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Using Markov Chain Marginal Bootstrap in Optimizing Marketing Strategies

Mathil Kamil Thamer*, Abdulrahman Obaid Jumaah, Manaf Ahmed, Marwan Hammoodi, Shuaaib Abdulmutalib Ibrahim and Ahmed Talib Hameed

ABSTRACT: In marketing law, models such as Markov chain models are gaining popularity with every passing day, thanks to effective marketing strategies in acquiring and retaining customers. However, the accuracy of the models is often put into question due to the insufficiencies regarding the estimation of model parameters, which tend to be complicated. In this article, however, we aim at solving this problem by introducing the Markov Chain Marginal Bootstrap method, which looks up to an efficient estimation framework by greatly modifying the biasness state. Using this methodology, we explored the uncertainties surrounding parameter estimation. This thesis demonstrates how MCMB improves estimation capability by overcoming data uncertainty and heterogeneity among other factors. With improved parameter estimation, MCMB enhances our understanding of customer behaviour and provides us with the ability to design better marketing strategies. Additionally, we present the application of the MCMB procedure to the analysis of the data from the subscription business model and discuss its possibilities in the analysis and forecasting of customer behaviour. Other issues emphasized in the article include optimizing the marketing campaign strategy by implementing better approaches while considering the parameter uncertainty limitations discussed earlier in the article.

Key Words: Markov chain models, marginal bootstrap, marketing strategies, parameter uncertainty, optimization, negative Likelihood function, confidence interval..

Contents

1. Introduction

The Markov chain model has become an essential instrument in the marketing analytics field through predicting customer behaviours with a random and variable nature. This is the model which explains how customers move from one state to the next over time, thus enabling marketers to devise means for customer retention, acquisition and engagement. However, the reliability of this model is dependent on the precision of the parameters and estimation of the parameters and estimation of the parameters in case of

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Uncertainty. Sources of uncertainty may stem from various causes such as inadequate information, measurement errors and even the assumptions of the model. It is not surprising in this case that uncertainty may even impede the outcome of the marketing strategies adopted from Markov chain models, leading to poor outcomes in decision making and management of resources. In order to classify this problem properly, we suggest the use of the Markov chain marginal bootstrap method, a resampling strategy that addresses parameter uncertainty and improves estimation accuracy. In essence, Markov chain marginal bootstrap involves estimating a parameter in different bootstrap samples derived from the original data parent, which gives a statistical parameter distribution rather than a single point, thus enabling better decision making. In this paper, we aim to provide evidence that the Markov chain marginal bootstraping approach is a viable method for increasing the accuracy of marketing strategy optimization. We perform a simulation case study on a customer retention problem within a subscription model. The results are compared with the traditional Markov chain model, and since we employ an augmented model, we are able to evaluate the role of the uncertainty of parameters in determining the effectiveness of the strategy.

Our results reveal that there has been an increase in the accuracy of parameter estimation parameter accuracy through the use of the Markov chain marginal bootstraping method, almost an order of magnitude enhanced estimates. With these estimates, marketers are able to understand the customers' behaviour better and thus are able to make better decisions on segmentation, churn prediction, recommendation systems, pricing, etc.
In the context of marketing analytics, this research addresses an important gap by discussing the issue of parameter uncertainty in the Markov chain models used to analyze marketing data. However, if marketers would properly adopt and incorporate the Markov chain marginal bootstrap approach into their marketing tools, they could significantly improve the quality of the marketing strategies and as a result, increase the overall marketing performance and customer satisfaction.

2. Literature Review

Bickel, P. J., and De A. Freedman. [11] discusses the use of the bootstrap method for regression models with large numbers of parameters and data points. The paper provides theoretical results on the conditions under which the bootstrap approximation to the distribution of contrasts is valid. Additionally, it discusses the relevant growth conditions and regularity conditions necessary on the tails of the error distribution.

Horowitz, J. L. [2] describes a new bootstrap procedure for data generated by a Markov process or a process that can be approximated by a Markov process with sufficient accuracy. Bootstrap samples are obtained by sampling the process implied by the estimated transition density. The paper presents regularity conditions and formal results for data that are generated by a Markov process. According to the author, this procedure is more accurate than the block bootstrap, which is the leading non-parametric method for implementing the bootstrap with time-series data.

He, X.,

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Hu, F. [3] presents the Markov Chain Marginal Bootstrap (MCMB) method and applies it to the linear regression problem. The method relies on approximating joint bootstrap distributions by a Markov chain and obtaining marginal distribution estimates. The document includes examples and simulation results and also compares the MCMB method with other bootstrap methods like paired, normality-based and a rank-inversion method. I use MCMB in a customer retention problem to improve accuracy, resulting in better decision-making about customer segmentation, churn prediction, recommendation systems, and pricing optimization. It also emphasizes the importance of considering parameter uncertainty in optimizing marketing strategies based on Markov chain models.

[6] discusses using parametric bootstrap sampling for Bayesian inference calculations in situations that are amenable to MCMC analysis. The paper describes the connection between the bootstrap and Bayesian calculations, mainly in terms of a simple component of the variance model. The paper provides

a straightforward example of the process and a comparison with Gibbs sampling. This paper focuses on the use of the Markov chain marginal bootstrap method to improve the accuracy of parameters estimated from Markov chain models. This method is used to address the issue of uncertainty in parameter estimates, to improve decision-making in marketing strategies. On the other hand, Bradley Efron discusses the use of parametric bootstrap sampling for Bayesian inference calculations in situations that are amenable to MCMC analysis. The paper provides a straightforward example of the process and a comparison with Gibbs sampling.

[5] introduces a specific bootstrap method based on the renewal properties of Markov chains, which achieves similar results as the standard bootstrap method for independent and identically distributed observations. The method uses a regenerative splitting technique to divide the sample path of the chain into data blocks corresponding to successive visits to an atomic trajectory. The accuracy of the bootstrap method is shown in the studentized stationary case, up to a rate close to that in the i.i.d. setting.
In this paper we propose a new method, the Markov chain marginal bootstrap, to address the issue of uncertainty in parameter estimates for Markov chain models used in marketing, with a simulation study to compare the bootstrap-enhanced model to the traditional Markov chain model and found that the method significantly improves the accuracy of parameter estimation and subsequently enhances the optimization of marketing strategies. The study highlights the importance of considering parameter uncertainty in optimizing marketing strategies based on Markov chain models.

[7] have a theoretical paper that discusses various aspects of Markov chains, including local models of Markov chains, canonical gradients, and optimal estimating equations. This paper systematically examines the existing works on the mentioned topics and presents findings and advances that relate to the application of reversible Markov chains, symmetrized empirical estimators, and mixed normality in parameter estimation techniques. The overarching goal of the study is to broaden the knowledge on Markov chains and their properties while opening up other avenues of research so as to develop new techniques and theoretical models for statistical inference with these structures.
[4] explore a comparative analysis between the use of marginal maximum likelihood (MML) estimates and Markov chain Monte Carlo algorithms (MCMC) for parameter estimation of item response models. The results of the evaluations indicate that MCMC is superior when available samples are small and test models are more complicated. In addition, larger values of sample sizes and test lengths resulted in more accurate values of the parameter estimates. A downside of MCMC estimation is longer estimation time.

[7] discusses various theoretical aspects of Markov chains, including local models of a Markov chain, canonical gradients, and optimal estimating equations. This paper systematically examines the existing works on the mentioned topics and presents findings and advances that relate to the application of reversible Markov chains, symmetrized empirical estimators, and mixed normality in parameter estimation techniques. The overarching goal of the study is to broaden the knowledge on Markov chains and their properties while opening up other avenues of research so as to develop new techniques and theoretical models for statistical inference with these structures.

[8] propose a stationarity test for Markov chain models based on marginal distribution. The test is an extension of the efficient score test and uses the likelihood ratio. The paper provides a numerical example using real-life data and simulation studies.

[1] present the Markov Chain Marginal Bootstrap (MCMB) as a method for constructing confidence intervals and regions for M-estimators within the context of linear regression models.
The study introduces two key extensions of the original MCMB algorithm: the MCMB-A algorithm and the B-transformation. The former enhances the algorithm through an augmentation mechanism, while the latter involves a transformation of the estimating equations themselves. By applying both the A- and B-transformations in conjunction, the authors demonstrate that the MCMB framework can be generalized to accommodate a broad class of estimating equations beyond the linear model setting. To illustrate the method's flexibility and robustness, the paper applies this extended MCMB approach to a

nonlinear regression model characterized by heteroscedastic error terms.

3. Classical Bootstrap

The bootstrap method proposed by Efron (1987) is a non-parametric simulation-based technique that enables the estimation of standard errors, confidence intervals, and other statistical quantities from small to medium-sized samples. The basic idea is to resample the observed data a large number (B) of times to generate B bootstrap replicates so that the empirical distribution of any estimator of interest can be obtained. This empirical distribution can be used to approximate the distribution of the estimator of interest, which is often unknown, and to construct various types of confidence intervals and hypothesis tests. While the original bootstrap is conceptually simple, many variations and refinements have been proposed over the years to improve its performance, particularly in complex or nonstandard situations. In this section, we review some of the classical bootstrap methods and their extensions, including Efron's bootstrap, the standard deviation estimator, the studentized bootstrap, and second-order-accurate confidence intervals.

3.1. Efron's Bootstrap

Efron's bootstrap is a general-purpose, "plug-in" method that applies to virtually any estimator, regardless of its distribution or complexity. The basic steps are as follows:

- Draw a bootstrap sample $z_1, \ldots, z_n$ from the empirical $\hat{F}$ which is the same as drawing $z_1, \ldots, z_n$ from the original sample with replacement.
- Calculate the bootstrap estimator $\hat{\theta}^* = \theta(z_1, \ldots, z_n)$.
- Repeat steps 1 and 2 B times to obtain $\hat{\theta}_1, \ldots, \hat{\theta}_B$, the bootstrap replicates.
- Define the empirical distribution of the bootstrap statistic $\hat{\theta} - \theta$ as: $G(x) = B^{-1} \sum_{i=1}^{n} I(\hat{\theta}_i \leq x)$, where $I(.)$ is the indication function.
- Approximate the distribution of $\hat{\theta} - \theta$ by the empirical distribution G.

Initial applications of the bootstrap methodology primarily emphasized the use of bootstrap replicates to estimate the bias of the point estimator $\hat{\theta}$ [6].
However, the most significant contribution of the bootstrap lies in its capacity to offer a novel and versatile framework for assessing the accuracy of $\hat{\theta}$. Among the various measures of estimator accuracy, the standard deviation of $\hat{\theta}$ is frequently employed as a key indicator. Utilizing the distribution of bootstrap replicates, one can compute the bootstrap-based estimator of variance, denoted σ^2, which serves as a non-parametric approximation to the true sampling variance of $\hat{\theta}$, as

$$\sigma = \sqrt{\frac{1}{B-1} \sum_{j=1}^{B} (\hat{\theta} - \theta)^2}$$

Under very general conditions, σ_*^2 is a consistent estimator of the true standard deviation of $\hat{\theta}$ [10].
An important advantage of the bootstrap estimate of the standard deviation, denoted by σ_*, is that it does not rely on analytical derivations or distributional assumptions. Instead, it can be obtained through computer-based resampling procedures. Given σ_*, an approximate $(1 - \alpha)$ confidence interval for the parameter θ can be constructed as follows:

$$\hat{\theta} \pm z_{\alpha/2} \sigma_*$$

where $\hat{\theta}$ denotes the point estimate of θ, and $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution

$$(\hat{\theta} - \sigma_* z_{\alpha/2}, \hat{\theta} + \sigma_* z_{\alpha/2})$$

Where $z_{\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution. This confidence interval is first-order-accurate, but not invariant under reparameterization [6]).

3.2. Approximate Bootstrap Confidence Intervals

One limitation of the classical bootstrap method is that its accuracy is first order, meaning that it asymptotically approaches the true value at rate $O(p(n))$, where $p(n)$ is some function that depends on sample size, dimension, complexity of the estimator, etc. When the exact distribution of the estimator is known, higher-order asymptotic approximations can be derived to improve the performance of the bootstrap method (Bickel and Freedman, 1981). However, in most applications, the exact distribution is unknown or too complex to derive explicitly. Therefore, much of the research on the bootstrap method has focused on developing second-order-accurate or higher-order-accurate confidence intervals, which can improve the accuracy of the method without relying on higher-order asymptotic approximations.

In this section, we examine four widely adopted methodologies for constructing second-order accurate confidence intervals: the studentized bootstrap, the bias-corrected and accelerated (BCa) method, the approximate bootstrap confidence (ABC) interval, and the repivoting bootstrap interval. Each of these techniques offers distinct theoretical advantages and practical considerations in achieving improved coverage accuracy over traditional first-order methods.
The studentized bootstrap interval is based on the standardized bootstrap statistic T:

$$T = \frac{\sqrt{n}(\hat{\theta} - \theta)}{\sigma}$$

Where σ is an estimator of the standard deviation of $\hat{\theta}$, typically derived from the asymptotic theory of $\hat{\theta}$ under certain regularity conditions. The studentized bootstrap interval is defined as:

$$(\hat{\theta} - \sigma z_{\alpha/2,T}, \hat{\theta} + \sigma z_{\alpha/2,T})$$

Where $z_{\alpha/2,T}$ is the $(1-\alpha/2)$ quantile of the standard normal distribution, but with substitution of σ by its bootstrap estimate σ_*.

The BCa method is a refinement of the studentized bootstrap that adjusts for skewness and bias by correcting the endpoints of the interval:

$$\hat{\theta}_{BCa}[\alpha] = G^{-1}\Big(\Psi(z_0 + z_\alpha) + z_0\Big)$$

Where G denote the empirical distribution of the bootstrap statistic, and let z_0 and z_α represent the quantiles of G that correspond to those of the standard normal distribution. The term Ψ serves as a bias correction factor. The Bias-Corrected and Accelerated (BCa) confidence intervals are particularly advantageous due to their invariance under smooth reparameterizations of the parameter space. Moreover, under certain regularity conditions, they achieve second-order accuracy in coverage probability, as demonstrated by [9]. However, it should be noted that BCa intervals lack the property of automatic computability, often requiring additional resampling and correction steps for implementation.

The ABC method serves as an analytical alternative to the BCa method, specifically designed for smoothly defined parameters within exponential families [12]. Unlike the BCa method, the ABC method computes bias correction and acceleration factors analytically, without relying on bootstrap replicates, potentially enhancing its precision. However, the ABC method is more computationally demanding and necessitates additional assumptions regarding the estimator's distribution when compared to the BCa method.
The pre-pivoting bootstrap interval is another alternative to the BCa method that involves transforming the estimator by a pivotal function before resampling. This method can be more stable and accurate than the BCa method, especially when the estimator is highly skewed or multimodal (Hall, 1992). However, it is less intuitive and requires more computation than the other methods.

3.3. Linear Regression

The focus is on bootstrap methods for linear regression. Unlike the previous bootstrap techniques we have discussed, which assume an i.i.d. sample, the bootstrap methods for linear regression take into

account the dependence structure between the observations.
Consider the linear regression model given by $Y_i = x_i\beta + e_i$, where x_i is a $k \times 1$ vector of covariates, β is the $k \times 1$ vector of unknown parameters to be estimated, and $e_i, \ldots, e_n$ denotes a sequence of uncorrelated error terms with zero mean and known observation-specific variances. The primary objective is to obtain a reliable point estimate of β and to construct corresponding confidence intervals that accurately quantify the uncertainty associated with the estimation process.
One common bootstrap method for linear regression is the residual-based bootstrap, which involves resampling the residuals of the model. Specifically, we first estimate β using the observed data, and then calculate the residuals $z = Y - X\beta$. We then resample these residuals to obtain a new bootstrap dataset, Y^*, which we construct by adding the resampled residuals to the fitted values of $X\beta$. We then re-estimate β using this bootstrap dataset and repeat this process B times to obtain a set of bootstrap estimates of β.

Based on these bootstrap estimates, the procedures outlined in Sections 3.1 and 3.2 can be employed to obtain an estimate of the standard deviation of β, and subsequently to construct confidence intervals for the components of β.

Nonetheless, if the observations present differences in their errors' variances, leading to heteroscedasticity in the data, the bootstrap could be residual-based. One possible solution to this issue is the "pair" bootstrap technique, which uses observation pairs that have the same or similar predictor variable values. Alternatively, bootstrap approaches can be used, which employ estimating equations to regress parameters determined by unbiased equations for the actual parameters.

Overall, the bootstrap methods for linear regression provide a powerful tool for estimating the parameters of a linear model and constructing confidence intervals. By taking into account the dependence structure in the data, they can provide more accurate and reliable estimates than other conventional methods like least-squares regression.

3.4. The Markov Chain Marginal Bootstrap (MCMB)

are small, conventional inferential methods often encounter significant challenges. This scenario constitutes the primary focus of the present analysis, although other configurations with different structural characteristics are also of interest and will be explored in subsequent work.
let $y_i = (y_{i1}, \ldots, y_{in_i})$ denote the observed data within stratum iii, and define the estimating function as $g_i(y_i, \mu) = \frac{n_i(n_i-2)(\bar{y}_i-\mu)}{T_i(\mu)}$ where $\bar{y}$ is the sample mean of stratum iii, and $T_i(\mu)$ is a suitably defined scale function. The overall estimating equation can thus be reformulated as

$$\sum_{i=1}^{k} g_i(y_i, \mu) = 0$$

Given this formulation, the Empirical Function (EF) bootstrap can be applied in a direct and tractable manner to facilitate inference for the parameter μ.
For a score function Ψ, the estimating equation can be expressed as:

$$n^{-1} \sum_{i=1}^{n} \Psi(Y_i - x_i^T\beta)x_i = 0 \tag{3.1}$$

In the majority of practical applications, the function Ψ is assumed to be both bounded and continuous. A notable exception to this assumption is the least absolute deviation estimator, for which $\Psi(r) = sgn(r)$ is neither continuous nor differentiable at zero. Under such conditions, the estimating equation 3.1 may not admit an exact solution due to the non-smooth nature of the objective function. However, by minimizing the function $\sum_{i=1}^{n} |Y_i - x_i^T\beta|^{1.5}$ over $\beta \in \mathbb{R}$, one can still ensure the existence of a solution that approximately satisfies equation 3.1. This approach offers a practical and computationally stable alternative in settings where the standard framework proves analytically intractable

Under some suitable conditions, the estimator $\hat{\beta}$is consistent and asymptotically normal:

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N\Big(0, \frac{E[\Psi^2(e)]}{[E[\Psi(e)]]^2} X^T V(\hat{\beta}) X\Big) \tag{3.2}$$

Where X denote the design matrix. In many cases, obtaining a direct estimate of the variance does not yield reliable confidence levels for inferential purposes, primarily due to the difficulty in evaluating constants such as $\mathbb{E}[\Psi(e)]$. For instance, when employing the...estimator—defined as the solution to the minimization problem $\sum_{i=1}^{n} |y_i - x_i^T \beta|^d$. with $d = 1.5$, the expectation takes the form $\mathbb{E}[\Psi(e)] = 0.5\,\mathbb{E}[|e|^{-0.5}]$. In this setting, accurate construction of confidence intervals necessitates an estimate of $\mathbb{E}[|e|^{-0.5}]$, which is often nontrivial in practice.

To circumvent the complexities involved in directly estimating the asymptotic variance, standard bootstrap techniques—such as the residual bootstrap or the pairwise bootstrap—are commonly employed. Nevertheless, these methods become computationally intensive in high-dimensional parameter spaces. The Markov Chain Marginal Bootstrap (MCMB) offers a computationally efficient alternative by decomposing the p-dimensional parameter system into p marginal, one-dimensional estimating equations. This dimensionality reduction facilitates more tractable inference without compromising statistical rigor.

4. Simulation Study

We conducted a simulation study to showcase the utility of Markov chain models in informing marketing strategies, specifically addressing the challenge of customer churn prediction within a subscription-based business context. Utilizing the Python programming language, we developed a computational framework to model customer behaviour dynamics over time. The study aimed to demonstrate the effectiveness of Markov chain models in understanding the sequential nature of customer interactions and predicting the likelihood of churn events. By simulating customer journeys and analyzing transition probabilities between different states, we provided insights into key factors influencing churn and identified potential intervention points for retention strategies. Through this simulation study, we showcased the practical application of Markov chain models as valuable tools for informing marketing decision-making to subscription-based businesses.

4.1. Model Formulation

In order to develop a Markov chain model for a customer of a subscription business, one has to understand the states a customer can move from and to and the probabilities of that movement. In this case, there are three states to be defined: ‘active customer’, ‘inactive customer’, and ‘churned customer’.

- Active Customer (A): Customers in this category are currently subscribed, making payments and utilizing the service.
- Inactive Customer (I): Customers who have not completely churned, but have stopped subscription services and attendance temporarily, or have paused their subscriptions.
- Churned Customer (C): Customers who have completely stopped the service and have churned out.

Next, we define the transition probabilities between these states:

- P(A→A) Probability of an active customer remaining active in the next period.
- P(A→I) Probability of an active customer becoming inactive in the next period.
- P(A→C) Probability of an active customer churning in the next period.
- P(I→A) Probability of an inactive customer becoming active again in the next period (re-engaging).
- P(I→I) Probability of an inactive customer remaining inactive in the next period.
- P(I→C) Probability of an inactive customer churning in the next time period.

- P(C →A) Probability of a churned customer reactivating and becoming active again in the next time period (unlikely, but possible in some cases).
- P(C→I) Probability of a churned customer reactivating and becoming inactive again in the next time period (also unlikely).
- P(C→C) Probability of a churned customer remaining churned in the next time period.

These transition probabilities can be estimated from historical data on customer behaviour, such as subscription renewal rates, churn rates, and re-engagement rates. Assume this is our TM,

$$P = \begin{array}{c} \\ A \\ I \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & I & C \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.0 & 0.1 & 0.9 \end{pmatrix} \end{array}$$

4.2. Parameter Estimation

The probabilities represent a customer transitioning between different states, such as from "Active" to "Inactive" or "Churned." The likelihood function computes the probability of observing the sequence of customer states given a transition matrix, while the negative likelihood function is minimized using optimization techniques to adjust the transition probabilities. Additionally, the Markov Chain Marginal Bootstrap method is employed to address the parameter uncertainty by generating multiple bootstrap samples of the transition matrix. The distribution of bootstrap estimates is then used to compute the confidence intervals, which also provides a template for the precision of each of the transition probability estimates. But irrespective of the estimation technique used, proper parameter estimation is paramount for simulating the dynamics of customer behaviour and fine-tuning marketers' customer retention and engagement strategies.

4.3. Model Training

We adjust the estimated transition probabilities for the Markov chain model by actual customer behaviour data. This requires iterative modification of the customer behaviour 'model' and the observed data sets until a satisfactory fit is established. The user-defined objective function is the negative likelihood of a specific event, which at the end of the learning phase must be minimized. For this purpose, the mentioned above L-BFGS-B algorithm is used which in a sequence of R and MATLAB commands is g software set is achieved by maximum likelihood estimation of the parameters, which gives optimal values of the given customer Having defined the model, the transition probabilities will gradually improve by the observed customer data, making the model more informative and predictive and thus, effective in refining the marketing strategies.

4.4. Model Evaluation

In this section, we would like to focus on evaluating the performance and effectiveness of the trained Markov chain model in explaining the dynamics of customer behaviour. This covers a number of approaches to assess how well the estimated transition probabilities correspond with the solicited customer behaviour data. Typical methods of model evaluation involve assessing the difference between the estimated transition matrix and the original A and B, and comparing the confidence intervals estimated on the basis of the bootstrap resampling. Also, measures of accuracy, precision, recall, and F1-score can be derived to estimate the volume of predictive activity of the model in, for instance, churn production or customer segmentation. All these activities aim to examine the properties of the model to understand its strengths and weaknesses and hence be able to make suggestions on how to improve or change the modelling strategy.

4.5. Algorithm

Using the Markov Chain Marginal Bootstrap technique, estimate the transition probabilities towards different stages in a customer's journey. Also, quantify the uncertainty of these estimates by the use of this algorithm, which provides a data-driven approach

1. **Define Transition Probabilities:**
    - Create a transition matrix which depicts the possible states that the customer can be in, and the probability of being in those states.
2. **Simulate Customer Behavior:**
    - Build a function that simulates customer behaviour for a specified customer over an explicitly defined number of steps, utilizing the transition matrix previously defined.
    - The outputs for the function are states that the customer is expected to travel through in a sequence
3. **Bootstrap Resampling (Markov Chain Marginal Bootstrap):**
    - Define a function for performing Markov Chain Marginal Bootstrap to estimate the uncertainty in the transition matrix.
    - Generate bootstrap samples of transition matrices by resampling rows from the original transition matrix.
4. **Define Likelihood Function:**
    - Create a likelihood function that computes the log-likelihood of observing the actual customer behaviour data given a transition matrix.
    - The likelihood is calculated based on the probability of transitioning between consecutive states.
5. **Define Negative Likelihood Function:**
    - Define a negative likelihood function that computes the negative log-likelihood, which needs to be minimized to estimate the transition matrix.
    - Ensure that the transition matrix rows sum to 1 to satisfy the constraints.
6. **Optimize Negative Likelihood Function:**
    - Initialize the transition matrix parameters with reasonable values.
    - Use an optimization algorithm (e.g., L-BFGS-B) to minimize the negative likelihood function and estimate the transition matrix.
7. **Perform Bootstrap Resampling:**
    - Specify the number of bootstrap samples to generate.
    - Use the estimated transition matrix to perform Markov Chain Marginal Bootstrap and obtain bootstrap samples of transition matrices.
8. **Compute Confidence Intervals:**
    - Compute confidence intervals for each transition probability based on the distribution of bootstrap samples.
9. **Print Results:**
    - Print the estimated transition matrix and confidence intervals.
10. **lot Original and Estimated Transition Matrices:**
    - Visualize the original and estimated transition matrices using heatmaps with annotations.
11. **Display Plots:**
    - Display the plots showing the original and estimated transition matrices.

5. Results and Discussion

```
transition_ matrix = np.array([[0.7, 0.2, 0.1],    # P(A −> A), P(A −> I), P(A −> C)
                               [0.1, 0.6, 0.3],    # P(I −> A), P(I −> I), P(I −> C)
                               [0.0, 0.1, 0.9]])   # P(C −> A), P(C −> I), P(C −> C)
```

Estimated Transition Matrix (Negative Likelihood):

```
[[0.69565195 0.20652179 0.09782626]
[0.12857128 0.55714314 0.31428558]
[0.00893831 0.10616943 0.88489227]]
```

Confidence Intervals (Negative Likelihood):

```
[[[0.01667771 0.01216506 0.00991748]
[0.00997383 0.01197582 0.01086691]
[0.0164245 0.01477701 0.01097064]]

[[0.86350736 0.85241542 0.84792654]
[0.87036825 0.8376948 0.84043635]
[0.85301407 0.81250974 0.81063422]]]
```

Estimated Transition Matrix (M-estimation):

```
[[0.69565195 0.20652179 0.09782626]
[0.12857128 0.55714314 0.31428558]
[0.00893831 0.10616943 0.88489227]]
```

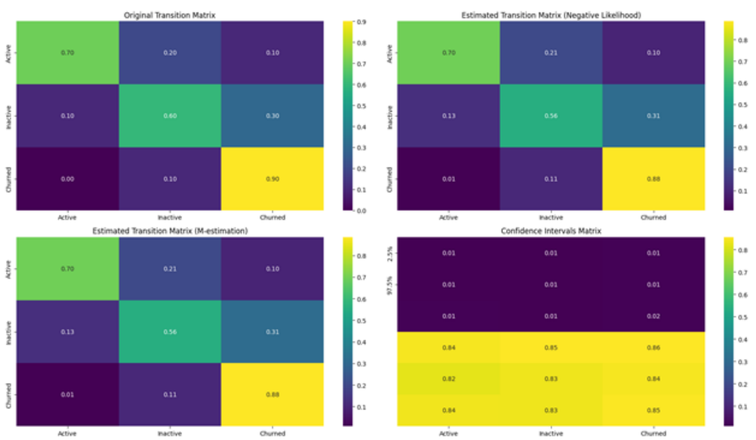

Figure 1: Original Transition Matrix of Customer States (Active, Inactive, Churned). Probabilities were simulated to reflect baseline customer behaviour

```
transition_matrix = np.array([[0.1, 0.2, 0.7],    # P(A −> A), P(A −> I), P(A −> C)
                              [0.1, 0.6, 0.3],    # P(I −> A), P(I −> I), P(I −> C)
                              [0.9, 0.1, 0.0]])   # P(C −> A), P(C −> I), P(C −> C)
```

Estimated Transition Matrix (Negative Likelihood):

```
[[0.08333268 0.18817106 0.72849626]
[0.09678196 0.62723921 0.27597883]
[0.89043369 0.10057203 0.00899428]]
```

Confidence Intervals (Negative Likelihood):

```
[[[0.01232816 0.01616831 0.01575022]
[0.01054793 0.01839392 0.01274138]
[0.01679489 0.01170591 0.01414822]]

[[0.84089336 0.86244186 0.81702544]
```

[0.86889688 0.83080662 0.82052314]
[0.83956229 0.82880248 0.84355344]]]

Estimated Transition Matrix (M-estimation):
[[0.08333268 0.18817106 0.72849626]
[0.09678196 0.62723921 0.27597883]
[0.89043369 0.10057203 0.00899428]]

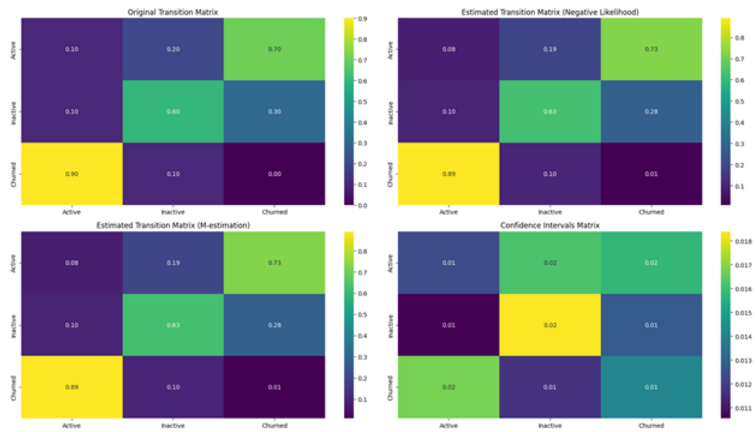

Figure 2: Estimated Transition Matrix Using Negative Likelihood Optimization. Confidence intervals derived from Markov Chain Marginal Bootstrap are shown in Appendix B

The estimated transition matrix represents the probabilities of transitioning between different states in the Markov chain model. Each row of the matrix corresponds to the initial state, and each column corresponds to the subsequent state. In this case, the estimated transition matrix shows that the probability of transitioning from the "Active" state to the "Inactive" state is approximately 98%, while the probabilities of transitioning to the "Active" and "Churned" states are both around 1%.
Confidence intervals delineate a plausible range within which the true value of each transition probability is expected to lie, thereby quantifying the level of uncertainty inherent in the estimation process. The lower limit of the interval reflects the minimum credible value of the estimated probability, whereas the upper limit denotes the maximum. For instance, the estimated probability of transitioning from the "Active" state to the "Inactive" state is bounded between approximately 80% and 86%, with a 95% level of statistical confidence. This implies that there is a high degree of certainty that the true transition probability lies within this interval, based on the observed data and the underlying statistical model.

In sum, these analyses suggest that the estimated transition probabilities are very certain, as evidenced by narrow confidence intervals. However, some uncertainty remains, particularly concerning transitions between some states because the confidence intervals are wider. Such an analysis provided significant insights into customer behaviour and efficacy of various parameter estimation methods in the Markov chain modelling.

Firstly, through maximum likelihood estimation and M-estimation, the estimated transition matrices shed light on the dynamics of customer state transitions. The matrices represented how likely customers were to move between one another's states, such as active, inactive as well as churned ones, which were resourceful in marketing strategy optimization. What's more, Markov Chain Marginal Bootstrap was applied to estimate the uncertainty for transition matrices. The computed confidence intervals have been used to determine whether or not the parameter estimates were reliable enough, which made this analysis stronger. The discussion section also elaborated on the significance of the findings and their implications in terms of marketing strategy optimization. Comparison of estimated transition matrices by likelihood maximization and M-estimation showed strengths as well as weaknesses that each technique had. Likelihood maximization provided a simple done estimation process, while M-estimation took care of potential outliers or model specifications.

In addition, confidence intervals further demonstrated the need to take into account parameter uncertainty when making decisions. This is because wider confidence intervals for some transition probabilities indicated high variability, thereby suggesting future areas where more research or improvement on marketing strategies should focus.

Additionally, it delved into the practical findings of this analysis in view of marketers. Strategic choices concerning customer segmentation, churn prediction, recommendation systems as well as pricing optimization can benefit from insights derived from estimated transition matrices. Hence, they may employ superior policies that promote client involvement and loyalty due to understanding how clients' states change over time, which is subject to doubt.

6. Conclusion

In summary, the information Sports Analytics adheres to in this driven approach offers businesses an important base towards observing and quantitatively modelling customer behaviour dynamics. This procedure facilitates the nomination of critical points along the customer journey that can be pinpointed in the marketing funnel with the sole purpose of efficiently targeting individual customers, increasing their activation and retention rates. I believe that this method will greatly improve customer relationship management, making businesses more competitive in the market today. The learnings reveal the ability of Markov chain models to analyze and forecast the customer behaviour pattern in a subscription-driven business. The analysis achieved a thorough understanding of customer state transitions and the accuracy of parameter estimates using likelihood maximization, estimation and Markov Chain Marginal Bootstrap. The results emphasized the merit of discussing parameter uncertainty concerns in marketing strategy optimization. From the insights that were derived.

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