Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 1** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.77984

Analytic study of the pine wilt disease epidemic model

A. Said, Y. Massoun, C. Cesarano, P. Agrawal

ABSTRACT: This paper presents an analytical study of the pine wilt disease epidemic model, a significant threat to forest ecosystems and agricultural economies. The model is formulated using Caputo fractional differential equations to account for memory-dependent dynamics and anomalous diffusion processes. We first establish the fundamental properties of the system, including existence, uniqueness, and stability criteria. By employing a semi-analytical approach combining the Laplace transform with iterative techniques, we derive approximate solutions and analyze their convergence. Numerical simulations for different fractional orders α demonstrate how fractional derivatives influence disease spread, providing valuable insights for more effective containment strategies.

Key Words: Pine wilt disease model, Laplace transform method, fractional epidemic modeling, homotopy analysis method, nonlinear biological systems.

Contents

1. Introduction

Pine wilt disease is a devastating phytosanitary threat affecting pine forests across East Asia, North America, and Europe. The disease is caused by the invasive pine wood nematode (Bursaphelenchus xylophilus), which colonizes and obstructs the vascular systems of host trees, leading to rapid mortality within 30 days of infection [1,2,3]. Dead trees subsequently become breeding grounds for nematodes and attract insect vectors (e.g., Monochamus beetles), which facilitate further dispersal. The ecological and economic severity of this disease has driven extensive modeling efforts to understand its dynamics and control its spread.

The evolution of pine wilt disease modeling reflects advancements in both mathematical biology and computational tools. Early studies (1970s–1980s) focused on empirical observations, while later epidemiological models (1980s–1990s) adapted compartmental frameworks like SIR models to tree populations [5,6]. By the 2000s, models incorporated environmental factors, spatial heterogeneity via GIS, and management strategies such as targeted tree removal. Recent approaches integrate climate change projections [7] and data-driven techniques, including machine learning and fractional calculus, to capture memory effects and anomalous diffusion in nematode dispersal [8]. Fractional operators, such as the Caputo derivative, have proven particularly effective in modeling complex biological processes with non-local dynamics, as demonstrated in analogous systems [4,9]. Recent advances in fractional modeling and numerical methods have shown significant potential in biomedical and diffusion processes like Hassani and al. [10,11,12] proposed various optimization techniques and polynomial-based approaches for solving nonlinear fractional optimal control problems, including applications in cancer treatment, tumor immune interaction, and HIV infection modeling. Avazzadeh et al. [13] applied the generalized Laguerre polynomials to approximate fractional order brain tumor models efficiently. Moreover, Irandoust-Pakchin et al.

2010 *Mathematics Subject Classification:* 34B15, 34A08, 35R11.
Submitted July 22, 2025. Published September 30, 2025

[14] developed an accurate and stable numerical method for distributed order for time fractional diffusion equations, providing valuable tools for modeling anomalous diffusion in biological systems.

This paper extends these efforts by formulating a fractional-order model of pine wilt disease transmission, analyzing its dynamics through semi-analytical methods, and evaluating the impact of fractional parameters on outbreak trajectories. Our work builds on recent advances in fractional calculus applications to ecological and epidemiological systems [15], aiming to provide actionable insights for forest management.

At present, many epidemic models have been represented using fractional differential equations, exemplified by the work of Muhammad Altaf and Yongjin Li. [16],Khan [17], and Khan, M [18].
The epidemic model is given as [16]

$$\begin{array}{rcl} S_h'(t) &=& \Pi_H - \varphi\alpha S_h(t)I_v(t) - K_1 S_h(t), \\ E_h'(t) &=& \varphi\alpha S_h(t)I_v(t) - (\beta + K_1)E_h(t), \\ I_h'(t) &=& \beta E_h(t) - K_1 I_h(t) \\ S_v'(t) &=& b_v - \gamma_1 S_v(t)I_h(t) - K_2 S_v(t), \\ I_v'(t) &=& \gamma_V I_h(t)S_v(t) - K_2 I_v(t), \end{array} \tag{1.1}$$

Here, we've utilized the homotopy analysis transform method (HATM) to obtain approximate solutions for the epidemic model under consideration.

In this section, we mathematically represent this epidemic model by formulating it as a system of fractional differential equations, as demonstrated below. [17] [19] :

$$\begin{array}{rcl} {}^C D^\alpha S_h(t) &=& \Pi_H - \varphi\alpha S_h(t)I_v(t) - K_1 S_h(t) \\ {}^C D^\alpha E_h(t) &=& \varphi\alpha S_h(t)I_v(t) - (\beta + K_1)E_h(t) \\ {}^C D^\alpha I_h(t) &=& \beta E_h(t) - K_1 I_h(t) \\ {}^C D^\alpha S_v(t) &=& b_v - \gamma_1 S_v(t)I_h(t) - K_2 S_v(t) \\ {}^C D^\alpha I_v(t) &=& \gamma_V I_h(t)S_v(t) - K_2 I_v(t) \end{array} \tag{1.2}$$

with initial condition

$$S_h(0) = \eta_1 \quad E_h(0) = \eta_2 \quad I_h(0) = \eta_3 \quad S_v(0) = \eta_4 \quad I_v(0) = \eta_5$$

Where $S_H(t)$ represents the total number of trees, while $E_H(t)$, $I_H(t)$, and $S_V(t)$ denote the susceptible, infected, and the total number of beetles in three different classes, respectively. Additionally, Π_H signifies the continuous growth rate for the tree population, and Π_V corresponds to the vector population. Finally, K_1 stands for the atrophy or natural death rate of pine trees, K_2 The natural death rate of beetles is considered as vectors. γ_H the natural death rates of pine trees, γ_V represents the death rate of the population, ψ signifies the natural death rate of uninfected pine trees. θ_1 and θ_2 denote the saturation constants, while δ_H indicates the rate at which exposed pine trees transition to the infected class. Similarly, δ_V represents the rate at which an exposed vector becomes an infected vector, and β_1 stands for the contact rate between susceptible vectors and infected pine trees.

2. Essential principles of fractional calculus

In this section, we provide fundamental definitions of the fractional calculus theory utilized in this paper:

Definition 2.1 *A real function $h(t)$ is said C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0,\infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$*

Definition 2.2 *The Riemann-Liouville fractional operator of order $\alpha \geq 0$, of a function $h \in C_\mu, \mu \geq -1$ is defined as [20]*

$$\begin{array}{rcl} I^\alpha h(t) &=& = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s)ds, \mu > 0, t > 0 \\ I^0 h(t) &=& h(t) \end{array}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.3 *The Caputo fractional derivative of h, $h \in C_{-1}^{m}$ is defined as*

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-\varsigma)^{(m-\alpha-1)}h^{(m)}(\varsigma)d\varsigma,$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}$

Definition 2.4 *The Laplace transform for the function $h(t)$ of the Caputo fractional derivative is [20]:*

$$L[^{C}D^{\alpha}h(t)] = s^{\alpha} - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}h^{(k)}(0), \quad n-1 < \alpha \leq n \tag{2.1}$$

3. Basic idea of the homotopy analysis transform method

We analyze the given fractional differential equation to exemplify the utilization of the HATM in solving fractional differential equations:

$$^{C}D^{\alpha}h(t) + \mathfrak{R}h(t) + \mathfrak{N}h(t) = f(t), \quad 0 < \alpha < 1 \tag{3.1}$$

Where $^{C}D^{\alpha}h(t)$ denotes the Caputo fractional derivative of $h(t)$ with order α, $\mathfrak{R}$ and $\mathfrak{N}$ represent the linear and nonlinear operators, respectively, and $f(t)$ is the source term. To start, upon applying the Laplace transform to equation (2.1), we acquire the following:

$$s^{\alpha}L[h(t)] - s^{\alpha-1}h(0) + L[\mathfrak{R}h(t)] + L[\mathfrak{N}h(t)] = L[f(t)], \quad 0 < \alpha < 1 \tag{3.2}$$

After simplifying, the result is:

$$L[h(t)] = \frac{1}{s}h(0) + \frac{1}{s^{\alpha}}(L[f(t)] - L[\mathfrak{R}h(t)] - L[\mathfrak{N}h(t)]), \quad 0 < \alpha < 1 \tag{3.3}$$

Utilizing the homotopy analysis method as demonstrated in [21], [24], [25], and [26], we define the nonlinear operator as follows:

$$N[\phi(t,q)] = L[\phi(t,q)] - \frac{1}{s}h(0) - \frac{1}{s^{\alpha}}(L[f(t)] + L[\mathfrak{R}h(t)] + L[\mathfrak{N}h(t)]) \tag{3.4}$$

where $\phi(t,q)$ is real function of t and $q \in [0,1]$, the zeroth order deformation [24] [25] is

$$(1-q)\mathcal{L}[\phi(t,q) - h_0(t)] = \hbar q H(t) N[\phi(t,q)] \tag{3.5}$$

Here, $\mathcal{L}$ represents a linear operator, with our choice being the Laplace operator denoted as L. $\hbar$ is a non-zero auxiliary parameter, and the auxiliary function $H(t)$ is non-zero as well. We define $h_0(t)$ as the initial guess for $h(t)$, and $\phi(t,q)$ is an unknown function. Notably, $\phi(t,0) = u_0(t)$ and $\phi(t,1) = h(t)$. We proceed by expanding $\phi(t,q)$ in a Taylor series with respect to q,

$$\phi(t,q) = \sum_{i=0}^{n} h_i(t)q^i$$

where

$$h_i(t) = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}|_{q=0}$$

Through differentiation(3.5) m times with respect q, and by performing $q=0$, we get the m-th order deformation equation

$$L[h_m(t) - \chi_m h_{m-1}(t)] = \hbar H(t) R_m(h_{m-1}(t)) \tag{3.6}$$

Applying L^{-1} in (3.6) we get:

$$h_m(t) = \chi_m h_{m-1}(t) + \hbar L^{-1}[R_m(h_{m-1}(t))] \tag{3.7}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

For our epidemic model (1.2), we establish a set of nonlinear operators as follows:

$$\begin{aligned}
N_1[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_1}{s} - \tfrac{1}{s^\alpha}(L[\Pi_H] - L[(\varphi\alpha\phi_1\phi_5 + K_1\phi_1) \\
N_2[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_2] - \tfrac{\eta_2}{s} - \tfrac{1}{s^\alpha}L[\varphi\alpha\phi_1\phi_5 - (\beta + K_1)\phi_2] \\
N_3[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_3] - \tfrac{\eta_3}{s} - \tfrac{1}{s^\alpha}L[\beta\phi_2 - K_1\phi_3] \\
N_4[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_4] - \tfrac{\eta_4}{s} - \tfrac{1}{s^\alpha}L[b_v - \gamma_1\phi_4\phi_3 - K_2\phi_4] \\
N_5[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_5] - \tfrac{\eta_5}{s} - \tfrac{1}{s^\alpha}L[(\gamma_V\phi_3\phi_4) - K_2\phi_5]
\end{aligned}$$

The corresponding mth order deformation equation is:

$$\begin{aligned}
S_{h,m}(t) &= \chi_m S_{h,m-1}(t) + hL^{-1}R_{1,m}[\overrightarrow{S}_{h,m}, \overrightarrow{E}_{h,m}, \overrightarrow{I}_{h,m}, \overrightarrow{S}_{v,m}, \overrightarrow{I}_{v,m}] \\
E_{h,m}(t) &= \chi_m E_{h,m}(t) + hL^{-1}R_{2,m}[\overrightarrow{S}_{h,m}, \overrightarrow{E}_{h,m}, \overrightarrow{I}_{h,m}, \overrightarrow{S}_{v,m}, \overrightarrow{I}_{v,m}] \\
I_{h,m}(t) &= \chi_m I_{h,m}(t) + hL^{-1}R_{3,m}[\overrightarrow{S}_{h,m}, \overrightarrow{E}_{h,m}, \overrightarrow{I}_{h,m}, \overrightarrow{S}_{v,m}, \overrightarrow{I}_{v,m}] \\
S_{v,m}(t) &= \chi_m S_{v,m}(t) + hL^{-1}R_{4,m}[\overrightarrow{S}_{h,m}, \overrightarrow{E}_{h,m}, \overrightarrow{I}_{h,m}, \overrightarrow{S}_{v,m}, \overrightarrow{I}_{v,m}] \\
I_{v,m}(t) &= \chi_m I_{v,m}(t) + hL^{-1}R_{6,m}[\overrightarrow{S}_{h,m}, \overrightarrow{E}_{h,m}, \overrightarrow{I}_{h,m}, \overrightarrow{S}_{v,m}, \overrightarrow{I}_{v,m}]
\end{aligned} \quad (3.8)$$

where L^{-1} represents the inverse Laplace transform operator, and it should be noted that each of $S_h(t)$, $E_h(t)$, $I_h(t)$, $S_v(t)$, and $I_v(t)$ can be expressed as a convergent series:

$$\begin{aligned}
&S_h(t) = \textstyle\sum_{i=0}^{\infty} S_{h,i}(t) \quad E_h(t) = \sum_{i=0}^{\infty} E_{h,i}(t) \quad I_h(t) = \sum_{i=0}^{\infty} I_{h,i}(t) \\
&S_v(t) = \textstyle\sum_{i=0}^{\infty} S_{v,i}(t) \quad I_v(t) = \sum_{i=0}^{\infty} I_{v,i}(t)
\end{aligned}$$

4. Analysis of Convergence

Theorem 4.1 *[28], [29] If the series solutions* $\sum_{i=0}^{\infty} S_{h,i}(t)$, $\sum_{i=0}^{\infty} E_{h,i}(t)$, $\sum_{i=0}^{\infty} I_{h,i}(t)$, $\sum_{i=0}^{\infty} S_{v,i}(t)$, *and* $\sum_{i=0}^{\infty} E_{v,i}(t)$, *determined by equation (3.8) converge, then they are indeed solutions of equation (1.2).*

Proof: Assuming that both $\sum_{i=0}^{\infty} S_{h,i}(t), \sum_{i=0}^{\infty} E_{h,i}(t), \sum_{i=0}^{\infty} I_{h,i}(t)$, $\sum_{i=0}^{\infty} S_{v,i}(t), \sum_{i=0}^{\infty} E_{v,i}(t)$ are convergent, i.e

$$\lim_{n\longrightarrow\infty} S_{H,n}(t) = \lim_{n\longrightarrow\infty} E_{H,n}(t) = \lim_{n\longrightarrow\infty} I_{H,n}(t) = \lim_{n\longrightarrow\infty} S_{V,n}(t) = \lim_{n\longrightarrow\infty} E_{V,n}(t) = 0$$

Based on equation (**??**), we obtain:

$$\begin{aligned}
\hbar H_1 \textstyle\sum_{m=1}^{\infty} R_{1,m} &= \lim_{k\longrightarrow\infty} \textstyle\sum_{m=0}^{k} L[S_{h,m} - \chi_m S_{h,m-1}] \\
&= L[\lim_{k\longrightarrow\infty} \textstyle\sum_{m=0}^{k}[S_{h,m} - \chi_m S_{h,m-1}]] \quad = \quad L[\lim_{k\longrightarrow\infty} S_{h,k}]
\end{aligned}$$

Given that L is a Laplace operator, and considering that $\lim_{k\to\infty} S_{h,k} = 0$, $H_1 \neq 0$, and $\hbar \neq 0$, we can conclude that $\sum_{m=1}^{\infty} R_{1,m} = 0$.

Now, let's expand $N_1[\Phi_i(t,q)], i = 1, ..., 5$, about $q = 0$, and then set $q = 1$.

$$N[\Phi_i(t,1)] = 0,$$

Indeed, it is evident that $S_h(t) = \Phi_1(t,1) = \sum_{i=0}^{\infty} S_{h,i}(t)$ is a solution to the equation (1.2). □

Theorem 4.2 *[30] The series solution* $\sum_{m=0}^{\infty} S_{h,m}(t)$ *converges if there exists a constant* $0 < \gamma < 1$ *such that for all* $m > m_0$*, where* m_0 *is a natural number, the following condition holds:*

$$\|S_{h,m+1}(t)\| \leq \gamma \|S_{h,m}(t)\|$$

where $|S_{h,m}(t)|$ *represents the norm or magnitude of the solution component* $S_{h,m}(t)$*.*

Proof: Define the sequence $(T_n)_{n\in\mathbb{N}}$ as: $T_n = S_{h,0} + S_{h,1} + ... + S_{h,n}$. We prove that T_n is a Cauchy sequence in $\mathbb{R}$.
For every $p, q \in \mathbb{N}, p > q \geq m_0$, we have

$$\begin{aligned} \parallel T_p - T_q \parallel &= \parallel (S_{h,p} - S_{h,p-1}) + (S_{h,p-1} - S_{h,p-2}) + ... + (S_{h,q+1} - S_{h,q}) \parallel \\ &\leq \parallel (S_{h,p} - S_{h,p-1}) \parallel + \parallel (S_{h,p-1} - S_{h,p-2}) \parallel + ... + \parallel (S_{h,q+1} - S_{h,q}) \parallel \\ &\leq \gamma^{p-m_0} \parallel S_{h,m_0} \parallel + \gamma^{p-m_0-1} \parallel S_{h,m_0} \parallel + ... + \gamma^{q-m_0+1} \parallel S_{h,m_0} \parallel \\ &\leq (\gamma^{p-m_0} + \gamma^{p-m_0-1} + ... + \gamma^{q-m_0+1}) \parallel S_{h,m_0} \parallel \\ &\leq \gamma^{q-m_0+1} \tfrac{\gamma^{p-q}}{1-\gamma} \parallel S_{h,m_0} \parallel \end{aligned} \tag{4.1}$$

Given that $0 < \gamma < 1$, we conclude

$$\lim_{p,q\to\infty} \parallel T_p - T_q \parallel = 0$$

Thus, (T_n) forms a Cauchy sequence in $\mathbb{R}$, signifying the convergence of (T_n). This, in turn, indicates the convergence of the series solution $\sum S_{h,i}(t)$ □

Theorem 4.3 *[30] Suppose that the series solution $\sum_{m=0}^{\infty} S_{h,m}(t)$ is convergent to the solution $S_h(t)$. If the truncated series $\sum_{m=0}^{k} S_{h,m}(t)$ is used as approximation to the solution $S_h(t)$, then the truncated error satisfies*

$$\parallel S_h(t) - \sum_{m=0}^{k} S_{h,m}(t) \parallel \leq \frac{1}{1-\gamma}\gamma^{k+1} \parallel S_{h,0}(t) \parallel .$$

Proof: Considering n and m as natural numbers, we apply the following inequality:

$$\parallel T_n - T_m \parallel \leq \gamma^{m+1} \frac{\gamma^{n-m}}{1-\gamma} \parallel S_{h,0} \parallel$$

□

5. Numerical result and discussion

In this section, we prove that HAM is a powerful semi-analytical method for solving nonlinear differential equations and has been applied to solve the above system of fractional differential equations. with the initial conditions,

$$\begin{array}{ccc} S_{h,0}(0) = \eta_1 = 300 & E_{h,0}(0) = \eta_2 = 40 & I_{h,0}(0) = \eta_3 = 20 \\ S_{v,0}(0) = \eta_4 = 70 & E_{v,0}(0) = \eta_5 = 20 & \beta = 0.0571 \end{array}$$

we solve (1.2) using (3.8), we obtain :

$$\begin{aligned} S_{h,1}(t) &= \tfrac{-1}{\Gamma[1+\tau]} ht^{\alpha}(-6000\alpha\varphi - 300K_1 + \Pi_h) \\ E_{h,1}(t) &= \tfrac{-1}{\Gamma[1+\tau]} ht^{\alpha}(6000\alpha\varphi - 40(\beta + K_1)) \\ I_{h,1}(t) &= \tfrac{-1}{\Gamma[1+\tau]} ht^{\alpha}(40\beta - 20K_1)) \\ S_{v,1}(t) &= \tfrac{-1}{\Gamma[1+\tau]} ht^{\alpha}(-1400\gamma_1 + b_v - 70K_2) \\ I_{v,1}(t) &= \tfrac{ht^{\tau} K_2}{\Gamma[1+\alpha]} - \tfrac{1400ht^{\alpha}\gamma_2}{\Gamma[1+\alpha]} \end{aligned}$$

and for

$$\beta = 0.0571, \gamma = 0.0405, K_1 = 0.003, \lambda = 0.01166, \varphi = 0.06, K_2 = 0.011, \Pi_h = 10, b_v = 4$$

and $h = -1$ we obtain:

$$\begin{aligned} S_{h,1}(t) &= \tfrac{1}{\Gamma[1+\alpha]} 4.9024ht^{\alpha} \\ E_{h,1}(t) &= -\tfrac{1}{\Gamma[1+\alpha]} 1.7936ht^{\alpha} \\ I_{h,1}(t) &= -\tfrac{1}{\Gamma[1+\alpha]} 2.224ht^{\alpha} \\ S_{v,1}(t) &= \tfrac{1}{\Gamma[1+\alpha]} 53.47ht^{\alpha} \\ I_{v,1}(t) &= -\tfrac{1}{\Gamma[1+\alpha]} 56.48ht^{\alpha} \end{aligned}$$

and :

$$\begin{aligned}
S_{h,2}(t) &= -\frac{1}{\Gamma[1+\alpha]}4.9024ht^{\alpha} + h(-\frac{1}{\Gamma[1+\alpha]}4.9024ht^{\alpha} - t^{\alpha}(\frac{10}{\Gamma[1+\alpha]} + \frac{1}{\Gamma[1+2\alpha]}11.9373ht^{\alpha})) \\
E_{h,2}(t) &= -\frac{1}{\Gamma[1+\alpha]}1.7936ht^{\alpha} + h(\frac{-1}{\Gamma[1+\alpha]}1.7936ht^{\alpha} + \frac{1}{\Gamma[1+2\alpha]}11.8148ht^{2\alpha}) \\
I_{h,2}(t) &= -\frac{1}{\Gamma[1+\alpha]}2.224ht^{\alpha} + h(\frac{-1}{\Gamma[1+\alpha]}2.224ht^{\alpha} - \frac{1}{\Gamma[1+2\alpha]}0.0670254ht^{2\alpha}) \\
S_{v,2}(t) &= \frac{1}{\Gamma[1+\alpha]}53.47ht^{\alpha} + h(\frac{1}{\Gamma[1+\alpha]}53.47ht^{\alpha} + \frac{1}{\Gamma[1+2\alpha]}37.5938ht^{2\alpha}) \\
I_{v,2}(t) &= -\frac{1}{\Gamma[1+\alpha]}56.48ht^{\alpha} - \frac{1}{\Gamma[1+\alpha]}56.48h^2t^{\alpha} - \frac{1}{\Gamma[1+2\alpha]}37.6269h^2t^{2\alpha}
\end{aligned}$$

The HAM solution of S_h, E_h, I_h, S_v and I_v for $\alpha = 1$ is presented in fig1, and for different values of α in the figure2 it's clear that the HAM solution is dependent on the fractional order α of the derivative.

Concluding remarks

By investigating the epidemic model and its transformation into a system of fractional differential equations, we have effectively employed the homotopy analysis transform method to solve the model. This has been demonstrated through the presentation of the first terms of the solution series, accompanied by graphical representations.

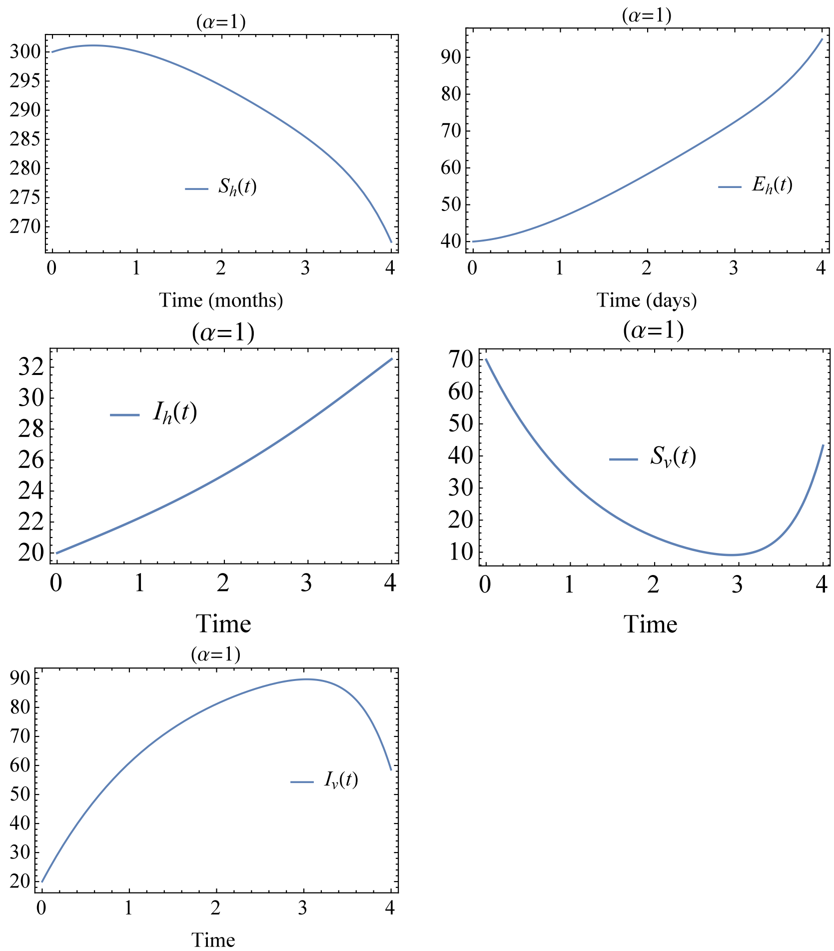

Figure 1: The HATM solution of S_h, E_h, I_h, S_v and I_v

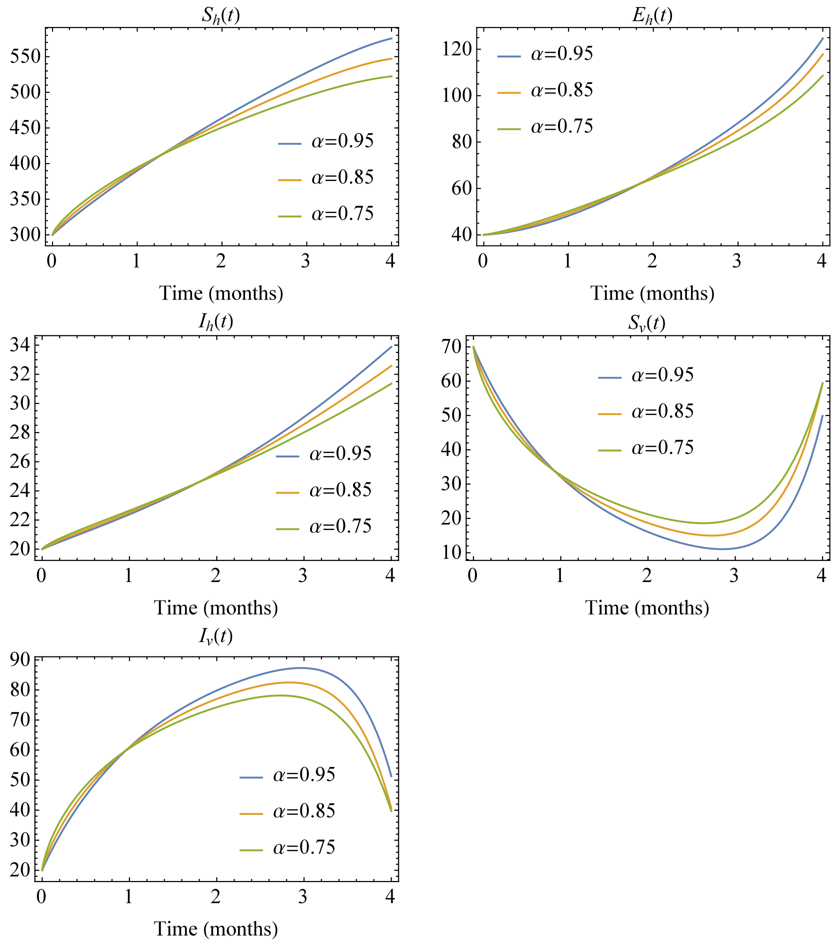

Figure 2: The HATM solution of S_h, E_h, I_h, S_v and I_v for different value of α

Acknowledgments

We think the referee by your suggestions.

References

1. Kottakkaran Sooppy Nisar, Muhammad Farman, Mahmoud Abdel-Aty, Jinde Cao; *A review on epidemic models in sight of fractional calculus*, Alexandria Engineering Journal, 10.1016/j.aej.2023.05.071, 75, (81-113), (2023).

2. Zubair Ahmad, Giuliano Bonanomi, Daniela di Serafino, Francesco Giannino; *Transmission dynamics and sensitivity analysis of pine wilt disease with asymptomatic carriers via fractal-fractional differential operator of Mittag-Leffler kernel*, Applied Numerical Mathematics, 10.1016/j.apnum.2022.12.004, 185, (446-465), (2023).

3. Y. Massoun; *Analytic study of pine wilt disease model with Caputo-Fabrizio fractional derivative*, Math. Methods Appl. Sci., 45 (11) (2022), pp. 7072-7080.

4. M Aychluh, DL Suthar, C Cesarano, SD Purohit; *Modeling and analysis of the dynamics of an excessive gambling problem with modified fractional operator*, An International Journal of Optimization and Control: Theories and Applications (2024).

5. Y Massoun, AK Alomari, C Cesarano; *Analytic solution for SIR epidemic model with multi-parameter fractional derivative*, Mathematics and Computers in Simulation 230, 484-492 (2023).

6. Y Massoun, AK Alomari, C Cesarano; *Analytic Approach Solution to Time-Fractional Phi-4 Equation with Two-Parameter Fractional Derivative*, Fractal and Fractional 8(10), 576 (2024).

7. TS Hassan, C Cesarano, MB Mesmouli, HN Zaidi, I Odinaev; *Iterative Hille-type oscillation criteria of half-linear advanced dynamic equations of second order*, Mathematical Methods in the Applied Sciences 47(7), 5651-5663 (2024).

8. FM Al-Askar, C Cesarano, WW Mohammed; *Effects of the Wiener process and beta derivative on the exact solutions of the Kadomtsev-Petviashvili equation*, Axioms 12(8), 748 (2023).

9. WW Mohammed, C Cesarano, EM Elsayed, FM Al-Askar; *The analytical fractional solutions for coupled Fokas system in fiber optics using different methods*, Fractal and Fractional 7(7), 556 (2023).

10. H. Hassani, J.A. Tenreiro Machado, S. Mehrabi; *An optimization technique for solving a class of nonlinear fractional optimal control problems: Application in cancer treatment*, Applied Mathematical Modelling, **93**, 868–884 (2021). doi:10.1016/j.apm.2021.01.004.

11. H. Hassani, Z. Avazzadeh, P. Agarwal, et al.; *A study on fractional tumor-immune interaction model related to lung cancer via generalized Laguerre polynomials*, BMC Medical Research Methodology, **23**, 189 (2023). doi:10.1186/s12874-023-02006-3.

12. H. Hassani, S. Mehrabi, E. Naraghirad, et al.; *An Optimization Method Based on the Generalized Polynomials for a Model of HIV Infection of $CD4^+$ T Cells*, Iranian Journal of Science and Technology, Transactions of Science, **44**, 407–416 (2020). doi:10.1007/s40995-020-00833-3.

13. Z. Avazzadeh, H. Hassani, M.J. Ebadi, et al.; *Optimal Approximation of Fractional Order Brain Tumor Model Using Generalized Laguerre Polynomials*, Iranian Journal of Science, **47**, 501–513 (2023). doi:10.1007/s40995-022-01388-1.

14. S. Irandoust-Pakchin, M.H. Derakhshan, S. Rezapour, M. Adel; *An efficient numerical method for the distributed-order time-fractional diffusion equation with the error analysis and stability properties*, Mathematical Methods in the Applied Sciences, **48**(3), 2743–2765 (2024). doi:10.1002/mma.10459.

15. FM Al-Askar, C Cesarano, WW Mohammed; *[Additional work on fractional dynamics]*

16. Y Li, F Haq, K Shah, M Shahzad, G ur Rahman; *Numerical analysis of fractional order Pine wilt disease model with bilinear incident rate*, J. Math. Computer Sci. 17, 420-428 (2017).

17. MA Khan, S Ullah, KO Okosun, K Shah; *A fractional order pine wilt disease model with Caputo-Fabrizio derivative*, Adv. Differ. Equ. (2018) doi.org/10.1186/s13662-018-1868-4.

18. Y Luchko, R Gorenflo; *An operational method for solving fractional differential equations with the Caputo derivatives*, Acta Math. Vietnamica 24, 207-233 (1999).

19. MA Khan, K Shah; *Mathematical modeling approach to the transmission dynamics of pine wilt disease with saturated incidence rate*, Int. J. Biomathematics 10(3), 1850035 (2018).

20. I. Podlubny ; *Fractional Differential Equations,* Academic Press, New York, 1999.

21. P Veeresha, DG Prakasha, HM Baskonus; *Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method*, doi.org/10.1007/s40096-019-0284-6 (2019).

22. D Baleanu, A Jajarmi, H Mohammadi, S Rezapour; *A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative*, Chaos Solitons Fractals 134, 109705 (2020).

23. S Kumar; *A new analytical modelling for fractional telegraph equation via Laplace transform*, Appl. Math. Model. 38, 3154-3163 (2014).

24. SJ Liao; *The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems*, PhD thesis, Shanghai Jiao Tong University (1992).

25. S Liao; *Notes on the homotopy analysis method: Some definitions and theorems*, Commun. Nonlinear Sci. Numer. Simulat. 14, 983-997 (2009).

26. S Liao; *Homotopy Analysis Method in Non Linear Differential Equations*, Springer Heidelberg Dordrecht London New York (2012).

27. J Losada, JJ Nieto; *Properties of a New Fractional Derivative without Singular Kernel*, Progr. Fract. Differ. Appl. 1(2), 87-92 (2015).

28. AK Alomari, GA Drabseh, MF Al-Jamal, RB Albadarneh; *Numerical simulation for fractional phi-4 equation using homotopy Sumudu approach*, Int. J. Simul. Process Model. (IJSPM).

29. Y Massoun, R Benzine, AK Alomari; *Comparative Numerical Study of Fornberg-Whitham Equation*, Int. J. Appl. Comput. Math 9, 10 (2023).

30. Y Massoun; *Analytic study of pine wilt disease model with Caputo-Fabrizio fractional derivative*, Math. Methods Appl. Sci. doi.org/10.1002/mma.8225.

Abderrzak Said

Department of Mathematics, Faculty of science,

university of Khemise-miliana,Algeria,

Laboratory of Nonlinear Partial Differential Equations, Department of Mathematics, E.N.S.,Kouba, Algiers, Algeria

E-mail address: `Abderrezaksaid@yahoo.fr`

and

Y. Massoun,
Department of Mathematics,
University of Algiers,
Laboratory of Mathematical Analysis and Applications, University of Algiers 1, Algiers, Algeria, Algeria.
E-mail address: youcef.ens@hotmail.fr

and

C. Cesarano,
Section of Mathematics,
International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma Italy.
E-mail address: clemente.cesarano@uninettuno.it

and

P. Agrawal,
Department of Mathematical Sciences,
Saveetha School of Engineering, Chennai, Tamil Nadu, 602105, India
Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
Department of Mathematics,
Anand International College of Engineering, Jaipur 303012, India
International Center for Basic and Applied Sciences, Jaipur-302029, India.
E-mail address: goyal.praveen2011@gmail.com