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Some Families of the Eulerian Integrals Involving Generalized Hypergeometric Functions

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ABSTRACT: In this paper, we established a family of unified Eulerian integrals involving generalized hypergeometric function by employing generalized summation theorems for the series ${}_3F_2$ such as generalized Watson's theorem, generalized Dixon's theorem and generalized Whipple's theorem. Various special cases are deduced as consequences of our main theorems.

Key Words: Eulerian integral, generalized hypergeometric function, generalized Watson's theorem, generalized Dixon's theorem and generalized Whipple's theorem.

Contents

1. Introduction

The generalized hypergeometric function ${}_rF_s[z]$ ($r,\ s \in \mathbb{N}_0$) for which the infinite series form reads [19]

$$ {}_rF_s\left[\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \middle| z\right] = \sum_{k\geq 0} \frac{\prod_{j=1}^{r}(a_j)_k}{\prod_{j=1}^{s}(b_j)_k} \frac{z^k}{k!}, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ signifies the Pochhammer symbol and $a_j \in \mathbb{C}, b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$. The series converges for all $z \in \mathbb{C}$ if $r \leq s$. It is divergent for all $z \neq 0$ when $r > s+1$. Finally, if $r = s+1$, the series converges on the unit circle $|z| = 1$ when $\Re\left(\sum b_j - \sum a_j\right) > 0$.

In the theory of hypergeometric and generalized hypergeometric functions, there exist a remarkably large number of hypergeometric summation formulas which can be expressed in terms of the Gamma functions. In particular, for specified values of the argument, usually, 1, -1 and $1/2$, the hypergeometric function ${}_2F_1$ and generalized hypergeometric function ${}_3F_2$, ${}_4F_3$ and ${}_5F_4$ reduce to the well-known classical summation theorems such as the Gauss, Gauss second, Bailey and Kummer ones for the ${}_2F_1$ series, as well as the Watson, Dixon, Whipple for the ${}_3F_2$, second Whipple for the ${}_4F_3$ and Dougall's summation for the ${}_5F_4$ series play an important role in the theory of generalized hypergeometric functions (cf. [19]).

In 1961, for $\Re(\varepsilon) > 0, \Re(\varrho) > 0$, MacRobert [14] defined the extension of the classical Beta integral as

$$ \int_0^1 z^{\varepsilon-1}(1-z)^{\varrho-1}\{1+mz+n(1-z)\}^{-\varepsilon-\varrho}dz = (1+m)^{-\varepsilon}(1+n)^{-\varrho}\frac{\Gamma(\varepsilon)\Gamma(\varrho)}{\Gamma(\varepsilon+\varrho)}, \tag{1.1} $$

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where the constants m and n are such that none of the expressions $1+m, 1+n$, and $1+mz+n(1-z)$ (where $z \in [0,1]$) are zero and the term $\Gamma(\mathrm{x})$ refers to the second-kind Eulerian integral, also known as the gamma function, and is defined as

$$\Gamma(\mathrm{x}) = \int_0^\infty e^{-z} z^{\mathrm{x}-1} dz \quad (\Re(\mathrm{x}) > 0). \tag{1.2}$$

By making use of the substitution $z = \frac{t-\varkappa}{\varsigma-\varkappa}$ such that $dz = \frac{dt}{\varsigma-\varkappa}$ in (1.1), we obtain

$$\int_\varkappa^\varsigma \frac{(t-\varkappa)^{\varepsilon-1}(\varsigma-t)^{\varrho-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{\varepsilon+\varrho}} dt = \frac{(1+m)^{-\varepsilon}(1+n)^{-\varrho}}{(\varsigma-\varkappa)} \frac{\Gamma(\varepsilon)\Gamma(\varrho)}{\Gamma(\varepsilon+\varrho)}, \tag{1.3}$$

$$(\Re(\varepsilon) > 0, \Re(\varrho) > 0, (\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t) \neq 0, t \in [\varkappa, \varsigma], \varkappa \neq \varsigma),$$

which is the well known generalization of the MacRobert integral [8, p.287, eq.(3.198)].

The well known Watson's theorem (see [1], [3, pp. 449, Eq. (1.1)], [7, Section 4.4]) for hypergeometric function of unit argument with the aid of the duplication formula can be written

$${}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{a+b+1}{2}, & 2c & \end{matrix} \middle| 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{a+b+1}{2})\Gamma(c+\frac{1-a-b}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})\Gamma(c+\frac{1-a}{2})\Gamma(c+\frac{1-b}{2})} \tag{1.4}$$

provided that $\Re(2c-a-b) > -1$ and all the parameters are such that the series ${}_3F_2$ in the left is defined.

For $\Re(a-2b-2c) > -2$, the classical Dixon's theorem [1, pp. 13, Eq. (1)], we have

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c & \end{matrix} \middle| 1\right) = \\ &\frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+\frac{a}{2}-c)\,\Gamma(1+a-b-c)}. \end{aligned} \tag{1.5}$$

The well known Whipple's theorem [1, pp. 16, Eq. (3.4.1)] for hypergeometric function of unit argument when $a+b=1$ and $e+f=2c+1$ and for $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(e) > 0, \Re(f) > 0$ can be written

$${}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix} \middle| 1\right) = \frac{\pi\,\Gamma(e)\Gamma(f)}{2^{2c-1}\,\Gamma(\frac{a+e}{2})\,\Gamma(\frac{a+f}{2})\,\Gamma(\frac{b+e}{2})\,\Gamma(\frac{b+f}{2})}. \tag{1.6}$$

In the present paper, we derive a family of unified Eulerian integrals involving generalized hypergeometric function by employing generalized summation theorems for the series ${}_3F_2$ such as generalized Watson's theorem, generalized Dixon's theorem and generalized Whipple's theorem. Various special cases are deduced as consequences of our main theorems.

2. Integrals Involving Hypergeometric Functions

In this section, we investigate the generalized Watson's theorem, generalized Dixon's theorem and generalizations of Whipple's theorem. Moreover, we present some integrals involving generalized hypergeometric functions.

The following generalization of the well-known classical Watson's theorem (1.4) on the sum of a ${}_3F_2$, was obtained in 1992 by Lavoie et al. [11] as

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+p+1), & 2c+q \end{matrix}\,\middle|\, 1\right) \\
&= \mathcal{A}_{p,q}\, 2^{a+b+p-2}\, \frac{\Gamma(\frac{a+b+p+1}{2})\,\Gamma(c+\lfloor\frac{q}{2}\rfloor+\frac{1}{2})\Gamma(c-\frac{a+b}{2}-\frac{|p+q|}{2}+\frac{q}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\quad\times\left\{\mathcal{B}_{p,q}\,\frac{\Gamma\left(\frac{a}{2}+\frac{(1-(-1)^p)}{4}\right)\Gamma(\frac{b}{2})}{\Gamma\left(c-\frac{a}{2}+\frac{1}{2}+\lfloor\frac{q}{2}\rfloor-\frac{(-1)^q(1-(-1)^p)}{4}\right)\Gamma(c-\frac{b}{2}+\frac{1}{2}+\lfloor\frac{q}{2}\rfloor)}\right. \\
&\qquad\left.+\,\mathcal{C}_{p,q}\,\frac{\Gamma\left(\frac{a}{2}+\frac{(1+(-1)^p)}{4}\right)\Gamma(\frac{b+1}{2})}{\Gamma\left(c-\frac{a}{2}+\lfloor\frac{q+1}{2}\rfloor+\frac{(-1)^q(1-(-1)^p)}{4}\right)\Gamma\left(c-\frac{b}{2}+\lfloor\frac{q+1}{2}\rfloor\right)}\right\} \\
&\quad= \Psi_{p,q},
\end{aligned}
\tag{2.1}
$$

for $p, q = -2, -1, 0, 1, 2$. The largest integer less than or equal to x is also $\lfloor x\rfloor$. $\mathcal{A}_{p,q}, \mathcal{B}_{p,q}$, and $\mathcal{C}_{p,q}$ have the coefficient tables provided in [11].

In the theory of the generalized hypergeometric series, closely related to Dixon's theorem (1.5), Lavoie et al. [12] obtained its generalization in 1994. The general form for $\Re(a-2b-2c) > -2-2\mu-\nu$ such that $\mu = -3, -2, -1, 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$, can be written

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ 1+\mu+a-b, & 1+\mu+\nu+a-c \end{matrix}\,\middle|\, 1\right) \\
&= \frac{2^{\mu+\nu-2c}\,\Gamma(1+\mu+a-b)\,\Gamma(1+\mu+\nu+a-c)}{\Gamma(a-2c+\mu+\nu+1)\,\Gamma(a-b-c+\mu+\nu+1)} \\
&\qquad\times\frac{\Gamma(b-\frac{\mu}{2}-\frac{|\mu|}{2})\,\Gamma(c-\frac{1}{2}(\mu+\nu+|\mu+\nu|))}{\Gamma(b)\Gamma(c)} \\
&\quad\times\left\{\mathcal{A}_{\mu,\nu}\,\frac{\Gamma(\frac{a}{2}-c+\frac{1}{2}+\lfloor\frac{\mu+\nu+1}{2}\rfloor)\,\Gamma(\frac{a}{2}-b-c+1+\mu+\lfloor\frac{\nu+1}{2}\rfloor)}{\Gamma(\frac{a}{2}+\frac{1}{2})\,\Gamma(\frac{a}{2}-b+1+\lfloor\frac{\mu}{2}\rfloor)}\right. \\
&\qquad\left.+\,\mathcal{B}_{\mu,\nu}\,\frac{\Gamma(\frac{a}{2}-c+1+\lfloor\frac{\mu+\nu}{2}\rfloor)\,\Gamma(\frac{a}{2}-b-c+\frac{3}{2}+\mu+\lfloor\frac{\nu}{2}\rfloor)}{\Gamma(\frac{a}{2})\,\Gamma(\frac{a}{2}-b+\frac{1}{2}+\lfloor\frac{\mu+1}{2}\rfloor)}\right\} \\
&\quad= \Xi_{\mu,\nu},
\end{aligned}
\tag{2.2}
$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x, and its modulus is denoted by $|x|$ and the coefficients $\mathcal{A}_{\mu,\nu}$ and $\mathcal{B}_{\mu,\nu}$ are appeared in [12].

Two years after the generalization of Dixon's theorem, in 1996, Lavoie, Grondin, and Rathie [13] generalized the classical Whipple's theorem (1.6) as following

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\,\middle|\, 1\right) \\
&= \frac{\Gamma(e)\,\Gamma(f)\Gamma\left(c-\frac{(v+|v|)}{2}\right)\Gamma\left(e-c-\frac{(u+|u|)}{2}\right)\Gamma\left(a-\frac{(u+v+|u+v|)}{2}\right)}{2^{2a-u-v}\,\Gamma(e-a)\,\Gamma(f-a)\,\Gamma(e-c)\,\Gamma(a)\Gamma(c)} \\
&\times\left\{\mathcal{A}_{u,v}\,\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{(1-(-1)^u)}{4}\right)\Gamma(\frac{f}{2}-\frac{a}{2})}{\Gamma(\frac{e+a-u}{2}+\lfloor\frac{-v}{2}\rfloor)\,\Gamma\left(\frac{f+a-u}{2}+\left(\frac{(-1)^v}{4}\right)((-1)^u-1)+\lfloor\frac{-v}{2}\rfloor\right)}\right. \\
&\left.+\,\mathcal{B}_{u,v}\,\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{(1+(-1)^u)}{4}\right)}{\Gamma(\frac{e+a-1-u}{2}+\lfloor\frac{1-v}{2}\rfloor)}\times\frac{\Gamma\left(\frac{f}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{f+a-1-u}{2}+\left(\frac{(-1)^v}{4}\right)(1-(-1)^u)+\lfloor\frac{1-v}{2}\rfloor\right)}\right\} \\
&\quad= \Theta_{u,v},
\end{aligned}
\tag{2.3}
$$

where u, v take values in a subset of $0, \pm 1, \pm 2, \pm 3$ and $a+b = 1+u+v$, $e+f = 2c+1+u$. Also $\lfloor x \rfloor$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $\mathcal{A}_{u,v}$ and $\mathcal{B}_{u,v}$ are given in [13]. In the same paper [13], for $u+v = 0, \pm 1, \pm 2, \pm 3, \pm 4$ after a simple change of variables, the limiting case of the generalized Whipple's theorem (2.3) constructed by

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a, & 1+k-a \\ & b \end{matrix} \middle| \frac{1}{2} \right) \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-k-1}\,\Gamma\Big(1-a+\frac{(k+|k|)}{2}\Big)} \left\{ \frac{\phi_k}{\Gamma(\frac{b}{2}-\frac{a}{2}+\frac{1}{2})\,\Gamma\big(\frac{b+a}{2}-\lfloor\frac{1+k}{2}\rfloor\big)} \right. \\ &\qquad \left. + \frac{\psi_k}{\Gamma(\frac{b}{2}-\frac{a}{2})\,\Gamma(\frac{b}{2}+\frac{a}{2}-\frac{1}{2}-\lfloor\frac{k}{2}\rfloor)} \right\}, \end{aligned} \tag{2.4}$$

where ϕ_k and ψ_k are given in [13] and the values have been extended to include $k = \pm 5$.

2.1. Main Results

Theorem 2.1 *Let $\Re(2\alpha - a - b) > -1 - p - 2q$ for $p, q = -2, -1, 0, 1, 2$ and $\Re(\alpha) > 0$ for $q = 0, 1, 2$ and $\Re(\alpha) > -q$ for $q = -2, -1$. Then the following integral holds*

$$\begin{aligned} &\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+q-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+q}} \\ &\quad\times\ {}_2F_1\left(\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+p+1) \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]} \right) dt \\ &= \frac{\Gamma(\alpha)\,\Gamma(\alpha+q)}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)} \quad \Psi_{p,q}, \end{aligned} \tag{2.5}$$

provided m and n are constants such that $1+m; 1+n \neq 0$ and $(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t) \neq 0$ where $t \in [\varkappa, \varsigma]$.

Proof: If we consider the MacRobert integral, which involves the hypergeometric function, and designate the left-hand side of (2.5) by Δ, express the ${}_2F_1$ function as a series, and alter the order of integration and summation, which is readily justified given the series' uniform convergence in the interval $(\varsigma - \varkappa)$, we conclude that

$$\Delta = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k\ (1+m)^k}{k!\ (\frac{1}{2}(a+b+p+1))_k} \int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha+k-1}(\varsigma-t)^{\alpha+q-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+q+k}}\ dt.$$

Following additional simplification and integral evaluation, we arrive at

$$\Delta = \frac{\Gamma(\alpha)\,\Gamma(\alpha+q)}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k\,(\alpha)_k}{k!\ (\frac{1}{2}(a+b+p+1))_k\,(2\alpha+q)_k}.$$

Summing up the series, we have

$$\Delta = \frac{\Gamma(\alpha)\,\Gamma(\alpha+q)}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)}\ {}_3F_2\left(\begin{matrix} a, & b, & \alpha \\ \frac{1}{2}(a+b+p+1), & 2\alpha+q \end{matrix} \middle| 1 \right).$$

Finally, by using the generalized Watson's theorem (defined in (2.1)) in the above expression, the right-hand side of (2.5) can easily be obtained. This completes the pf of (2.5). □

Theorem 2.2 *The more general integral also holds with its twenty-five results hold for $y \in \mathbb{Z}$ and $p, q = 0, \pm 1, \pm 2$ and $\Re(2\alpha - a - b) > -1 - p - 2q$; $\Re(\alpha) > 0$ for $y = 0, 1, 2, 3, \ldots$ and $\Re(\alpha) > -y$ for $y =$*

$-1,-2,-3,\ldots$

$$\begin{aligned}&\int_{\varkappa}^{\varsigma}\frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+1}}\\&\times\;{}_4F_3\left(\begin{array}{c}a,\ b,\ 2\alpha+y+1,\ c\\ \frac{1}{2}(a+b+p+1),\ \alpha,\ 2c+q\end{array}\middle|\frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right)dt\\&=\frac{\Gamma(\alpha)\Gamma(\alpha+y+1)}{(1+m)^{\alpha}(1+n)^{\alpha+y+1}(\varsigma-\varkappa)\Gamma(2\alpha+y+1)}\quad\Psi_{p,q},\end{aligned}\tag{2.6}$$

provided m and n are constants such that $1+m; 1+n\neq 0$ and $(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)\neq 0$ where $t\in[\varkappa,\varsigma]$.

Proof: If we consider the MacRobert integral, which involves the hypergeometric function, and designate the left-hand side of (2.6) by Ω, express the ${}_4F_3$ function as a series, and alter the order of integration and summation, which is readily justified given the series' uniform convergence in the interval $(\varsigma-\varkappa)$, we conclude that

$$\begin{aligned}\Omega=&\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(2\alpha+y+1)_k(c)_k\,(1+m)^k}{k!\,(\frac{1}{2}(a+b+p+1))_k(\alpha)_k(2c+q)_k}\\&\times\int_{\varkappa}^{\varsigma}\frac{(t-\varkappa)^{\alpha+k-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+k+1}}\,dt.\end{aligned}$$

Following the integral evaluation with additional simplification and summing up the series, we obtain

$$\Omega=\frac{\Gamma(\alpha)\Gamma(\alpha+y+1)}{(1+m)^{\alpha}(1+n)^{\alpha+y+1}(\varsigma-\varkappa)\Gamma(2\alpha+y+1)}\,{}_3F_2\left(\begin{array}{ccc}a, & b, & c\\ \frac{1}{2}(a+b+p+1), & 2c+q\end{array}\middle|1\right).$$

By using the generalized Watson's theorem (defined in (2.1)) in the above expression, the right-hand side of (2.6) can easily be obtained. This completes the pf of (2.6) □

Theorem 2.3 *Let $\Re(a-2b-2c)>-2-2\mu-\nu$ such that $\mu=-3,-2,-1,0,1,2,3$ and $\nu=0,1,2,3$ and $\Re(a)>0,\Re(b)>0$. Then the following integral holds*

$$\begin{aligned}&\int_{\varkappa}^{\varsigma}\frac{(t-\varkappa)^{c-1}(\varsigma-t)^{a+\mu+\nu-2c}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{a+\mu+\nu+1-c}}\\&\times\;{}_2F_1\left(\begin{array}{c}a,\quad b\\ 1+\mu+a-b\end{array}\middle|\frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right)dt\\&=\frac{(1+m)^{-c}(1+n)^{2c-a-\mu-\nu-1}}{(\varsigma-\varkappa)}\quad\frac{\Gamma(c)\Gamma(a+\mu+\nu+1-2c)}{\Gamma(a+\mu+\nu+1-c)}\quad\Xi_{\mu,\nu},\end{aligned}\tag{2.7}$$

provided m and n are constants such that $1+m; 1+n\neq 0$ and $(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)\neq 0$ where $t\in[\varkappa,\varsigma]$.

Proof: If we consider the MacRobert integral, which involves the hypergeometric function, and designate the left-hand side of (2.7) by Λ, express the ${}_2F_1$ function as a series, and alter the order of integration and summation, which is readily justified given the series' uniform convergence in the interval $(\varsigma-\varkappa)$, we conclude that

$$\Lambda=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r\,(1+m)^r}{r!\,(1+\mu+a-b)_r}\int_{\varkappa}^{\varsigma}\frac{(t-\varkappa)^{c+r-1}(\varsigma-t)^{a+\mu+\nu-2c}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{a+\mu+\nu+1-c+r}}\,dt.$$

Following the integral evaluation with additional simplification and summing up the series, we obtain

$$\begin{aligned}\Lambda=&\frac{(1+m)^{-c}\,(1+n)^{2c-a-\mu-\nu-1}}{\varsigma-\varkappa}\frac{\Gamma(c)\Gamma(a+\mu+\nu+1-2c)}{\Gamma(a+\mu+\nu+1-c)}\\&\times\;{}_3F_2\left(\begin{array}{ccc}a, & b, & c\\ 1+\mu+a-b, & 1+\mu+\nu+a-c\end{array}\middle|1\right).\end{aligned}$$

Using generalized Dixon's theorem (defined in (2.2)) in the above expression, the desired result (2.7) can easily be obtained. □

Theorem 2.4 *Let $\beta = i-j+1+u$ and $a+b = 1+u+v$ such that $\Re(i) > 0, \Re(j) > 0, \Re(a) > 0, \Re(b) > 0$ for which $u+v = 0, \pm 1, \pm 2, \pm 3, \pm 4$. Then the following integral holds*

$$\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{i-1}(\varsigma-t)^{j-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{i+j}} \times {}_2F_1\left(\begin{matrix} a, & b \\ & \beta \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]} \right) dt$$
$$= \frac{\Gamma(i)\Gamma(j)}{(1+m)^i(1+n)^j(\varsigma-\varkappa)\Gamma(i+j)} \times \frac{\Gamma(\beta)\Gamma(i+j)\Gamma\left(i-\frac{(v+|v|)}{2}\right)\Gamma\left(\beta-i-\frac{(u+|u|)}{2}\right)\Gamma\left(a-\frac{(u+v+|u+v|)}{2}\right)}{2^{2a-u-v}\,\Gamma(\beta-a)\,\Gamma(i+j-a)\,\Gamma(\beta-i)\,\Gamma(a)\Gamma(i)}$$
$$\times \Bigg\{ \mathcal{A}_{u,v} \frac{\Gamma\left(\frac{\beta}{2}-\frac{a}{2}+\frac{(1-(-1)^u)}{4}\right)\Gamma(\frac{i+j}{2}-\frac{a}{2})}{\Gamma\left(\frac{\beta+a-u}{2}+\lfloor\frac{-v}{2}\rfloor\right)\Gamma\left(\frac{i+j+a-u}{2}+\left(\frac{(-1)^v}{4}\right)((-1)^u-1)+\lfloor\frac{-v}{2}\rfloor\right)}$$
$$+\mathcal{B}_{u,v} \frac{\Gamma\left(\frac{\beta}{2}-\frac{a}{2}+\frac{(1+(-1)^u)}{4}\right)}{\Gamma\left(\frac{\beta+a-1-u}{2}+\lfloor\frac{1-v}{2}\rfloor\right)} \times \frac{\Gamma(\frac{i+j}{2}-\frac{a}{2}+\frac{1}{2})}{\Gamma\left(\frac{i+j+a-1-u}{2}+\left(\frac{(-1)^v}{4}\right)(1-(-1)^u)+\lfloor\frac{1-v}{2}\rfloor\right)}\Bigg\}, \tag{2.8}$$

where $\lfloor x\rfloor$ is the greatest integer less than or equal to x, and its modulus is denoted by $|x|$ and provided m and n are constants such that $1+m; 1+n \neq 0$ and $(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)\neq 0$ where $t\in[\varkappa,\varsigma]$.

Proof: If we consider the MacRobert integral, which involves the hypergeometric function, and designate the left-hand side of (2.8) by Υ, express the ${}_2F_1$ function as a series, and alter the order of integration and summation, which is readily justified given the series' uniform convergence in the interval $(\varsigma-\varkappa)$, we conclude that

$$\Upsilon = \sum_{l=0}^{\infty} \frac{(a)_l(b)_l(1+m)^l}{l!\ (\beta)_l} \int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{i+l-1}(\varsigma-t)^{j-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{i+j+l}}.$$

Following additional simplification and integral evaluation, we arrive at

$$\Upsilon = \frac{\Gamma(i)\Gamma(j)}{(1+m)^i(1+n)^j(\varsigma-\varkappa)\Gamma(i+j)} \sum_{l=0}^{\infty} \frac{(a)_l(b)_l(i)_l}{l!\ (\beta)_l(i+j)_l}.$$

Summing up the series, we obtain

$$\Upsilon = \frac{\Gamma(i)\Gamma(j)}{(1+m)^i(1+n)^j(\varsigma-\varkappa)\Gamma(i+j)}\, {}_3F_2\left(\begin{matrix} a, & b, & i \\ \beta, & i+j & \end{matrix} \middle| 1 \right).$$

By making use of the generalized Whipple's theorem (defined in (2.3)) in the above expression, the desired result (2.8) can easily be obtained. □

Remark 2.1 If we set $\varkappa = 0, \varsigma = 1$ such that $t\in[0,1]$ in (2.5) and (2.6), then we are led to the results found in [16] and [10], respectively.

3. Applications of Integrals

More than fifty intriguing special cases in the form of integrals that are also general in nature will be discussed in this section.

3.1. Special Cases of (2.5)

Let $b=-2n$ and substitute a with $a+2n$ or let $b=-2n-1$ and substitute a with $a+2n+1$, where n is either zero or a positive integer. Each of the two words that occur on the right-hand sides of (2.5) will disappear in each instance, giving us fifty intriguing special cases that are shown below as two corollaries under the identical convergence conditions.

Corollary 3.1 *For $p,q=-2,-1,0,1,2$ and $\Re(\alpha)>0$ for $q=0,1,2$ and $\Re(\alpha)>-q$ for $q=-2,-1$ the following twenty-five results hold true,*

$$\begin{aligned}\int_{\varkappa}^{\varsigma} & \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+q-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+q}} \\ & \times \;{}_2F_1\left(\begin{matrix}-2n, \quad a+2n \\ \frac{1}{2}(a+p+1)\end{matrix}\,\middle|\,\frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right)dt \\ = & \frac{\mathcal{D}_{p,q}\,\Gamma(\alpha)\,\Gamma(\alpha+q)\,\left(\frac{1}{2}\right)_k\left(\frac{a}{2}-\alpha+\frac{3}{4}-\frac{(-1)^p}{4}-\left[\frac{q}{2}+\frac{(1-(-1)^p)}{4}\right]\right)_k}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)\,\left(\alpha+\frac{1}{2}+[\frac{q}{2}]\right)_k\left(\frac{a}{2}+\frac{1}{4}(1+(-1)^p)\right)_k},\end{aligned} \tag{3.1}$$

where the coefficients $\mathcal{D}_{p,q}$ are given in [11] and the Pochhammer symbol $(\chi)_k=\frac{\Gamma(\chi+k)}{\Gamma(\chi)}$.

Corollary 3.2 *For $p,q=-2,-1,0,1,2$ and $\Re(\alpha)>0$ for $q=0,1,2$ and $\Re(\alpha)>-q$ for $q=-2,-1$ the following twenty-five results hold true,*

$$\begin{aligned}\int_{\varkappa}^{\varsigma} & \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+q-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+q}} \\ & \times \;{}_2F_1\left(\begin{matrix}-2n-1, \quad a+2n+1 \\ \frac{1}{2}(a+p+1)\end{matrix}\,\middle|\,\frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right)dt \\ = & \frac{\mathcal{E}_{p,q}\,\Gamma(\alpha)\,\Gamma(\alpha+q)\,\left(\frac{3}{2}\right)_k\left(\frac{a}{2}-\alpha+\frac{5}{4}+\frac{(-1)^p}{4}-\left[\frac{q}{2}+\frac{(1-(-1)^p)}{4}\right]\right)_k}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)\,\left(\alpha+\frac{1}{2}+[\frac{q+1}{2}]\right)_k\left(\frac{a}{2}+\frac{1}{4}(3-(-1)^p)\right)_k},\end{aligned} \tag{3.2}$$

where the coefficients $\mathcal{E}_{p,q}$ are given in [11].

Remark 3.1 If we set $\varkappa=0,\varsigma=1$ such that $t\in[0,1]$ in (3.1) and (3.2), then we are led to the results found in [16].

Corollary 3.3 *Let $b=a-\lambda+1$ and substitute p with λ where $\lambda=0,\pm1,\pm2$. For $q=-2,-1,0,1,2$ and $\Re(\alpha)>0$ for $q=0,1,2$ and $\Re(\alpha)>-q$ for $q=-2,-1$ then the following limiting case of (2.5) holds true*

$$\begin{aligned}\int_{\varkappa}^{\varsigma} & \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+q-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+q}} \\ & \times \;{}_2F_1\left(\begin{matrix}a, \quad a-\lambda+1 \\ a+1\end{matrix}\,\middle|\,\frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right)dt \\ = & \frac{\Gamma(\alpha)\,\Gamma(\alpha+q)}{(1+m)^{\alpha}(1+n)^{\alpha+q}(\varsigma-\varkappa)\,\Gamma(2\alpha+q)}\;{}_3F_2\left(\begin{matrix}a, \quad a-\lambda+1, \quad \alpha \\ a+1, \quad 2\alpha+q\end{matrix}\,\middle|\,1\right).\end{aligned} \tag{3.3}$$

3.2. Special Cases of (2.6)

Let $b=-2n$ and substitute a with $a+2n$ or let $b=-2n-1$ and substitute a with $a+2n+1$, where n is either zero or a positive integer. Each of the two words that occur on the right-hand sides of (2.6) will disappear in each instance, giving us fifty intriguing special cases that are shown below as two corollaries under the identical convergence conditions.

Corollary 3.4 *For $y \in \mathbb{Z}$ and $p, q = 0, \pm 1, \pm 2$ the integral having twenty-five results hold true,*

$$\begin{aligned}&\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+1}} \\ &\times \quad {}_4F_3\left(\begin{matrix} -2n,\ a+2n,\ 2\alpha+y+1,\ c \\ \frac{1}{2}(a+p+1),\ \alpha,\ 2c+q \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right) dt \\ &= \frac{\mathcal{D}_{p,q}\ \Gamma(\alpha)\,\Gamma(\alpha+y+1)\ (\frac{1}{2})_k \left(\frac{a}{2}-c+\frac{3}{4}-\frac{(-1)^p}{4}-\left[\frac{q}{2}+\frac{(1-(-1)^p)}{4}\right]\right)_k}{(1+m)^{\alpha}(1+n)^{\alpha+y+1}(\varsigma-\varkappa)\,\Gamma(2\alpha+y+1)\ \left(c+\frac{1}{2}+[\frac{q}{2}]\right)_k\ \left(\frac{a}{2}+\frac{1}{4}(1+(-1)^p)\right)_k},\end{aligned} \tag{3.4}$$

where the coefficients $\mathcal{D}_{p,q}$ are given in [11].

Corollary 3.5 *For $y \in \mathbb{Z}$ and $p, q = 0, \pm 1, \pm 2$ the integral having twenty-five results hold true,*

$$\begin{aligned}&\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+1}} \\ &\times {}_4F_3\left(\begin{matrix} -2n-1,\ a+2n+1,\ 2\alpha+y+1,\ c \\ \frac{1}{2}(a+p+1),\ \alpha,\ 2c+q \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right) dt \\ &= \frac{\mathcal{E}_{p,q}\ \Gamma(\alpha)\,\Gamma(\alpha+y+1)\ (\frac{1}{2})_k \left(\frac{a}{2}-c+\frac{5}{4}-\frac{(-1)^p}{4}-\left[\frac{q}{2}+\frac{(1+(-1)^p)}{4}\right]\right)_k}{(1+m)^{\alpha}(1+n)^{\alpha+y+1}(\varsigma-\varkappa)\,\Gamma(2\alpha+y+1)\ \left(c+\frac{1}{2}+[\frac{q+1}{2}]\right)_k\ \left(\frac{a}{2}+\frac{1}{4}(3-(-1)^p)\right)_k},\end{aligned} \tag{3.5}$$

where the coefficients $\mathcal{E}_{p,q}$ are given in [11].

Specifically, if we let $p = q = 0$ in (3.4), we obtain the following result:

$$\begin{aligned}&\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+1}} \\ &\times \quad {}_4F_3\left(\begin{matrix} -2n,\ a+2n,\ 2\alpha+y+1,\ c \\ \frac{1}{2}(a+1),\ \alpha,\ 2c \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right) dt \\ &= \frac{\Gamma(\alpha)\Gamma(\alpha+y+1)}{(1+m)^{\alpha}(1+n)^{\alpha+y+1}(\varsigma-\varkappa)\Gamma(2\alpha+y+1)}\ \frac{\left(\frac{1}{2}\right)_k \left(\frac{a}{2}-c+\frac{1}{2}\right)_k}{\left(c+\frac{1}{2}\right)_k \left(\frac{a}{2}+\frac{1}{2}\right)_k}.\end{aligned} \tag{3.6}$$

Further, if we substitute $y = -1$, it reduces to

$$\begin{aligned}&\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha}} \\ &\times \quad {}_4F_3\left(\begin{matrix} -2n,\ a+2n,\ 2\alpha,\ c \\ \frac{1}{2}(a+1),\ \alpha,\ 2c \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right) dt \\ &= \frac{\Gamma(\alpha)\Gamma(\alpha)}{(1+m)^{\alpha}(1+n)^{\alpha}(\varsigma-\varkappa)\Gamma(2\alpha)}\ \frac{\left(\frac{1}{2}\right)_k \left(\frac{a}{2}-c+\frac{1}{2}\right)_k}{\left(c+\frac{1}{2}\right)_k \left(\frac{a}{2}+\frac{1}{2}\right)_k}.\end{aligned} \tag{3.7}$$

Similarly, in (3.5), if we set $p = q = 0$, we obtain the following proficient result,

$$\begin{aligned}&\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{\alpha-1}(\varsigma-t)^{\alpha+y}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{2\alpha+y+1}} \\ &\times {}_4F_3\left(\begin{matrix} -2n-1,\ a+2n+1,\ 2\alpha+y+1,\ c \\ \frac{1}{2}(a+1),\quad \alpha,\quad 2c \end{matrix} \middle| \frac{(1+m)(t-\varkappa)}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]}\right) dt \\ &= 0.\end{aligned} \tag{3.8}$$

We note that result (3.8) is remarkable.

Remark 3.2 If we set $\varkappa = 0, \varsigma = 1$ such that $t \in [0, 1]$ in (3.4), (3.5), (3.6) and (3.7), then we are led to the results found in [10].

3.3. Limiting Case of (2.8)

Let $\Re(i) > 0, \Re(j) > 0, \Re(a) > 0$ and $\Re(b) > 0$ such that $j = a + i - 1 - k$. Then the following limiting case of (2.8) holds

$$\begin{aligned} &\int_{\varkappa}^{\varsigma} \frac{(t-\varkappa)^{i-j-1}(\varsigma-t)^{j-1}}{[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]^{i}} \\ &\quad \times {}_1F_0\left(\begin{array}{c} a \\ - \end{array} \middle| \frac{(1+m)(t-\varkappa)}{2[(\varsigma-\varkappa)+m(t-\varkappa)+n(\varsigma-t)]} \right) dt \\ &= \frac{\Gamma(i-j)\Gamma(j)}{(1+m)^{i-j}(1+n)^{j}(\varsigma-\varkappa)\Gamma(i)}\, {}_2F_1\left(\begin{array}{c} a,\ 1+k-a \\ i \end{array} \middle| \frac{1}{2} \right). \end{aligned} \tag{3.9}$$

4. Conclusion

In the theory of hypergeometric and generalized hypergeometric functions, there exist a remarkably large number of hypergeometric summation formulas which can be expressed in terms of the Gamma functions. The importance of the generalized hypergeometric function lies in the fact that almost all elementary functions such as exponential, binomial, trigonometric, hyperbolic, logarithmic etc. are special case of this function. Thus the well known classical summation theorems such as those of generalized Watson's theorem, generalized Dixon's theorem and generalized Whipple's theorem for the series ${}_3F_2$. In our present investigation, we established a family of unified Eulerian integrals involving generalized hypergeometric function by employing generalized summation theorems for the series ${}_3F_2$ such as generalized Watson's theorem, generalized Dixon's theorem and generalized Whipple's theorem. Various special cases are deduced as consequences of our main theorems. The results derived in this papers can be applied to some other Euler type integrals and other special functions [2,4,5,6,9,17,18,20,21].

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