $n \in \mathbb{N}$, we have

$$\begin{aligned} \{|\nabla u_n - \nabla u| > t\} &\subset \{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\} \\ &\cup \{|\nabla T_k(u)| > k\} \cup \{|u_n - u| > l\} \cup B, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} B = \{|\nabla u_n - \nabla u| > t, |u_n| \leq k, |u| \leq k, |\nabla T_k(u_n)| \leq k, \\ |\nabla T_k(u)| \leq k, |u_n - u| \leq l\}. \end{aligned}$$

Using ((3.12)) and lemma (3.1), there exists an $n_1 \in \mathbb{N}$ and for k sufficiently large we obtain that

$$meas(\{|u_n| > k\} \cup \{|u| > k\} \cup \{|\nabla T_k(u_n)| > k\}) \leq \frac{\varepsilon}{4}, \tag{3.14}$$

$$meas(\{|\nabla T_k(u)| > k\} \leq \frac{\varepsilon}{4}, \tag{3.15}$$

and

$$meas(\{|u_n - u| > l\}) \leq \frac{\varepsilon}{4} \quad \text{for all } \ n \geq n_1. \tag{3.16}$$

Remark that we have

$$T_{k+l}(u_n) \rightharpoonup T_{k+l}(u) \text{ in } W_0^{1,\varphi}(\Omega), \tag{3.17}$$

then by using (H_3), we have that

$$\theta(T_{k+l}(u_n)) \to \theta(T_{k+l}(u)) \ \text{ in } (L^{\varphi}(\Omega))^N, \tag{3.18}$$

and

$$\nabla T_l(T_{k+l}(u_n)) \to \nabla T_l(T_{k+l}(u)) \ \text{ in } (L^{\varphi}(\Omega))^N. \tag{3.19}$$

Substituting (3.17)–(3.19) into our argument gives

$$\begin{aligned} \lim_{n\to+\infty} \int_\Omega \Phi_{p,q}(x, \nabla T_k(u), \theta(T_{k+l}(u_n)))\nabla T_l(T_{k+l}(u_n) - T_k(u))dx &= \\ \int_\Omega \Phi_{p,q}(x, \nabla T_k(u), \theta(T_{k+l}(u)))\nabla T_l(T_{k+l}(u) - T_k(u))dx &. \end{aligned} \tag{3.20}$$

Since the

$$\lim_{l\to 0} \nabla T_l(T_{k+l}(u_n) - T_k(u)) = 0. \tag{3.21}$$

Let $l < 1$, from (H_3) there exist a constant positive C_8 such that

$$\begin{aligned} \Phi_{p,q}(x, \nabla T_k(u), \theta(T_{k+l}(u_n)))\nabla T_l(T_{k+l}(u) - T_k(u)) \leq \\ C_8(|T_{k+1}(u)|^{p-1} + |\nabla T_k(u_n)|^{p-1})|\nabla T_1(T_{k+1}(u) - T_k(u))| \end{aligned} \tag{3.22}$$

with

$$C_8(|T_{k+1}(u)|^{p-1} + |\nabla T_k(u_n)|^{p-1})|\nabla T_1(T_{k+1}(u) - T_k(u))| \in L^1(\Omega)$$

Then, we get by using the Dominated Convergence Theorem that

$$\lim_{n\to+\infty}\int_\Omega \Phi_{p,q}(x,\nabla T_k(u),\theta(T_{k+l}(u_n)))\nabla T_l(T_{k+l}(u_n)-T_k(u))dx=0. \tag{3.23}$$

Let δ be a strictly positive number such that $\delta<\frac{\varepsilon}{8}$, there exists $n_2\in\mathbb{N}$ such that for all $n\geq n_2$, we have

$$-\int_\Omega \Phi_{p,q}(x,\nabla T_k(u),\theta(T_{k+l}(u_n)))\nabla T_l(T_{k+l}(u_n)-T_k(u))dx\leq\frac{\delta}{2}. \tag{3.24}$$

Now, the mapping

$$\mathcal{B}:(s,\xi_1,\xi_2)\to\Phi_{pq}(\xi_1-\theta(s))-\Phi(\xi_2-\theta(s)))(\xi_1-\xi_2). \tag{3.25}$$

is continuous and the set

$$\mathbf{L}:=\{(s,\xi_1,\xi_2)\in\Re:|s|\leq k,|\xi_1|\leq k,|\xi_2|\leq k,|\xi_1-\xi_2|>t\} \tag{3.26}$$

is a compact. Such that

$$\Re:=\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N.$$

Moreover, we have

$$\left(\Phi_{p,q}(x,\xi_1,\theta(\xi_1))-\Phi_{p,q}\left(x,\xi_2,\theta(\xi_2)\right)(\xi_1-\xi_2\right)>0,\ \ \forall\ \ \xi_1\neq\xi_2 \tag{3.27}$$

Then, the application $\mathcal{B}$ attains its minimum on $\mathbf{L}$, we shall note it by β, we have easily that $\beta>0$. Then using ((3.2)) and ((3.17)), we deduce that

$$\begin{aligned}
meas(B) &= \frac{1}{\beta}\int_B \beta dx, \qquad (3.28)\\
&\leq \int_\Omega[\Phi_{p,q}(x,\nabla u_n,\theta(u_n))-\Phi_{p,q}(x,\nabla T_k(u),\theta(T_{k+l}(u_n)))]\\
&\times \nabla T_l(T_{k+l}(u_n)-T_k(u))dx\\
&\leq \int_\Omega \Phi_{p,q}(x,\nabla u_n,\theta(u_n))\nabla T_l(T_{k+l}(u_n)-T_k(u))dx\\
&- \int_\Omega \Phi_{p,q}(x,\nabla T_k(u),\theta(T_{k+l}(u_n)))\nabla T_l(T_{k+l}(u_n)-T_k(u))dx\\
&\leq l\Big(\|f\|_1+\sum_{i=1}^N\Big\|\frac{\partial F}{\partial x_i}\Big\|_{\varphi^*}\Big)+\frac{\delta}{2},\\
&\leq l\Big(C_9+\sum_{i=1}^N\Big\|\frac{\partial F}{\partial x_i}\Big\|_{\varphi^\star}\Big)+\frac{\delta}{2},\\
&\leq l\,C_{10}+\frac{\delta}{2},\\
&\leq 2\delta\\
&\leq \frac{\varepsilon}{4}
\end{aligned}$$

Where

$$\begin{aligned}
l &< \frac{\delta}{2C_{10}},\\
C_{10} &= C_9+\sum_{i=1}^N\Big\|\frac{\partial F}{\partial x_i}\Big\|_{p'(.)}
\end{aligned}$$

By (3.14) - (3.16) and (3.28), we conclude that

$$meas\{|\nabla u_n - \nabla u| > t\} \leq \varepsilon. \tag{3.29}$$

This implies that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to ∇u.

Now by passing to the limit we show that limit function u is an entropy solution of our problem (1.1). Let $\varphi \in W_0^{1,p(.)}(\Omega,\omega) \cap L^\infty(\Omega)$ and take $v = T_k(u_n - \varphi)$ in equality (3.2), we get

$$\begin{aligned}\int_\Omega \Phi_{p,q}(x,\nabla u_n,\theta(u_n))\nabla T_k(w_n)dx &+ \int_\Omega T_n(|u_n|^{p-2}u_n)T_k(w_n)dx \\ &= \int_\Omega T_n(f)T_k(w_n)dx \\ &+ \int_\Omega F\nabla T_k(w_n(u,\varphi))dx.\end{aligned} \tag{3.30}$$

where

$$w_n = u_n - \varphi.$$

Define $\overline{k} = k + |\varphi|\infty$, and consider the set $\Omega_{n,\overline{k}} = \{|T_{\overline{k}}(u_n - \varphi)| < k\}$. Then we have

$$\begin{aligned}&\int_\Omega \Phi_{p,q}(x,\nabla u_n,\theta(u_n))\nabla T_k(u_n-\varphi)dx \\ = &\int_\Omega \Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}u_n))\nabla T_k(T_{\overline{k}}(u_n)-\varphi)dx, \\ = &\int_\Omega \Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n)\chi_{\Omega_{n,\overline{k}}}dx \\ - &\int_\Omega \Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla\varphi\chi_{\Omega_{n,\overline{k}}}dx\end{aligned}$$

$$\begin{aligned}&\int_\Omega (\Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n) + \frac{1}{p}|\theta(T_{\overline{k}}(u_n))|^p)\chi_{\Omega_{n,\overline{k}}}dx \\ - &\int_\Omega \Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla\varphi\chi_{\Omega_{n,\overline{k}}}dx \\ + &\int_\Omega T_n(|u_n|^{p-2}u_n)T_k(u_n-\varphi)dx = \int_\Omega T_n(f)T_k(u_n-\varphi)dx \\ + &\int_\Omega F\nabla T_k(u_n-\varphi)dx \\ + &\int_\Omega \left(\frac{1}{p}|\theta(T_{\overline{k}}(u_n))|^p\right)\chi_{\Omega_{n,\overline{k}}}dx\end{aligned} \tag{3.31}$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$.
Given that the sequence $T_{\overline{k}}(u_n)$ is bounded in the weighted variable-exponent Sobolev space $W_0^{1,\varphi}(\Omega)$, hypothesis (H_3) ensures the boundedness of $\theta(T_{\overline{k}}u_n)$ in $L^\varphi(\Omega)^N$. This boundedness property, combined with the structure of $\Phi_{p,q}(.,.,.)$, guarantees that the operator $\Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}u_n))$ is bounded and converges weakly in $L^\varphi(\Omega)$. Recalling that $(L^\varphi(\Omega))'$ represents the dual space, we nevertheless find that

$$\begin{aligned}u_n &\to u \text{ a.e in } \Omega, \\ \nabla u_n &\to \nabla u \text{ a.e in } \Omega.\end{aligned} \tag{3.32}$$

Hence,

$$\begin{aligned}\theta(T_{\overline{k}}(u_n)) &\to \theta(T_{\overline{k}}(u) \text{ a.e in } \Omega, \\ \nabla T_{\overline{k}}(u_n) &\to \nabla T_{\overline{k}}(u) \text{ a.e in } \Omega.\end{aligned} \tag{3.33}$$

This implies that

$$\Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)) \to \Phi_{p,q}(x,\nabla T_{\overline{k}}(u),\theta(T_{\overline{k}}(u)) \quad \text{in} \ \ (L^{\varphi}(\Omega))' \tag{3.34}$$

As,

$$\nabla\varphi\chi_{\Omega_{n,\overline{k}}} \ \text{ converges in } \ (L^{\varphi}(\Omega))'. \tag{3.35}$$

Via the estimates contained in (3.24)–(3.26), we obtain

$$\int_\Omega \Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n)\chi_{\Omega_{n,\overline{k}}}dx \to \int_\Omega g_k(u)\chi_{\Omega_{\overline{k}}}dx,$$

where,

$$g_k(u) = \Phi_{p,q}(x,\nabla T_{\overline{k}}(u),\theta(T_{\overline{k}}(u)))\nabla T_{\overline{k}}(u).$$

Now, by using (H_3) and properties of the truncated function, there exist a positive constant C_9 such that

$$|\theta(T_{\overline{k}}(u_n))|^p \leq C_{11}, \tag{3.36}$$

with

$$C_{11} = max((\overline{k}C_9)^p,(\overline{k}C_9)^q)$$

This follows from applying (3.24) - (3.27), and the Dominated Convergence Theorem.

$$\frac{1}{p}\int_\Omega |T_{\overline{k}}(u_n)|^p\chi_{\Omega_{n,\overline{k}}}dx \to \frac{1}{p}\int_\Omega |T_{\overline{k}}(u))|^p\chi_{\Omega_{\overline{k}}}dx \tag{3.37}$$

Now, by using lemma (2.1) we obtain that

$$\left(\Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n) + \frac{1}{p}|T_{\overline{k}}(u_n))|^p\chi_{\Omega_{n,\overline{k}}}\right) \geq 0 \quad \text{a.e in } \Omega. \qquad (40)$$

Finally by using ((3.24)), ((3.25)) and Fatou's lemma, we have

$$\begin{aligned}\int_\Omega \left(\Phi_{p,q}(x,\nabla T_{\overline{k}}(u)),\theta(T_{\overline{k}}(u)))\nabla T_{\overline{k}}(u) + \frac{1}{p}|T_{\overline{k}}(u))|^p\chi_{\Omega_{\overline{k}}}\right)dx \\ \leq \liminf_{n\to\infty}\int_\Omega \big(\Phi_{p,q}(x,\nabla T_{\overline{k}}(u_n),\theta(T_{\overline{k}}(u_n)))\nabla T_{\overline{k}}(u_n) \\ +\tfrac{1}{p}|(T_{\overline{k}}(u_n))|^p\chi_{\Omega_{n,\overline{k}}}\Big)dx.\end{aligned}$$

Taking limits as n goes to infinity in (3.22) we have

$$\begin{aligned}\int_\Omega \Phi_{p,q}(x,\nabla u-\theta(u))\nabla T_k(u),\varphi)dx \quad &+ \quad \int_\Omega |u|^{p-2}uT_k(u-\varphi)dx \\ &\leq \quad \int_\Omega fT_k(u-\varphi)dx \\ &+ \quad \int_\Omega F\nabla T_k(u-\varphi)dx,\end{aligned} \tag{3.38}$$

Therefore, there exists at least one entropy solution to our problem. □

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