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Reversible Cyclic Codes Over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q$ and Applications to DNA Codes

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ABSTRACT: In this paper, we investigate the structure of reversible and reversible-complement cyclic codes of length n over the ring $\mathbf{R} = \mathbb{F}_q + uF_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q$, $u^4 = 0$, when n and q are coprime. We give the necessary and sufficient conditions for the cyclic codes of length n over $\mathbf{R}$ to be reversible and reversible-compliment cyclic codes, when $(n, q) = 1$ over $\mathbf{R}$. We also study the dual of cyclic codes over $\mathbf{R}$ when $(n, q) = 1$ and obtain the reversibility condition for dual of cyclic codes. Additionally, we explore cyclic DNA codes over nucleotide 4-base pair. First, we establish a one-to-one correspondence between $\mathbf{R_1}$, where $\mathbf{R_1} = \mathbb{F}_4 + uF_4 + u^2\mathbb{F}_4 + u^3\mathbb{F}_4$, $u^4 = 0$ and 4-mers and then cyclic DNA codes constructed as the images of reversible-complement cyclic codes over $\mathbf{R_1}$.

Key Words: Reversible cyclic codes, Watson-Crick complement rule, reversible-Complement cyclic codes.

Contents

1. Introduction

Numerous combinatorial problems are addressed by DNA computing, such as the Maximal Clique problem [16], the Hamiltonian path problem [4] and others. It was demonstrated by Mansuripur et al. [13] the use of DNA codes for storage media. Adleman et al. [5] cracked the Data Encryption Standard (DES) cryptosystem using DNA computing techniques.

In the area of error-correcting codes, cyclic codes over finite rings played a vital role [1,3,9,18,22]. Since then, good error-correcting codes have been constructed using the DNA structure as a model, and error-correcting codes with properties similar to DNA structure have also been utilized for understanding DNA. The linear construction of DNA codes was studied by Gaborit and King [11]. Abualrub et al. studied the DNA codes over the finite field of four elements [2]. Later, Siap et al. discussed DNA codes over the ring $F_2[u]/\langle u^2 - 1\rangle$ with four elements [21]. DNA codes over the ring $F_2[u]/\langle u^4 - 1\rangle$ with 16 elements were studied by Yildiz and Siap [23]. Liang and Wang [12] studied cyclic DNA codes over the ring $F_2 + uF_2$, $u^2 = 0$. Subsequently, Mostafanasab and Darani [14] explored the cyclic DNA codes over the ring $R = F_2 + uF_2 + u^2F_2$, $u^3 = 0$. One of the major problems in the theory of DNA coding is the reversibility problem. The reversibility problem has been addressed by many authors over different algebraic structures such as Oztas et al. [17] identified the DNA k-bases with the elements of the ring $\mathbf{F}_2[u]/\langle u^{2k} - 1\rangle$ and used

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coterm polynomials for solving the reversibility problem. In [8] Ashraf et al. addressed the reversibility problem over a non-chain ring. In [10] Dinh et al. studied the reversibility problem over the ring $\mathbf{Z_4}[u,v]/\langle u^2-2, uv-2, v^2, 2u, 2v\rangle$. Further, Alali [6] and coauthors addressed the reversibility problem over a non-chain ring $\mathcal{R}_k = \mathbf{Z_4}[u_1, u_2, \ldots, u_k]/\langle u_i^2 - u_i, u_iu_j - u_ju_i\rangle$, where $1 \leq i, j \leq k,\ k \geq 1$. Motivated by these works, we study reversible cyclic codes over $\mathbf{R} = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q,\ u^4 = 0$, when the length of the code is coprime to q and construct DNA codes over the set of all DNA strings of length 4 as images of cyclic reversible-complement codes over $\mathbf{R_1} = \mathbb{F}_4 + u\mathbb{F}_4 + u^2\mathbb{F}_4 + u^3\mathbb{F}_4,\ u^4 = 0$. Moreover, we study dual $\mathbf{C}^\perp$ of cyclic code $\mathbf{C}$ over $\mathbf{R}$ of length n and obtain the conditions for the dual of cyclic code to be reversible when $(n, q) = 1$. This paper is organized as follows: In Section 2, we discuss some basic definitions and results which are used in later sections and a relationship between the elements of the ring $\mathbf{R_1}$ and DNA strings of length 4 is presented. In Section 3, we discuss the structure of cyclic codes over $\mathbf{R}$. In Section 4, reversiblilty conditions for cyclic codes of length n over $\mathbf{R}$ are discussed, when $(n, q) = 1$. Section 5 presents the study of dual of cyclic codes of length n over $\mathbf{R}$ and reversiblity condition for dual of cyclic codes is discussed. Reversibile-complement codes over $\mathbf{R_1}$ are discussed in Section 6. In Section 7, some examples are presented. Finally, Section 8 concludes the paper.

2. Preliminaries

Consider the set $\mathbf{R} = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q,\ u^4 = 0$, where $q = p^k,\ p$ is a prime, $k \in \mathbb{N}$. Then R is a finite commutative ring with q^4 elements and characteristic p. The ring R is isomorphic to the quotient ring $\mathbb{F}_q[u]/\langle u^4\rangle$ and $\langle 0\rangle$, $\langle 1\rangle$, $\langle u\rangle$, $\langle u^2\rangle$, $\langle u^3\rangle$ are ideals of $\mathbf{R}$. Also, $\langle 0\rangle = \langle u^4\rangle \subseteq \langle u^3\rangle \subseteq \langle u^2\rangle \subseteq \langle u\rangle \subseteq \langle 1\rangle = \mathbf{R}$. We can see that $\mathbf{R}$ is a finite local chain ring with maximal ideal $\langle u\rangle$. A linear code $\mathbf{C}$ over $\mathbf{R}$ of length n is an $\mathbf{R}-$submodule of $\mathbf{R}^n$. Elements of $\mathbf{C}$ are called codewords. A code of length n is cyclic if it is closed under cyclic shift i.e., $(c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathbf{C}$ whenever $(c_0, c_1, \ldots, c_{n-1}) \in \mathbf{C}$. It is well known that a cyclic code of length n over a ring $\mathbf{R}$ can be identified with an ideal in the quotient ring $\mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle$ via the $\mathbf{R}-$module isomorphism as follows:

$$\mathbf{R}^n \longrightarrow \mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle$$
$$(c_0, c_1, \ldots, c_{n-1}) = c_0 + c_1\varkappa + \cdots + c_{n-1}\varkappa^{n-1}.$$

For any vector $v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbf{R}^n$, the reverse of v denoted as v^r is the reversal of the components of v i.e., $v^r = (v_{n-1}, v_{n-2}, \ldots, v_1, v_0)$. The number of non-zero components in any codeword $c \in \mathbf{C}$ is the hamming weight of c denoted as $w_H(c)$ and minimum of weight of all codewords is defined as the hamming weight of the code i.e., $w_H(\mathbf{C}) = min\{w_H(c)|c \in \mathbf{C}\}$. The hamming distance (d_H) between two codewords is the number of components in which they differ. The distance of code is defined as $d(\mathbf{C}) = min\{d_H(c, c')|c, c' \in \mathbf{C}, c \neq c'\}$. For a linear code, distance of the code is weight of the code. The Lee weight of $x = a + ub + u^2c + u^3d \in \mathbf{R}$ is defined as $w_L(x) = w_L(a + ub + u^2c + u^3d) = w_H(a + b + c + d, c + d, b + d, d)$. This definition of Lee weight immediately leads to a Gray map ϕ from $\mathbf{R}$ to $\mathbb{F}_q^4$ defined as

$$\phi : \mathbf{R} \to \mathbb{F}_q^4$$
$$\phi(a + ub + u^2c + u^3d) = (a + b + c + d, c + d, b + d, d).$$

This Gray map can naturally be extended to $\mathbf{R}^n$ as

$$\Phi : \mathbf{R}^n \to \mathbb{F}_q^{4n}$$
$$\Phi(a_0, a_1, \ldots, a_{n-1}) = (\phi(a_0), \phi(a_1), \ldots, \phi(a_{n-1})).$$

The Lee distance between any two codewords $c, c' \in \mathbf{C}$ is given by $d_L(c_1, c_2) = d_H(\Phi(c_1), \Phi(c_2))$ and minimum of Lee distance of all codewords is defined as the Lee distance of the code i.e., $d_L(\mathbf{C}) = min\{d_L(c, c')|c, c' \in \mathbf{C}, c \neq c'\}$.

Theorem 2.1 *The Gray map Φ is a linear isometry from $(\mathbf{R}^n, Lee\ distance)$ to $(\mathbb{F}_q^{4n}, Hamming\ distance)$.*

Theorem 2.2 *If $\mathbf{C}$ is a linear $[n, k, d_L]$ code over $\mathbf{R}$, then $\Phi(\mathbf{C})$ is a linear code $[4n, k, d_H]$ over $\mathbb{F}_q$.*

Table 1 Correspondence of DNA base pair with elements of the ring $\mathbf{R_1}$

Elements of $\mathbf{R_1}$	DNA string	Elements of $\mathbf{R_1}$	DNA string
0	$AAAA$	u^3	$TTTT$
α	$AAAT$	$u^3+\alpha$	$TTTA$
α^2	$AAAG$	$u^3+\alpha^2$	$TTTC$
1	$AAAC$	u^3+1	$TTTG$
αu	$ATCA$	$u^3+\alpha u$	$TAGT$
$\alpha u+\alpha$	$ATCT$	$u^3+\alpha u+\alpha$	$TAGA$
$\alpha u+\alpha^2$	$ATCG$	$u^3+\alpha u+\alpha^2$	$TAGC$
$\alpha u+1$	$ACGG$	$u^3+\alpha u+1$	$TGCC$
$\alpha^2 u$	$ATCC$	$u^3+\alpha^2 u$	$TAGG$
$\alpha^2 u+\alpha$	$ACGA$	$u^3+\alpha^2 u+\alpha$	$TGCT$
$\alpha^2 u+\alpha^2$	$ACGC$	$u^3+\alpha^2 u+\alpha^2$	$TGCG$
$\alpha^2 u+1$	$AATA$	$u^3+\alpha^2 u+1$	$TTAT$
u	$AATT$	u^3+u	$TTAA$
$u+\alpha$	$AATG$	$u^3+u+\alpha$	$TTAC$
$u+\alpha^2$	$AATC$	$u^3+u+\alpha^2$	$TTAG$
$u+1$	$AGAA$	u^3+u+1	$TCTT$
αu^2	$ATTA$	$u^3+\alpha u^2$	$TAAT$
$\alpha u^2+\alpha$	$TAAA$	$u^3+\alpha u^2+\alpha$	$ATTT$
$\alpha u^2+\alpha^2$	$AGAG$	$u^3+\alpha u^2+\alpha^2$	$TCTC$
αu^2+1	$AGAC$	$u^3+\alpha u^2+1$	$TCTG$
$\alpha u^2+\alpha u$	$ACCT$	$u^3+\alpha u^2+\alpha u$	$TGGA$
$\alpha u^2+\alpha u+\alpha$	$ACCG$	$u^3+\alpha u^2+\alpha u+\alpha$	$TGGC$
$\alpha u^2+\alpha u+\alpha^2$	$ACCC$	$u^3+\alpha u^2+\alpha u+\alpha^2$	$TGGG$
$\alpha u^2+\alpha u+1$	$TTGA$	$u^3+\alpha u^2+\alpha u+1$	$AACT$
$\alpha u^2+\alpha^2 u$	$TTGT$	$u^3+\alpha u^2+\alpha^2 u$	$AACA$
$\alpha u^2+\alpha^2 u+\alpha$	$TTGG$	$u^3+\alpha u^2+\alpha^2 u+\alpha$	$AACC$
$\alpha u^2+\alpha^2 u+\alpha^2$	$TTGC$	$u^3+\alpha u^2+\alpha^2 u+\alpha^2$	$AACG$
$\alpha u^2+\alpha^2 u+1$	$AAGT$	$u^3+\alpha u^2+\alpha^2 u+1$	$TTCA$
αu^2+u	$ATAT$	$u^3+\alpha u^2+u$	$TATA$
$\alpha u^2+u+\alpha$	$CATT$	$u^3+\alpha u^2+u+\alpha$	$GTAA$
$\alpha u^2+u+\alpha^2$	$AAGG$	$u^3+\alpha u^2+u+\alpha^2$	$TTCC$
αu^2+u+1	$CAGG$	$u^3+\alpha u^2+u+1$	$GTCC$
$\alpha^2 u^2$	$ACCA$	$u^3+\alpha^2 u^2$	$TGGT$
$\alpha^2 u^2+\alpha$	$CAGC$	$u^3+\alpha^2 u^2+\alpha$	$GTCG$
$\alpha^2 u^2+\alpha^2$	$GAAA$	$u^3+\alpha^2 u^2+\alpha^2$	$CTTT$
$\alpha^2 u^2+1$	$CAGA$	$u^3+\alpha^2 u^2+1$	$GTCT$
$\alpha^2 u^2+\alpha u$	$GTGC$	$u^3+\alpha^2 u^2+\alpha u$	$CACG$
$\alpha^2 u^2+\alpha u+\alpha$	$CACC$	$u^3+\alpha^2 u^2+\alpha u+\alpha$	$GTGG$
$\alpha^2 u^2+\alpha u+\alpha^2$	$GGTC$	$u^3+\alpha^2 u^2+\alpha u+\alpha^2$	$CCAG$
$\alpha^2 u^2+\alpha u+1$	$GAAC$	$u^3+\alpha^2 u^2+\alpha u+1$	$CTTG$
$\alpha^2 u^2+\alpha^2 u$	$TACA$	$u^3+\alpha^2 u^2+\alpha^2 u$	$ATGT$
$\alpha^2 u^2+\alpha^2 u+\alpha$	$GGTA$	$u^3+\alpha^2 u^2+\alpha^2 u+\alpha$	$CCAT$
$\alpha^2 u^2+\alpha^2 u+\alpha^2$	$CACT$	$u^3+\alpha^2 u^2+\alpha^2 u+\alpha^2$	$GTGA$
$\alpha^2 u^2+\alpha^2 u+1$	$CTCA$	$u^3+\alpha^2 u^2+\alpha^2 u+1$	$GAGT$
$\alpha^2 u^2+u$	$ACGT$	$u^3+\alpha^2 u^2+u$	$TGCA$
$\alpha^2 u^2+u+\alpha$	$GCTC$	$u^3+\alpha^2 u^2+u+\alpha$	$CGAG$

Elements of $\mathbf{R_1}$	DNA string	Elements of $\mathbf{R_1}$	DNA string
$\alpha^2u^2+u+\alpha^2$	$GATT$	$u^3+\alpha^2u^2+u+\alpha^2$	$CTAA$
α^2u^2+u+1	$CCTG$	$u^3+\alpha^2u^2+u+1$	$GGAC$
u^2	$GGGG$	u^3+u^2	$CCCC$
$u^2+\alpha$	$CGAC$	$u^3+u^2+\alpha$	$GCTG$
$u^2+\alpha^2$	$GAGA$	$u^3+u^2+\alpha^2$	$CTCT$
u^2+1	$CAAA$	u^3+u^2+1	$GTTT$
$u^2+\alpha u$	$GAGG$	$u^3+u^2+\alpha u$	$GAGG$
$u^2+\alpha u+\alpha$	$CGCC$	$u^3+u^2+\alpha u+\alpha$	$GCGG$
$u^2+\alpha u+\alpha^2$	$GATG$	$u^3+u^2+\alpha u+\alpha^2$	$CTAC$
$u^2+\alpha u+1$	$GCCC$	$u^3+u^2+\alpha u+1$	$CGGG$
$u^2+\alpha^2u$	$CGTA$	$u^3+u^2+\alpha^2u$	$GCAT$
$u^2+\alpha^2u+\alpha$	$CACA$	$u^3+u^2+\alpha^2u+\alpha$	$GTGT$
$u^2+\alpha^2u+\alpha^2$	$GATA$	$u^3+u^2+\alpha^2u+\alpha^2$	$CTAT$
$u^2+\alpha^2u+1$	$GACT$	$u^3+u^2+\alpha^2u+1$	$CTGA$
u^2+u	$AGCT$	u^3+u^2+u	$TCGA$
$u^2+u+\alpha$	$GAGC$	$u^3+u^2+u+\alpha$	$CTCG$
$u^2+u+\alpha^2$	$CCTT$	$u^3+u^2+u+\alpha^2$	$GGAA$
u^2+u+1	$TTCT$	u^3+u^2+u+1	$AAGA$
αu^3	$AGGA$	α^2u^3	$TCCT$
$\alpha u^3+\alpha$	$CTGC$	$\alpha^2u^3+\alpha$	$GACG$
$\alpha u^3+\alpha^2$	$AAGC$	$\alpha^2u^3+\alpha^2$	$TTCG$
αu^3+1	$AGTA$	α^2u^3+1	$TCAT$
$\alpha u^3+\alpha u$	$ACTA$	$\alpha^2u^3+\alpha u$	$TGAT$
$\alpha u^3+\alpha u+\alpha$	$GCCA$	$\alpha^2u^3+\alpha u+\alpha$	$CGGT$
$\alpha u^3+\alpha u+\alpha^2$	$CTGG$	$\alpha^2u^3+\alpha u+\alpha^2$	$GACC$
$\alpha u^3+\alpha u+1$	$CCCG$	$\alpha^2u^3+\alpha u+1$	$GGGC$
$\alpha u^3+\alpha^2u$	$GGAT$	$\alpha^2u^3+\alpha^2u$	$CCTA$
$\alpha u^3+\alpha^2u+\alpha$	$CCAA$	$\alpha^2u^3+\alpha^2u+\alpha$	$GGTT$
$\alpha u^3+\alpha^2u+\alpha^2$	$AGTG$	$\alpha^2u^3+\alpha^2u+\alpha^2$	$TCAC$
$\alpha u^3+\alpha^2u+1$	$AGTC$	$\alpha^2u^3+\alpha^2u+1$	$TCAG$
αu^3+u	$CATG$	α^2u^3+u	$GTAC$
$\alpha u^3+u+\alpha$	$TAAG$	$\alpha^2u^3+u+\alpha$	$ATTC$
$\alpha u^3+u+\alpha^2$	$TAAC$	$\alpha^2u^3+u+\alpha^2$	$ATTG$
αu^3+u+1	$AGGG$	α^2u^3+u+1	$TCCC$
$\alpha u^3+\alpha u^2$	$CAAC$	$\alpha^2u^3+\alpha u^2$	$GTTG$
$\alpha u^3+\alpha u^2+\alpha$	$CGTC$	$\alpha^2u^3+\alpha u^2+\alpha$	$GCAG$
$\alpha u^3+\alpha u^2+\alpha^2$	$AGGC$	$\alpha^2u^3+\alpha u^2+\alpha^2$	$TCCG$
$\alpha u^3+\alpha u^2+1$	$TATG$	$\alpha^2u^3+\alpha u^2+1$	$ATAC$
$\alpha u^3+\alpha u^2+\alpha u$	$TCCA$	$\alpha^2u^3+\alpha u^2+\alpha u$	$AGGT$
$\alpha u^3+\alpha u^2+\alpha u+\alpha$	$TCTA$	$\alpha^2u^3+\alpha u^2+\alpha u+\alpha$	$AGAT$
$\alpha u^3+\alpha u^2+\alpha u+\alpha^2$	$GTAG$	$\alpha^2u^3+\alpha u^2+\alpha u+\alpha^2$	$CATC$
$\alpha u^3+\alpha u^2+\alpha u+1$	$CAAG$	$\alpha^2u^3+\alpha u^2+\alpha u+1$	$GTTC$
$\alpha u^3+\alpha u^2+\alpha^2u$	$ACAA$	$\alpha^2u^3+\alpha u^2+\alpha^2u$	$TGTT$
$\alpha u^3+\alpha u^2+\alpha^2u+\alpha$	$TCGT$	$\alpha^2u^3+\alpha u^2+\alpha^2u+\alpha$	$AGCA$
$\alpha u^3+\alpha u^2+\alpha^2u+\alpha^2$	$TATC$	$\alpha^2u^3+\alpha u^2+\alpha^2u+\alpha^2$	$ATAG$
$\alpha u^3+\alpha u^2+\alpha^2u+1$	$TGAG$	$\alpha^2u^3+\alpha u^2+\alpha^2u+1$	$ACTC$
$\alpha u^3+\alpha u^2+u$	$CTAG$	$\alpha^2u^3+\alpha u^2+u$	$GATC$

Elements of $\mathbf{R_1}$	DNA string	Elements of $\mathbf{R_1}$	DNA string
$\alpha u^3+\alpha u^2+u+\alpha$	$CTTA$	$\alpha^2u^3+\alpha u^2+u+\alpha$	$GAAT$
$\alpha u^3+\alpha u^2+u+\alpha^2$	$TGAC$	$\alpha^2u^3+\alpha u^2+u+\alpha^2$	$ACTG$
$\alpha u^3+\alpha u^2+u+1$	$AGCG$	$\alpha^2u^3+\alpha u^2+u+1$	$TCGC$
$\alpha u^3+\alpha^2u^2$	$CGGC$	$\alpha^2u^3+\alpha^2u^2$	$GCCG$
$\alpha u^3+\alpha^2u^2+\alpha$	$AGCC$	$\alpha^2u^3+\alpha^2u^2+\alpha$	$TCGG$
$\alpha u^3+\alpha^2u^2+\alpha^2$	$CGAA$	$\alpha^2u^3+\alpha^2u^2+\alpha^2$	$GCTT$
$\alpha u^3+\alpha^2u^2+1$	$GTAT$	$\alpha^2u^3+\alpha^2u^2+1$	$CATA$
$\alpha u^3+\alpha^2u^2+\alpha u$	$CGTG$	$\alpha^2u^3+\alpha^2u^2+\alpha u$	$GCAC$
$\alpha u^3+\alpha^2u^2+\alpha u+\alpha$	$CCGC$	$\alpha^2u^3+\alpha^2u^2+\alpha u+\alpha$	$GGCG$
$\alpha u^3+\alpha^2u^2+\alpha u+\alpha^2$	$GCTA$	$\alpha^2u^3+\alpha^2u^2+\alpha u+\alpha^2$	$CGAT$
$\alpha u^3+\alpha^2u^2+\alpha u+1$	$AGTT$	$\alpha^2u^3+\alpha^2u^2+\alpha u+1$	$TCAA$
$\alpha u^3+\alpha^2u^2+\alpha^2u$	$TGTA$	$\alpha^2u^3+\alpha^2u^2+\alpha^2u$	$ACAT$
$\alpha u^3+\alpha^2u^2+\alpha^2u+\alpha$	$TGTG$	$\alpha^2u^3+\alpha^2u^2+\alpha^2u+\alpha$	$ACAC$
$\alpha u^3+\alpha^2u^2+\alpha^2u+\alpha^2$	$GCGT$	$\alpha^2u^3+\alpha^2u^2+\alpha^2u+\alpha^2$	$CGCA$
$\alpha u^3+\alpha^2u^2+\alpha^2u+1$	$ACTT$	$\alpha^2u^3+\alpha^2u^2+\alpha^2u+1$	$TGAA$
$\alpha u^3+\alpha^2u^2+u$	$CCGG$	$\alpha^2u^3+\alpha^2u^2+u$	$GGCC$
$\alpha u^3+\alpha^2u^2+u+\alpha$	$TGTC$	$\alpha^2u^3+\alpha^2u^2+u+\alpha$	$ACAG$
$\alpha u^3+\alpha^2u^2+u+\alpha^2$	$GTTA$	$\alpha^2u^3+\alpha^2u^2+u+\alpha^2$	$CAAT$
$\alpha u^3+\alpha^2u^2+u+1$	$CGCT$	$\alpha^2u^3+\alpha^2u^2+u+1$	$GCGA$
αu^3+u^2	$GAAG$	$\alpha^2u^3+u^2$	$CTTC$
$\alpha u^3+u^2+\alpha$	$CCGA$	$\alpha^2u^3+u^2+\alpha$	$GGCT$
$\alpha u^3+u^2+\alpha^2$	$CGGA$	$\alpha^2u^3+u^2+\alpha^2$	$GCCT$
αu^3+u^2+1	$ATGA$	$\alpha^2u^3+u^2+1$	$TACT$
$\alpha u^3+u^2+\alpha u$	$GGAG$	$\alpha^2u^3+u^2+\alpha u$	$CCTC$
$\alpha u^3+u^2+\alpha u+\alpha$	$CCAC$	$\alpha^2u^3+u^2+\alpha u+\alpha$	$GGTG$
$\alpha u^3+u^2+\alpha u+\alpha^2$	$CCCA$	$\alpha^2u^3+u^2+\alpha u+\alpha^2$	$GGGT$
$\alpha u^3+u^2+\alpha u+1$	$GGCA$	$\alpha^2u^3+u^2+\alpha u+1$	$CCGT$
$\alpha u^3+u^2+\alpha^2u$	$TACG$	$\alpha^2u^3+u^2+\alpha^2u$	$ATGC$
$\alpha u^3+u^2+\alpha^2u+\alpha$	$TACC$	$\alpha^2u^3+u^2+\alpha^2u+\alpha$	$ATGG$
$\alpha u^3+u^2+\alpha^2u+\alpha^2$	$GCAA$	$\alpha^2u^3+u^2+\alpha^2u+\alpha^2$	$CGTT$
$\alpha u^3+u^2+\alpha^2u+1$	$TATT$	$\alpha^2u^3+u^2+\alpha^2u+1$	$ATAA$
αu^3+u^2+u	$CGCG$	$\alpha^2u^3+u^2+u$	$GCGC$
$\alpha u^3+u^2+u+\alpha$	$GACA$	$\alpha^2u^3+u^2+u+\alpha$	$CTGT$
$\alpha u^3+u^2+u+\alpha^2$	$GTCA$	$\alpha^2u^3+u^2+u+\alpha^2$	$CAGT$
αu^3+u^2+u+1	$CCCT$	$\alpha^2u^3+u^2+u+1$	$GGGA$

3. Structure of Cyclic Codes Over R

This Section provides the algebraic structure of cyclic codes over the ring $\mathbf{R}$. Al-Ashker and Chen [14] obtained the structure of cyclic codes over general ring $\mathbb{F}_q[u]/\langle u^k\rangle$. Cyclic codes over $\mathbf{R}$ of length n are precisely the ideals in the quotient ring $\mathbf{R}[\varkappa]/\langle\varkappa^n-1\rangle$. Let $\mathfrak{R}=\mathbb{F}_q+u\mathbb{F}_q$, where $u^2=0$ and $\mathfrak{S}=\mathbb{F}_q+u\mathbb{F}_q+u^2\mathbb{F}_q$, where $u^3=0$ be subrings of $\mathbf{R}$. Let $\theta:\mathfrak{S}\to\mathfrak{R}$ be a map defined as $\theta(a+ub+u^2c)=a+ub$ for $a,b,c\in\mathbb{F}_q$. Then, θ is a ring homomorphism. Let D be a cyclic code over $\mathfrak{S}$. Then, we can extend this map to $\Theta: D\to\mathfrak{R}[\varkappa]/\langle\varkappa^n-1\rangle$ such that $\Theta(a_0+a_1\varkappa+\cdots+a_{n-1}\varkappa^{n-1})=\theta(a_0)+\theta(a_1)\varkappa+\cdots+\theta(a_{n-1})\varkappa^{n-1}$, where $a_i\in\mathfrak{S}$ and $ker\Theta=\{u^2t(\varkappa)|t(\varkappa)\in\mathbb{F}_q[\varkappa]\}$. Let $I=\{t(\varkappa)\in\mathbb{F}_q[\varkappa]|u^2t(\varkappa)\in ker\Theta\}$. Then, I is a cyclic code over $\mathbb{F}_q$ and hence $I=\langle a_2(\varkappa)\rangle$ and $ker\Theta=\langle u^2a_2(\varkappa)\rangle$, also $\Theta(D)$ is cyclic code over $\mathfrak{R}$ and hence, $\Theta(D)$ is given by [19]

$$\Theta(D)=\langle g(\varkappa)+up_1(\varkappa),ua_1(\varkappa)\rangle,$$

and therefore, D is given by

$$D=\langle g(\varkappa)+up_1(\varkappa)+u^2p_2(\varkappa),ua_1(\varkappa)+u^2q_1(\varkappa),u^2a_2(\varkappa)\rangle,$$

where $a_i(\varkappa), p_i(\varkappa), g(\varkappa), q_1(\varkappa) \in \mathbb{F}_q[\varkappa]$ and $a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)|\varkappa^n - 1$.

Let $\varphi : \mathbf{R} \to \mathfrak{S}$ defined as $\varphi(a+ub+u^2c+u^3d) = a+ub+u^2c$ for $a, b, c, d \in \mathbb{F}_q$ be a ring homomorphism. Suppose, $\mathbf{C}$ be a cyclic code over $\mathbf{R}$. Then we can extend this map to $\psi : \mathbf{C} \to \mathfrak{S}[\varkappa]/\langle \varkappa^n - 1\rangle$ such that $\psi(c_0 + c_1\varkappa + \cdots + c_{n-1}\varkappa^{n-1}) = \varphi(c_0) + \varphi(c_1)\varkappa + \cdots + \varphi(c_{n-1})\varkappa^{n-1}$, where $c_i \in \mathbf{R}$. The $ker\psi = \{u^3l(\varkappa)|l(\varkappa) \in \mathbb{F}_q[\varkappa]\}$. Let $J = \{l(\varkappa) \in \mathbb{F}_q[\varkappa]|u^3l(\varkappa) \in ker\psi\}$. Then, J is a cyclic code over $\mathbb{F}_q$ and hence $J = \langle a_3(\varkappa)\rangle$ and $ker\psi = \langle u^3a_3(\varkappa)\rangle$; also, $\psi(\mathbf{C})$ is cyclic code over $\mathfrak{S}$ and hence $\mathbf{C}$ is given by the following results:

Theorem 3.1 [7, Theorem 3.6] *Consider that $\mathbf{C}$ is a cyclic code over $\mathbf{R}$ of length n. Then,*
$\mathbf{C} = \langle g(\varkappa) + up_1(\varkappa) + u^2p_2(\varkappa) + u^3p_3(\varkappa), ua_1(\varkappa) + u^2q_1(\varkappa) + u^3q_2(\varkappa), u^2a_2(\varkappa) + u^3l_1(\varkappa), u^3a_3(\varkappa)\rangle$ *with*

$a_3(\varkappa)|a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)|(\varkappa^n - 1)(mod\ q)$, $a_1(\varkappa)|p_1(\varkappa)(\frac{\varkappa^n-1}{g(\varkappa)})$, $a_2(\varkappa)|q_1(\varkappa)(\frac{\varkappa^n-1}{a_1(\varkappa)})$ *and*

$a_2(\varkappa)|p_2(\varkappa)(\frac{\varkappa^n-1}{g(\varkappa)})(\frac{\varkappa^n-1}{a_1(\varkappa)})$, $a_3(\varkappa)|l_1(\varkappa)(\frac{\varkappa^n-1}{a_2(\varkappa)})$, $a_3(\varkappa)|q_2(\varkappa)(\frac{\varkappa^n-1}{q_1(\varkappa)})(\frac{\varkappa^n-1}{a_1(\varkappa)})$ *and*

$a_3(\varkappa)|p_3(\varkappa)(\frac{\varkappa^n-1}{g(\varkappa)})(\frac{\varkappa^n-1}{a_2(\varkappa)})(\frac{\varkappa^n-1}{a_1(\varkappa)})$. *Moreover,* $deg(p_3(\varkappa)) < deg(a_3(\varkappa)), deg(q_2(\varkappa)) \leq deg(a_3(\varkappa))$, $deg(l_1(\varkappa)) < deg(a_3(\varkappa))$, $deg(p_2(\varkappa)) < deg(a_2(\varkappa))$, $deg(q_1(\varkappa)) < deg(a_2(\varkappa)), deg(p_1(\varkappa)) < deg(a_1(\varkappa))$.

Lemma 3.1 [7, Lemma 3.4] *Let* $\mathbf{C} = \langle g(\varkappa)+up_1(\varkappa)+u^2p_2(\varkappa)+u^3p_3(\varkappa), ua_1(\varkappa)+u^2q_1(\varkappa)+u^3q_2(\varkappa), u^2a_2(\varkappa)+u^3l_1(\varkappa), u^3a_3(\varkappa)\rangle$ *over* $\mathbf{R}$ *and* $a_3(\varkappa) = g(\varkappa)$. *Then* $\mathbf{C} = \langle g(\varkappa)+up_1(\varkappa)+u^2p_2(\varkappa)+u^3p_3(\varkappa)\rangle$ *and* $(g(\varkappa)+up_1(\varkappa)+u^2p_2(\varkappa)+u^3p_3(\varkappa))|\varkappa^n - 1$ *in* $\mathbf{R}$.

Lemma 3.2 [7, Lemma 3.5] *If n is relatively prime to q, then $\mathbf{C}$ can be written as*

$$\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle = \langle g(\varkappa) + ua_1(\varkappa) + u^2a_2(\varkappa) + u^3a_3(\varkappa)\rangle.$$

Lemma 3.3 *Suppose $f_i(\varkappa), g_i(\varkappa) \in \mathbb{F}_q[\varkappa]$ for $i = 1, 2, 3, 4$ if*

$$f_1(\varkappa) + uf_2(\varkappa) + u^2f_3(\varkappa) + u^3f_4(\varkappa) = g_1(\varkappa) + ug_2(\varkappa) + u^2g_3(\varkappa) + u^3g_4(\varkappa),$$

then $f_i(\varkappa) = g_i(\varkappa)$ for $i = 1, 2, 3, 4$.

Definition 3.1 *Suppose that $\mathbf{C}$ is a cyclic code over $\mathbf{R}$. Define a subcode of C as*

$$\mathbf{C}_{u^3} = \{k(\varkappa)|\ u^3k(\varkappa) \in \mathbf{C}\}.$$

Theorem 3.2 *Consider $\mathbf{C} = \langle g(\varkappa) + up_1(\varkappa) + u^2p_2(\varkappa) + u^3p_3(\varkappa), ua_1(\varkappa) + u^2q_1(\varkappa) + u^3q_2(\varkappa), u^2a_2(\varkappa) + u^3l_1(\varkappa), u^3a_3(\varkappa)\rangle$ be a cyclic code over R with conditions in Theorem 3.1. Then, $\mathbf{C}_{u^3} = \langle a_3(\varkappa)\rangle$ and $w_H(\mathbf{C}) = w_H(\mathbf{C}_{u^3})$.*

Proof: Since $u^3a_3(\varkappa) \in \mathbf{C}$, we have $\langle a_3(\varkappa)\rangle \subseteq \mathbf{C}_{u^3}$.

Conversely, suppose $k(\varkappa) \in \mathbf{C}_{u^3}$. Then, $u^3k(\varkappa) \in \mathbf{C}$ and hence we must have polynomials $b(\varkappa), c(\varkappa), d(\varkappa)$ and $e(\varkappa)$ in $\mathbb{F}_q[\varkappa]$ such that

$$u^3k(\varkappa) = u^3b(\varkappa)g(\varkappa) + u^3c(\varkappa)a_1(\varkappa) + u^3d(\varkappa)a_2(\varkappa) + u^3e(\varkappa)a_3(\varkappa).$$

Since $a_3(\varkappa)$ divides $a_2(\varkappa), a_1(\varkappa)$ and $g(\varkappa)$, we have, $k(\varkappa) = t(\varkappa)a_3(\varkappa)$ for some $t(\varkappa) \in \mathbb{F}_q[\varkappa]$. This means $\mathbf{C}_{u^3} \subseteq \langle a_3(\varkappa)\rangle$ hence, $\mathbf{C}_{u^3} = \langle a_3(\varkappa)\rangle$.

Further $u^3\mathbf{C}$ is a subcode of $\mathbf{C}$ with $w_H(u^3\mathbf{C}) \leq w_H(\mathbf{C})$ it is sufficient to focus on the subcode $u^3\mathbf{C}$ to compute hamming weight of of $\mathbf{C}$. Since, $u^3\mathbf{C} = \langle u^3a_3(\varkappa)\rangle$, thus $w_H(\mathbf{C}) = w_H(\mathbf{C}_{u^3})$. □

Lemma 3.4 [2, Lemma 4] *Let $f(\varkappa), g(\varkappa)$ be two polynomials in $R[\varkappa]$ with $deg(g(\varkappa)) \leq deg(f(\varkappa))$. Then,*

(i) $(f(\varkappa)g(\varkappa))^* = f^*(\varkappa)g^*(\varkappa)$,

(ii) $(f(\varkappa) + g(\varkappa))^* = f^*(\varkappa) + \varkappa^{deg(f(\varkappa))-deg(g(\varkappa))}g^*(\varkappa)$.

Lemma 3.5 *Let $f_i(\varkappa) \in \mathbb{F}_q[\varkappa]$ for $i = 1,2,3,4$. Assume that, $deg(f_1(\varkappa)) = r$, $deg(f_2(\varkappa)) = s, deg(f_3(\varkappa)) = t, deg(f_4(\varkappa)) = v$ where $r > max\{s,t,v\}$. Then*

$$(f_1(\varkappa) + uf_2(\varkappa) + u^2f_3(\varkappa) + u^3f_4(\varkappa))^* = f_1^*(\varkappa) + u\varkappa^{r-s}f_2^*(\varkappa) + u^2\varkappa^{r-t}f_3^*(\varkappa) + u^3\varkappa^{r-v}f_4^*(\varkappa).$$

Proof: Without loss of generality, we may assume that $s \geq t \geq v$. From the Lemma 3.4, we deduce that

$$\begin{aligned}(f_1(\varkappa) + uf_2(\varkappa) + u^2f_3(\varkappa) + u^3f_4(\varkappa))^* =& f_1^*(\varkappa) + \varkappa^{r-s}(uf_2(\varkappa) + u^2f_3(\varkappa) + u^3f_4(\varkappa))^* \\ =& f_1^*(\varkappa) + \varkappa^{r-s}(uf_2^*(\varkappa) + \varkappa^{s-t}(u^2f_3(\varkappa) + u^3f_4(\varkappa))^*) \\ =& f_1^*(\varkappa) + u\varkappa^{r-s}uf_2^*(\varkappa) + \varkappa^{r-t}(u^2f_3^*(\varkappa) + \varkappa^{t-v}u^3f_4^*(\varkappa)) \\ =& f_1^*(\varkappa) + u\varkappa^{r-s}uf_2^*(\varkappa) + \varkappa^{r-t}u^2f_3^*(\varkappa) + \varkappa^{r-v}u^3f_4^*(\varkappa).\end{aligned}$$

□

4. Reversible Codes

In this section, we give necessary and sufficient condition for a cyclic code $\mathbf{C}$ over $\mathbf{R}$ to be reversible when the length of the code is coprime with q.

Definition 4.1 *For a linear code $\mathbf{C}$ over a ring $\mathbf{R}$ and length n, if the reverse of every codeword is again in $\mathbf{C}$, then $\mathbf{C}$ is said to be reversible i.e., $Y^r \in \mathbf{C}$ for all $Y \in \mathbf{C}$.*

For any polynomial $f(\varkappa) = f_0 + f_1\varkappa + \cdots + f_k\varkappa^k$ with $f_k \neq 0$, $f^*(\varkappa) = \varkappa^k f(1/\varkappa) = f_k + f_{k-1}\varkappa + \cdots + f_0\varkappa^k$ is its reciprocal polynomial. Clearly, $deg(f^*(\varkappa)) \leq deg(f(\varkappa))$ and if $f_0 \neq 0$, then $deg(f^*(\varkappa)) = deg(f(\varkappa))$. If $f^*(\varkappa) = f(\varkappa)$, then $f(\varkappa)$ is said to be self-reciprocal. It is easy to check that if $f(\varkappa), g(\varkappa)$ are two polynomials such that $f(\varkappa)|g(\varkappa)$, then $f^*(\varkappa)|g^*(\varkappa)$. Also, let R be a commutative ring and $f(\varkappa) = f_0 + f_1\varkappa + \cdots + f_s\varkappa^s$ be a polynomial in $R[\varkappa]/\langle\varkappa^n - 1\rangle$. Then the reverse of $f(\varkappa)$ is defined as polynomial $f(\varkappa)^r = f_s\varkappa^{n-s-1} + \cdots + f_1\varkappa^{n-2} + f_0\varkappa^{n-1}$.

Lemma 4.1 *[14] Suppose that R is a commutative ring and $f(\varkappa) \in R[\varkappa]/\langle\varkappa^n - 1\rangle$ with $deg(f(\varkappa)) = m$. Then,*

(1) $\varkappa^{m+1}f(\varkappa)^r = f^*(\varkappa)$ *in* $R[\varkappa]/\langle\varkappa^n - 1\rangle$,

(2) $\varkappa^{n-m-1}f^*(\varkappa) = f(\varkappa)^r$ *in* $R[\varkappa]/\langle\varkappa^n - 1\rangle$.

Proposition 4.1 *Let $\mathbf{C}$ be a cyclic code over a commutative ring R with length n and $f(\varkappa) \in R[\varkappa]/\langle\varkappa^n - 1\rangle$. Then, $f(\varkappa)^r \in \mathbf{C}$ iff $f^*(\varkappa) \in \mathbf{C}$.*

Proof: The proof is straightforward using the Lemma 4.1. □

Corollary 4.1 *Suppose that $\mathbf{C}$ is a cyclic code over R. Then, $\mathbf{C}$ is reversible if and only if $f^*(\varkappa) \in \mathbf{C}$ for all $f(\varkappa) \in \mathbf{C}$.*

Lemma 4.2 *[15] The cyclic code over $\mathbb{F}_q$ generated by the monic polynomial $f(\varkappa)$ is reversible if and only if $f(\varkappa)$ is self-reciprocal.*

Lemma 4.3 *[19] Let $\mathfrak{C} = \langle f(\varkappa) + um(\varkappa), ua(\varkappa)\rangle$ be a reversible cyclic code over $\mathfrak{R}$. Then $\langle f(\varkappa)\rangle$ and $\langle a(\varkappa)\rangle$ are reversible cyclic codes over $\mathbb{F}_q$.*

Lemma 4.4 *[20] Let D be a reversible cyclic code of length n over $\mathfrak{S}$ and $\Theta : D \to \mathfrak{R}[\varkappa]/\langle\varkappa^n - 1\rangle$ be a homomorphism as defined earlier in Section 3. Then $\Theta(D)$ is also a reversible cyclic code over $\mathfrak{R}$.*

Lemma 4.5 *Let $D = \langle g(\varkappa) + um_1(\varkappa) + u^2m_2(\varkappa), ua_1(\varkappa) + u^2q_1(\varkappa), u^2a_2(\varkappa)\rangle$ be a reversible cyclic code over $\mathfrak{S}$. Then $\langle g(\varkappa)\rangle$, $\langle a_1(\varkappa)\rangle$ and $\langle a_2(\varkappa)\rangle$ are reversible cyclic codes over $\mathbb{F}_q$.*

Proof: Let $\Theta : D \to \Re[\varkappa]/\langle \varkappa^n - 1\rangle$ be the homomorphism defined in Section 3. Then, from Lemma 4.4, $\Theta(D)$ is a reversible cyclic code and from Lemma 4.3, $\langle g(\varkappa)\rangle$ and $\langle a_1(\varkappa)\rangle$ are reversible cyclic codes over $\mathbb{F}_q$. From the discussion in the previous section, we have $ker\Theta = \{u^2t(\varkappa)|t(\varkappa) \in \mathbb{F}_q[\varkappa]\}$ and $I = \{t(\varkappa) \in \mathbb{F}_q[\varkappa]|u^2t(\varkappa) \in ker\Theta\} = \langle a_2(\varkappa)\rangle$. Let $t(\varkappa) = t_0 + t_1\varkappa + \cdots + t_{n-1}\varkappa^{n-1} \in I$. Then, we have $t(\varkappa) \in \mathbb{F}_q[\varkappa]$ is a polynomial in D. Since D is reversible $t^*(\varkappa) \in D$ and $u^2t^*(\varkappa) \in ker\Theta$, this means $t^*(\varkappa) \in I$. Therefore, from Corollary 4.1, $\langle a_2(\varkappa)\rangle$ is reversible. □

Theorem 4.1 *Let* $\mathbf{C}$ *be a reversible cyclic code over* $\mathbf{R}$ *and* $\psi : \mathbf{C} \to \mathfrak{S}[\varkappa]/\langle \varkappa^n - 1\rangle$ *be the ring homomorphism defined in Sec 3. Then* $\psi(\mathbf{C})$ *is also a reversible cyclic code.*

Proof: Let $c = (c_0, c_1, \ldots, c_{n-1}) \in C$. Then, from the definition of ψ in Section 3, we have

$$\psi(c) = (\varphi(c_0), \varphi(c_1), \ldots, \varphi(c_{n-1})) \in \psi(\mathbf{C}).$$

Since $\mathbf{C}$ is reversible, $c^r = (c_{n-1}, c_{n-2}, \ldots, c_0) \in C$. Let

$$\begin{aligned}(\psi(c))^r =&(\varphi(c_0), \varphi(c_1), \ldots, \varphi(c_{n-1}))^r\\ =&(\varphi(c_{n-1}), \varphi(c_{n-2}), \ldots, \varphi(c_0))\\ =&\psi(c_{n-1}, c_{n-2}, \ldots, c_0) \in \psi(\mathbf{C}).\end{aligned}$$

Hence, $\psi(\mathbf{C})$ is a reversible cyclic code. □

Theorem 4.2 *Let* $\mathbf{C} = \langle g(\varkappa) + up_1(\varkappa) + u^2p_2(\varkappa) + u^3p_3(\varkappa), ua_1(\varkappa) + u^2q_1(\varkappa) + u^3q_2(\varkappa), u^2a_2(\varkappa) + u^3l_1(\varkappa), u^3a_3(\varkappa)\rangle$ *be a reversible cyclic code of length* n *over* $\mathbf{R}$. *Then* $\langle g(\varkappa)\rangle$, $\langle a_1(\varkappa)\rangle$, $\langle a_2(\varkappa)\rangle$ *and* $\langle a_3(\varkappa)\rangle$ *are also reversible cyclic codes over* $\mathbb{F}_q$.

Proof: Consider the homomorphism $\psi : \mathbf{C} \to \mathfrak{S}[\varkappa]/\langle \varkappa^n - 1\rangle$ in Section 3. Then, from Theorem 4.1, $\psi(\mathbf{C})$ is also a reversible cyclic code and from Lemma 4.5, $\langle g(\varkappa)\rangle$, $\langle a_1(\varkappa)\rangle$ and $\langle a_2(\varkappa)\rangle$ are reversible cyclic codes over $\mathbb{F}_q$. From the discussion in the previous section, we have $ker\psi = \{u^3l(\varkappa)|l(\varkappa) \in \mathbb{F}_q[\varkappa]\}$ and $J = \{l(\varkappa) \in \mathbb{F}_q[\varkappa]|u^3l(\varkappa) \in ker\psi\} = \langle a_3(\varkappa)\rangle$. Let $l(\varkappa) = l_0 + l_1\varkappa + \cdots + l_{n-1}\varkappa^{n-1} \in J$. Then $l(\varkappa) \in \mathbb{F}_q[\varkappa]$ is a polynomial in $\mathbf{C}$. Since $\mathbf{C}$ is reversible $l^*(\varkappa) \in C$ and $u^3l^*(\varkappa) \in ker\psi$, we have $l^*(\varkappa) \in J$. Therefore, from Corollary 4.1, $\langle a_3(\varkappa)\rangle$ is reversible. □

Theorem 4.3 *Let* $\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle$ *be a cyclic code of length* n *over* $\mathbf{R}$, *where* $(n, q) = 1$, $g(\varkappa), a_i(\varkappa) \in \mathbb{F}_q[\varkappa]/\langle \varkappa^n - 1\rangle$ *and* $a_3(\varkappa)|a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)$. *Then* $\mathbf{C}$ *is reversible cyclic code if and only if* $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$ *and* $a_3(\varkappa)$ *are self-reciprocal polynomials.*

Proof: Let $\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle$ be a cyclic code of length n over $\mathbf{R}$, where $(n, q) = 1$. Then by Lemma 4.2 and Theorem 4.2, $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$ and $a_3(\varkappa)$ are self-reciprocal polynomials.

Conversely suppose that $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$ and $a_3(\varkappa)$ are self-reciprocal polynomials. Then for any $c(\varkappa) \in \mathbf{C}$, there exist $f_1(\varkappa), f_2(\varkappa), f_3(\varkappa)$ and $f_4(\varkappa) \in \mathbb{F}_q[\varkappa]$ such that

$$c(\varkappa) = f_1(\varkappa)g(\varkappa) + uf_2(\varkappa)a_1(\varkappa) + u^2f_3(\varkappa)a_2(\varkappa) + u^3f_4(\varkappa)a_3(\varkappa).$$

Now, from Lemma 3.4 and 3.5, we have

$$\begin{aligned}c^*(\varkappa) =&(f_1(\varkappa)g(\varkappa) + uf_2(\varkappa)a_1(\varkappa) + u^2f_3(\varkappa)a_2(\varkappa) + u^3f_4(\varkappa)a_3(\varkappa))^*\\ =&f_1^*(\varkappa)g^*(\varkappa) + u\varkappa^if_2^*(\varkappa)a_1^*(\varkappa) + u^2\varkappa^jf_3^*(\varkappa)a_2^*(\varkappa) + u^3\varkappa^kf_4^*(\varkappa)a_3^*(\varkappa),\end{aligned}$$

where $i = deg(g(\varkappa)) - deg(a_1(\varkappa)), j = deg(g(\varkappa)) - deg(a_2(\varkappa))$ and $k = deg(g(\varkappa)) - deg(a_3(\varkappa))$. Since $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$ and $a_3(\varkappa)$ self-reciprocal, we have

$$c^*(\varkappa) = f_1^*(\varkappa)g(\varkappa) + u\varkappa^if_2^*(\varkappa)a_1(\varkappa) + u^2\varkappa^jf_3^*(\varkappa)a_2(\varkappa) + u^3\varkappa^kf_4^*(\varkappa)a_3(\varkappa).$$

This implies that $c^*(\varkappa) \in \mathbf{C}$, hence $\mathbf{C}$ is a reversible cyclic code. □

5. Dual of Cyclic Code Over R

In this Section, we find the set of generators for dual cyclic codes of length n over the ring $\mathbf{R}$ and then find the condition for dual cyclic codes to be reversible when the length of the code is coprime to q i.e., $(n, q) = 1$.

Consider two vectors, $X = x_0x_1 \dots x_n$ and $Y = y_1y_2 \dots y_n$ in $\mathbf{R}^n$. $\langle X, Y\rangle_E = x_0y_0 + x_1y_1 + \cdots + x_ny_n$ denotes the Euclidean inner product in $\mathbf{R}^n$. With respect to the Euclidean inner product, two vectors X and Y are orthogonal if $\langle X, Y\rangle_E = 0$. Similarly, $\langle X, Y\rangle_H = x_0\bar{y_0} + x_1\bar{y_1} + \cdots + x_n\bar{y_n}$ is the Hermitian inner product and two vectors X and Y are orthogonal if $\langle X, Y\rangle_H = 0$ with respect to the Hermitian inner product. The dual code $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined as

$$\mathbf{C}^\perp = \{X \in \mathbf{R}^n | \langle X, Y\rangle_E = 0 \text{ for all } \ Y \in \mathbf{C}\}.$$

Likewise, the Hermitian dual code $\mathbf{C}^{\perp H}$ of $\mathbf{C}$ is defined as

$$\mathbf{C}^{\perp H} = \{X \in \mathbf{R}^n | \langle X, Y\rangle_H = 0 \text{ for all } \ Y \in \mathbf{C}\}.$$

Now suppose that $\mathcal{K}$ is an ideal in the quotient ring $\mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle$, then the annihilator of $\mathcal{K}$ in $\mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle$ is defined as

$$\mathcal{A}(\mathcal{K}) = \{f(\varkappa) \in \mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle | g(\varkappa)f(\varkappa) = 0 \ \forall g(\varkappa) \in \mathcal{K}\}.$$

It is easy to verify that $\mathcal{A}(\mathcal{K})$ itself is an ideal in $\mathbf{R}[\varkappa]/\langle \varkappa^n - 1\rangle$. Further, if $\mathcal{K}$ generates a cyclic code $\mathbf{C}$ of length n over $\mathbf{R}$, then the dual code of $\mathbf{C}$ is given by

$$\mathbf{C}^\perp = \mathcal{A}^*(\mathcal{K}) = \{h^*(\varkappa) | h(\varkappa) \in \mathcal{A}(\mathcal{K})\},$$

where $h^*(\varkappa)$ is the reciprocal polynomial of $h(\varkappa)$.
Next theorem gives the annihilator of cyclic code $\mathbf{C}$ of length n over $\mathbf{R}$, when $(n, q) = 1$.

Theorem 5.1 *Let $\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle$ be a cyclic code of length n over $\mathbf{R}$, where $(n, q) = 1$, $g(\varkappa), a_i(\varkappa) \in \mathbb{F}_q[\varkappa]/\langle \varkappa^n - 1\rangle$ and $a_3(\varkappa)|a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)$. Then, annihilator of $\mathbf{C}$ is given as*

$$\mathcal{A}(\mathbf{C}) = \left\langle \frac{\varkappa^n - 1}{a_3(\varkappa)}, u\frac{\varkappa^n - 1}{a_2(\varkappa)}, u^2\frac{\varkappa^n - 1}{a_1(\varkappa)}, u^3\frac{\varkappa^n - 1}{g(\varkappa)} \right\rangle.$$

Proof: Let us suppose that

$$\mathcal{M} = \left\langle \frac{\varkappa^n - 1}{a_3(\varkappa)}, u\frac{\varkappa^n - 1}{a_2(\varkappa)}, u^2\frac{\varkappa^n - 1}{a_1(\varkappa)}, u^3\frac{\varkappa^n - 1}{g(\varkappa)} \right\rangle.$$

Since $g(\varkappa) + ua_1(\varkappa) + u^2a_2(\varkappa) + u^3a_3(\varkappa) \in \mathbf{C}$, we get

$$\frac{\varkappa^n - 1}{a_3(\varkappa)}\big(g(\varkappa) + ua_1(\varkappa) + u^2a_2(\varkappa) + u^3a_3(\varkappa)\big) = 0.$$

Therefore,

$$\frac{\varkappa^n - 1}{a_3(\varkappa)} \in \mathcal{A}(\mathbf{C})$$

and hence,

$$\left\langle \frac{\varkappa^n - 1}{a_3(\varkappa)} \right\rangle \in \mathcal{A}(\mathbf{C}).$$

Likewise, $\left\langle \frac{\varkappa^n-1}{a_2(\varkappa)} \right\rangle, \left\langle \frac{\varkappa^n-1}{a_1(\varkappa)} \right\rangle$ and $\left\langle \frac{\varkappa^n-1}{g(\varkappa)} \right\rangle \in \mathcal{A}(\mathbf{C})$.

This means that $\left\langle \frac{\varkappa^n-1}{a_3(\varkappa)}, u\frac{\varkappa^n-1}{a_2(\varkappa)}, u^2\frac{\varkappa^n-1}{a_1(\varkappa)}, u^3\frac{\varkappa^n-1}{g(\varkappa)} \right\rangle \in \mathcal{A}(\mathbf{C})$, that is, $\mathcal{M} \in \mathcal{A}(\mathbf{C})$.
Since, $\mathcal{A}(\mathbf{C})$ itself is a cyclic code over $\mathbf{R}$ of length n, where $(n, q) = 1$, we have

$$\mathcal{A}(\mathbf{C}) = \langle l_1(\varkappa), ul_2(\varkappa), u^2l_3(\varkappa), u^3l_4(\varkappa)\rangle,$$

where $l_i(\varkappa) \in \mathbb{F}_q[\varkappa]/\langle \varkappa^n - 1\rangle$ and $l_4(\varkappa)|l_3(\varkappa)|l_2(\varkappa)|l_1(\varkappa)$. Since $u^3 a_3(\varkappa) \in \mathbf{C}$ and $l_1(\varkappa) \in \mathcal{A}(\mathbf{C})$, we have,

$$u^3 a_3(\varkappa) \cdot l_1(\varkappa) \equiv 0 \bmod (\varkappa^n - 1).$$

This means $l_1(\varkappa) = \frac{\varkappa^n-1}{a_3(\varkappa)}\alpha(\varkappa)$, for some polynomial $\alpha(\varkappa) \in \mathbb{F}_q[\varkappa]$, this means $l_1(\varkappa) \in \left\langle \frac{\varkappa^n-1}{a_3(\varkappa)} \right\rangle$. Similarly, $l_2(\varkappa) \in \left\langle \frac{\varkappa^n-1}{a_2(\varkappa)} \right\rangle$, $l_3(\varkappa) \in \left\langle \frac{\varkappa^n-1}{a_1(\varkappa)} \right\rangle$ and $l_4(\varkappa) \in \left\langle \frac{\varkappa^n-1}{g(\varkappa)} \right\rangle$, ultimately $\mathcal{A}(\mathbf{C}) \subseteq \mathcal{M}$. Hence,

$$\mathcal{A}(\mathbf{C}) = \left\langle \frac{\varkappa^n-1}{a_3(\varkappa)}, u\frac{\varkappa^n-1}{a_2(\varkappa)}, u^2\frac{\varkappa^n-1}{a_1(\varkappa)}, u^3\frac{\varkappa^n-1}{g(\varkappa)} \right\rangle.$$

□

Corollary 5.1 *Let $\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle$ be a cyclic code of length n over $\mathbf{R}$, where $(n, q) = 1$, $g(\varkappa), a_i(\varkappa) \in \mathbb{F}_q[\varkappa]/\langle \varkappa^n - 1\rangle$ and $a_3(\varkappa)|a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)$. Then,*

$$\mathbf{C}^{\perp} = \left\langle \left(\frac{\varkappa^n-1}{a_3(\varkappa)}\right)^*, u\left(\frac{\varkappa^n-1}{a_2(\varkappa)}\right)^*, u^2\left(\frac{\varkappa^n-1}{a_1(\varkappa)}\right)^*, u^3\left(\frac{\varkappa^n-1}{g(\varkappa)}\right)^* \right\rangle.$$

Theorem 5.2 *Let $\mathbf{C} = \langle g(\varkappa), ua_1(\varkappa), u^2a_2(\varkappa), u^3a_3(\varkappa)\rangle$ be a reversible cyclic code of length n over $\mathbf{R}$, where $(n, q) = 1$, $g(\varkappa), a_i(\varkappa) \in \mathbb{F}_q[\varkappa]/\langle \varkappa^n - 1\rangle$ and $a_3(\varkappa)|a_2(\varkappa)|a_1(\varkappa)|g(\varkappa)$. Then $\mathbf{C}^{\perp}$ is a revesible cyclic code over $\mathbf{R}$.*

Proof: Since $\mathbf{C}$ is a reversible cyclic code over $\mathbf{R}$ of length n with $(n, q) = 1$, from Theorem 4.3, $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$, and $a_3(\varkappa)$ are self-reciprocal polynomials. Consider $\frac{\varkappa^n-1}{a_3(\varkappa)} = \lambda_1(\varkappa)$, $\frac{\varkappa^n-1}{a_2(\varkappa)} = \lambda_2(\varkappa)$, $\frac{\varkappa^n-1}{a_1(\varkappa)} = \lambda_3(\varkappa)$ and $\frac{\varkappa^n-1}{g(\varkappa)} = \lambda_4(\varkappa)$. Therefore $\frac{(\varkappa^n-1)^*}{(a_3(\varkappa))^*} = \lambda_1^*(\varkappa)$, $\frac{(\varkappa^n-1)^*}{(a_2(\varkappa))^*} = \lambda_2^*(\varkappa)$, $\frac{(\varkappa^n-1)^*}{(a_1(\varkappa))^*} = \lambda_3^*(\varkappa)$ and $\frac{(\varkappa^n-1)^*}{(g(\varkappa))^*} = \lambda_4^*(\varkappa)$. Since $g(\varkappa), a_1(\varkappa), a_2(\varkappa)$, and $a_3(\varkappa)$ are self-reciprocal polynomials, we have $\frac{-(\varkappa^n-1)}{a_3(\varkappa)} = \lambda_1^*(\varkappa)$, $\frac{-(\varkappa^n-1)}{a_2(\varkappa)} = \lambda_2^*(\varkappa)$, $\frac{-(\varkappa^n-1)}{a_1(\varkappa)} = \lambda_3^*(\varkappa)$ and $\frac{-(\varkappa^n-1)}{(g(\varkappa))} = \lambda_4^*(\varkappa)$. Thus, $\lambda_1^*(\varkappa) = -\lambda_1(\varkappa)$, $\lambda_2^*(\varkappa) = -\lambda_2(\varkappa)$, $\lambda_3^*(\varkappa) = -\lambda_3(\varkappa)$ and $\lambda_4^*(\varkappa) = -\lambda_4(\varkappa)$. Let $\hat{c}(\varkappa) \in \mathbf{C}^{\perp}$. Then there exist polynomials $q_1(\varkappa), q_2(\varkappa), q_3(\varkappa), q_4(\varkappa) \in \mathbf{R}[\varkappa]$ such that

$$\hat{c}(\varkappa) = \left(\frac{\varkappa^n-1}{a_3(\varkappa)}\right)^* q_1(\varkappa) + u\left(\frac{\varkappa^n-1}{a_2(\varkappa)}\right)^* q_2(\varkappa) + u^2\left(\frac{\varkappa^n-1}{a_1(\varkappa)}\right)^* q_3(\varkappa) + u^3\left(\frac{\varkappa^n-1}{g(\varkappa)}\right)^* q_4(\varkappa)$$

and

$$\begin{aligned}
\hat{c}(\varkappa)^* =& \left[\left(\frac{\varkappa^n-1}{a_3(\varkappa)}\right)^* q_1(\varkappa) + u\left(\frac{\varkappa^n-1}{a_2(\varkappa)}\right)^* q_2(\varkappa) + u^2\left(\frac{\varkappa^n-1}{a_1(\varkappa)}\right)^* q_3(\varkappa) + u^3\left(\frac{\varkappa^n-1}{g(\varkappa)}\right)^* q_4(\varkappa)\right]^* \\
=& [\lambda_1^* q_1(\varkappa) + u\lambda_2^* q_2(\varkappa) + u^2\lambda_3^* q_3(\varkappa) + u^3\lambda_4^* q_4(\varkappa)]^* \\
=& [-\lambda_1 q_1(\varkappa) + u(-\lambda_2) q_2(\varkappa) + u^2(-\lambda_3) q_3(\varkappa) + u^3(-\lambda_4) q_4(\varkappa)]^* \\
=& -\lambda_1^* q_1^*(\varkappa) - u\varkappa^{j_1}\lambda_2^* q_2^*(\varkappa) - u^2\varkappa^{j_2}\lambda_3^* q_3(\varkappa) - u^3\varkappa^{j_3}\lambda_4^* q_4^*(\varkappa) \\
=& \lambda_1^* f_1(\varkappa) + u\lambda_2^*\varkappa^{j_1} f_2(\varkappa) + u^2\lambda_3^*\varkappa^{j_2} f(\varkappa) + u^3\lambda_4^*\varkappa^{j_3} f_4(\varkappa),
\end{aligned}$$

where $j_1 = deg(\lambda_1(\varkappa)) - deg(\lambda_2(\varkappa))$, $j_2 = deg(\lambda_1(\varkappa)) - deg(\lambda_3(\varkappa))$ and $j_3 = deg(\lambda_1(\varkappa)) - deg(\lambda_4(\varkappa))$ and $f_i(\varkappa) = -q_i^*(\varkappa)$ for $i = 1, 2, 3, 4$. This implies that $\hat{c}(\varkappa)^* \in \mathbf{C}^{\perp}$. Therefore $\mathbf{C}^{\perp}$ is a reversible cyclic code. □

6. Reversible-Complement Codes

This section discusses the reversible-complement cyclic codes of odd length over the ring $\mathbf{R_1} = \mathbb{F}_4 + u\mathbb{F}_4 + u^2\mathbb{F}_4 + u^3\mathbb{F}_4$, where $u^4 = 0$. Necessary and sufficient conditions are given for cyclic codes over $\mathbf{R_1}$ to be reversible-complement codes. Moreover, reversibility problem is also addressed in this Section.

Deoxyribonucleic acid or DNA, is a type of nucleic acid found in all living organisms, carrying the genetic information called genes. DNA sequences contain two long polymers, which are called strands, composed of four nucleotide bases, namely Adenine(A), Guanine(G), Thymine(T) and Cytosine(C). Two strands are so twisted, forming a double helix, running in opposite directions to each other and joined together by hydrogen bonds between nucleotide bases. This attachment follows the Watson-Crick Complement rule. A pairs with T and G pairs with C, as per the Watson-Crick Complement rule. A and G are called the complements of T and C, respectively, and vice versa. The Complement of a base X is denoted by $\bar{X}$. $\bar{G} = C$, for instance, is the complement of G. Thus, if $X = AGATT$ is a DNA strand, then $\bar{X} = TCTAA$ would be its complement. According to the Watson-Crick Complement rule, a DNA strand $Y = y_1y_2 \dots y_l$ will pair up with $Y^{rc} = \bar{y_l}\bar{y_{l-1}} \dots \bar{y_2}\bar{y_1}$, the reverse-complement of Y. For instance, a DNA strand $5' - TCTAAGT - 3'$ will pair up with $3' - ACTTAGA - 5'$. A DNA code $\mathcal{C}$ with minimum distance d may satisfy some or all the following constraints:

1. **The Hamming constraint:** $d_H(s_1, s_2) \geq d$, where $s_1, s_2 \in \mathcal{C}$ and $s_1 \neq s_2$.
2. **The Reverse constraint:** $d_H(s_1, s_2^r) \geq d$, where $s_1, s_2 \in \mathcal{C}$ and s_2^r is the reverse of s_2.
3. **The Reverse-Complement constraint:** $d_H(s_1^r, s_2^c) \geq d$, where $s_1, s_2 \in \mathcal{C}$ and s_2^c is the complement of s_2.
4. **The GC-content constraint:** Each codeword $s \in \mathcal{C}$ has the same number of G or C.

First three constraints ensure to reduce the probability of non-specific hybridization. Fixed GC-content constraint ensures the similar melting point.

In Table 1, we provide a relationship between the elements of $\mathbf{R_1}$ and DNA 4-mers such that for any $a \in \mathbf{R_1}$ the complement of a denoted by $\overline{a}$ is obtained as $\overline{a} = a + u^3$. Notice from Table 1 for any $x \in \mathbf{R_1}$ the corresponding DNA string is $b_1b_2b_3b_4$ (say), then the reverse $b_4b_3b_2b_1$ of $b_1b_2b_3b_4$ is obtained by multiplying x with $1 + u^2$ that is $(1 + u^2)x = b_4b_3b_2b_1$. Observing this fact, we try to solve the reversibility problem. To explain reversibility problem, let us consider that $\eta = (\eta_1, \eta_2, \eta_3)$ be any codeword over $\mathbf{R_1}$ and the DNA correspondence of η is $ATAGGCGGCCCT$ such that η_1 corresponds to $ATAG$, η_2 corresponds to $GCGG$ and η_3 corresponds to $CCCT$. Now, the reverse of η is given by $\eta^r = (\eta_3, \eta_2, \eta_1)$, then the DNA correspondence of η^r is given by $CCCTGCGGATAG$ but $CCCTGCGGATAG$ is not the reverse of $ATAGGCGGCCCT$, the reverse of $ATAGGCGGCCCT$ is $TCCCGGCGGATA$. To solve this reversibility problem the following lemma is useful.

Lemma 6.1 *Let $d = (d_0, d_1, \dots, d_{n-1})$ be some codeword over $\mathbf{R_1}$ of length n and $X = b_1b_2 \cdots b_{4n-1}b_{4n}$ is the DNA correspondence of d. Then, the DNA sequence corresponding to the codeword $(1 + u^2)d^r$ is the reverse of X that is $(1 + u^2)d^r = b_{4n}b_{4n-1} \cdots b_2b_1$.*

Definition 6.1 *A linear code $\mathbf{C}$ over a ring R and length n is called complement if the complement of every codeword is again in $\mathbf{C}$ that is $Y^c \in \mathbf{C}$, for all $Y \in \mathbf{C}$ and reversible-complement if $Y^{rc} \in \mathbf{C}$, for all $Y \in \mathbf{C}$.*

Definition 6.2 *[23] For a linear code $\mathbf{C}$ of length n over R, $\mathbf{C}$ is said to be cyclic DNA code if $\mathbf{C}$ is a reversible-complement cyclic code.*

Lemma 6.2 *The following are true:*

(1) *For any $a, b, c, d \in \mathbb{F}_q$, $\overline{a + ub + u^2c + u^3d} = \bar{a} + ub + u^2c + u^3d$,*

(2) *For any $x \in \mathbf{R_1}$, $x + \bar{x} = u^3$,*

(3) *For any $x, y \in \mathbf{R_1}$, $\overline{x + y} = \bar{x} + \bar{y} + u^3$.*

Proof: (1), (2) are straightforward from the Table 1.
(3) Let $x, y \in \mathbf{R_1}$. Then, $\overline{x + y} = x + y + u^3 = (\bar{x} + u^3) + (\bar{y} + u^3) + u^3 = \bar{x} + \bar{y} + u^3$. □

Theorem 6.1 *Let $g(\varkappa) \in R[\varkappa]$. Then $g(\varkappa)^{rc} + u^3(\frac{\varkappa^n-1}{\varkappa-1}) = g(\varkappa)^r$.*

Proof: Assume that

$$\begin{aligned}
g(\varkappa) =& g_0 + g_1\varkappa + ... + g_{t-1}\varkappa^{t-1} + g_t\varkappa^t \\
g(\varkappa)^{rc} =& \bar{0} + \bar{0}\varkappa + ... + \bar{0}\varkappa^{n-t-2} + \bar{g}_t\varkappa^{n-t-1} + \bar{g_{t-1}}\varkappa^{n-t} + ... + \bar{g}_1\varkappa^{n-2} + \bar{g}_0\varkappa^{n-1} \\
=& u^3 + u^3\varkappa + ... + u^3\varkappa^{n-t-2} + (g_t + u^3)\varkappa^{n-t-1} + (g_{t-1} + u^3)\varkappa^{n-t} + ... + (g_1 + u^3)\varkappa^{n-2} \\
& + (g_0 + u^3)\varkappa^{n-1} \\
=& g_t\varkappa^{n-t-1} + g_{t-1}\varkappa^{n-t} + ... + g_1\varkappa^{n-2} + g_0\varkappa^{n-1} + u^3(1 + \varkappa + ... + \varkappa^{n-t-2} + \varkappa^{n-t-1} + ... + \varkappa^{n-1}) \\
=& g(\varkappa)^r + u^3(\frac{\varkappa^n - 1}{\varkappa - 1}).
\end{aligned}$$

This implies that $g(\varkappa)^{rc} + u^3(\frac{\varkappa^n-1}{\varkappa-1}) = g(\varkappa)^r$. □

Theorem 6.2 *Suppose that $\mathbf{C}$ is a cyclic code with odd length n over $\mathbf{R_1}$. Then, $\mathbf{C}$ is a reversible-complement code iff $\mathbf{C}$ is reversible and $u^3(\frac{\varkappa^n-1}{\varkappa-1}) \in \mathbf{C}$.*

Proof: Consider C to be a reversible-complement code. Since,

$$\begin{aligned}
0 + 0\varkappa + ... + 0\varkappa^{n-1} \in& \mathbf{C}, \text{ we have} \\
\bar{0} + \bar{0}\varkappa + ... + \bar{0}\varkappa^{n-1} =& u^3(1 + \varkappa + ... + \varkappa^{n-1}) \in \mathbf{C} \\
=& u^3(\frac{\varkappa^n - 1}{\varkappa - 1}) \in \mathbf{C}.
\end{aligned}$$

Now, from Theorem 6.1 the result holds. □

For any two cyclic codes $\mathbf{C}_1$ and $\mathbf{C}_2$ over $\mathbf{R_1}$, consider

$$\mathbf{C_1} + \mathbf{C_2} = \{c_1 + c_2 | c_1 \in \mathbf{C_1}, c_2 \in \mathbf{C_2}\}$$

and

$$\mathbf{C_1} \cap \mathbf{C_2} = \{c | c \in \mathbf{C_1} \text{ and } c \in \mathbf{C_2}\}.$$

Then we have the following result:

Theorem 6.3 *Let $\mathbf{C}_1, \mathbf{C}_2$ be two cyclic DNA codes of length n over R. Then, $\mathbf{C}_1 + \mathbf{C}_2$ and $\mathbf{C}_1 \cap \mathbf{C}_2$ are cyclic DNA codes.*

Proof: Let $f(\varkappa) = f_0 + f_1\varkappa + ...f_s\varkappa^s \in \mathbf{C}_1, g(\varkappa) = g_0 + g_1\varkappa + ...g_t\varkappa^t \in \mathbf{C}_2$ be two arbitrary codewords. We may assume that $s \geq t$ without loss of generality, then

$$\begin{aligned}
(f(\varkappa) + g(\varkappa))^{rc} =& ((f_0 + g_0) + (f_1 + g_1)\varkappa + ...(f_t + g_t)\varkappa^t + f_{t+1} + ... + f_s\varkappa^s)^{rc} \\
=& \bar{0} + \bar{0}\varkappa + ... + \bar{0}\varkappa^{n-s-2} + \bar{f}_s\varkappa^{n-s-1} + \bar{f_{s-1}}\varkappa^{n-s} + ... + \bar{f_{t+1}}\varkappa^{n-t-2} + \\
& (\overline{f_t + g_t})\varkappa^{n-t-1} + ... + (\overline{f_1 + g_1})\varkappa^{n-2} + (\overline{f_0 + g_0})\varkappa^{n-1} \\
=& f(\varkappa)^{rc} + g_t\varkappa^{n-t-1} + g_{t-1}\varkappa^{n-t} + ... + g_1\varkappa^{n-2} + g_0\varkappa^{n-1} \\
=& f(\varkappa)^{rc} + g(\varkappa)^r \\
=& f(\varkappa)^{rc} + g(\varkappa)^{rc} + u^3(\frac{\varkappa^n - 1}{\varkappa - 1}) \in \mathbf{C}_1 + \mathbf{C}_2.
\end{aligned}$$

Which proves that $\mathbf{C}_1 + \mathbf{C}_2$ is reversible-compliment. Clearly $\mathbf{C}_1 \cap \mathbf{C}_2$ is reversible-complement. □

7. Examples

We present some examples of reversible and reversible-complement cyclic codes over different rings, some of them are optimal according to online database Grassl http://www.codetables.de/.

Example 7.1 *Let $\mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha = \alpha^2\}$ be the finite field of four elements. Consider $\varkappa^7 - 1 \in \mathbb{F}_4[\varkappa]$ then factorization of $\varkappa^7 - 1$ over $\mathbb{F}_4$ is*

$$\varkappa^7 - 1 = (\varkappa - 1)(\varkappa^3 + \varkappa + 1)(\varkappa^3 + \varkappa^2 + 1).$$

Take $g(\varkappa) = 1+\varkappa+\varkappa^2+\varkappa^3+\varkappa^4+\varkappa^5+\varkappa^6$ then, $g(\varkappa)|\varkappa^7-1$ and $g(\varkappa)$ is a self-reciprocal polynomial. Then Theorem 6.2 implies that $\mathbf{C} = \langle g(\varkappa)\rangle$ is a reversible-complement cyclic code of length 7 as $u^3\left(\frac{\varkappa^7-1}{\varkappa-1}\right) \in \mathbf{C}$. The ζ image of $\mathbf{C}$ is a DNA code of length 28 and size 256, given in table 2.

Example 7.2 *let $n = 17$, consider the factorization over $\mathbb{F}_2$*

$$\begin{aligned}\varkappa^{17} - 1 =& (\varkappa + 1)(\varkappa^8 + \varkappa^5 + \varkappa^4 + \varkappa^3 + 1)(\varkappa^8 + \varkappa^7 + \varkappa^6 + \varkappa^4 + \varkappa^2 + \varkappa + 1)\\ =& f_1 f_2 f_3.\end{aligned}$$

Then f_1, f_2, f_3 are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over $\mathbf{R_2} = \mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2 + u^3\mathbb{F}_2$ listed in table 3.

Table 3 *Reversible cyclic codes of length 17 over $\mathbf{R_2}$*

Non zero generator polynomial	*Parameter*	*Remark*
f_1	$[17, 16, 2]$	*Optimal*
f_2	$[17, 9, 5]$	*Optimal*
f_3	$[17, 9, 5]$	*Optimal*
$f_1 f_2$	$[17, 8, 6]$	*Optimal*
$f_1 f_3$	$[17, 8, 6]$	*Optimal*
$f_2 f_3$	$[17, 1, 17]$	*Optimal*
f_3, uf_3	$[17, 9, 5]$	*Optimal*
$f_1 f_3, u^3 f_3$	$[17, 9, 5]$	*Optimal*

Example 7.3 *Let $\mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha = \alpha^2\}$ be the finite field of four elements. Consider $\varkappa^{17} - 1 \in \mathbb{F}_4[\varkappa]$ then factorization of $\varkappa^{17} - 1$ over $\mathbb{F}_4$ is*

$$\begin{aligned}\varkappa^{17} - 1 =& (\varkappa - 1)(\varkappa^4 + \varkappa^3 + \alpha\varkappa^2 + \varkappa + 1)(\varkappa^4 + \varkappa^3 + \alpha^2\varkappa^2 + \varkappa + 1)\\ & (\varkappa^4 + \alpha\varkappa^3 + \varkappa^2 + \alpha\varkappa + 1)(\varkappa^4 + \alpha^2\varkappa^3 + \varkappa^2 + \alpha^2\varkappa + 1)\\ =& f_1 f_2 f_3 f_4 f_5.\end{aligned}$$

Then f_1, f_2, f_3, f_4 and f_5 are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over $\mathbf{R_4} = \mathbb{F}_4 + u\mathbb{F}_4 + u^2\mathbb{F}_4 + u^3\mathbb{F}_4$ listed in table 4.

Table 4 *Reversible cyclic codes of length 17 over $\mathbf{R_4}$*

Non zero generator polynomial	*Parameter*	*Remark*
f_2	$[17, 13, 4]$	*Optimal*
$f_1 f_2$	$[17, 12, 4]$	*Optimal*
$f_1 f_5$	$[17, 12, 4]$	*Optimal*
$f_1 f_2 f_5$	$[17, 8, 8]$	*Optimal*
$f_1 f_2 f_3 f_5$	$[17, 4, 12]$	*Optimal*
$f_1 f_2 f_3 f_5, u^3 f_2$	$[17, 13, 4]$	*Optimal*
$f_1 f_2 f_3 f_5, uf_1 f_2 f_5, u^2 f_1 f_5, u^3 f_5$	$[17, 13, 4]$	*Optimal*

Example 7.4 *let $n = 13$, consider the factorization over $\mathbb{F}_5$*

$$\begin{aligned}\varkappa^{13} - 1 =&(\varkappa + 4)(\varkappa^4 + \varkappa^3 + 4\varkappa^2 + \varkappa + 1)(\varkappa^4 + 2\varkappa^3 + \varkappa^2 + 2\varkappa + 1)(\varkappa^4 + 3\varkappa^3 + 3\varkappa + 1)\\ =&f_1 f_2 f_3 f_4.\end{aligned}$$

Then f_2, f_3, f_4 are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over $\mathbf{R_5} = \mathbb{F}_5 + u\mathbb{F}_5 + u^2\mathbb{F}_5 + u^3\mathbb{F}_5$ listed in table 5.

Table 5 *Reversible cyclic codes of length 13 over $\mathbf{R_5}$*

Non zero generator polynomial	*Parameter*	*Remark*
f_2	$[13, 9, 4]$	*Optimal*
f_3	$[13, 9, 4]$	*Optimal*
f_4	$[13, 9, 4]$	*Optimal*
$f_2 f_3$	$[13, 5, 7]$	*Optimal*
$f_2 f_3 f_4$	$[13, 1, 13]$	*Optimal*

Example 7.5 *let $n = 41$, consider the factorization over $\mathbb{F}_4$*

$$\begin{aligned}\varkappa^{41} - 1 =&(\varkappa + 1)(\varkappa^{10} + \alpha\varkappa^8 + \alpha^2\varkappa^7 + \varkappa^5 + \alpha^2\varkappa^3 + \alpha\varkappa^2 + 1)(\varkappa^{10} + \alpha^2\varkappa^8 + \alpha\varkappa^7 + \varkappa^5 + \alpha\varkappa^3 + \alpha^2\varkappa^2 + 1)\\ &(\varkappa^{10} + \alpha\varkappa^9 + \alpha\varkappa^8 + \alpha\varkappa^7 + \varkappa^6 + \alpha^2\varkappa^5 + \varkappa^4 + \alpha\varkappa^3 + \alpha\varkappa^2 + \alpha\varkappa + 1)\\ &(\varkappa^{10} + \alpha^2\varkappa^9 + \alpha^2\varkappa^8 + \alpha^2\varkappa^7 + \varkappa^6 + \alpha\varkappa^5 + \varkappa^4 + \alpha^2\varkappa^3 + \alpha^2\varkappa^2 + \alpha^2\varkappa + 1)\\ =&f_1 f_2 f_3 f_4 f_5.\end{aligned}$$

Then $f_1 f_2, f_3, f_4$ and f_5 are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over $\mathbf{R_4}$ listed in table 6.

Table 6 *Reversible cyclic codes of length 41 over $\mathbf{R_4}$*

Non zero generator polynomial	*Parameter*	*Remark*
f_1	$[41, 40, 2]$	*Optimal*
f_2	$[41, 31, 6]$	*Optimal*
f_5	$[41, 31, 6]$	*Optimal*
$f_1 f_2$	$[41, 30, 7]$	*Optimal*
$f_1 f_5$	$[41, 30, 7]$	*Optimal*
$f_1 f_2 f_3$	$[41, 20, 10]$	
$f_2 f_3 f_4$	$[41, 11, 20]$	*Optimal*
$f_1 f_2, u^3 f_2$	$[41, 31, 6]$	*Optimal*
$f_1 f_2 f_3, u f_1 f_2 f_3, u^2 f_1 f_2 f_3, u^3 f_2$	$[41, 31, 6]$	*Optimal*
$f_2 f_3 f_4 f_5$	$[41, 1, 41]$	*Optimal*

Example 7.6 *Let $\mathbb{F}_8$ be the finite field of order 8 and ω be a primitive element of $\mathbb{F}_8$. Consider $\varkappa^9 - 1 \in \mathbb{F}_8[\varkappa]$ then factorization of $\varkappa^9 - 1$ over $\mathbb{F}_8$ is*

$$\begin{aligned}\varkappa^9 - 1 =&(\varkappa - 1)(\varkappa^2 + \varkappa + 1)(\varkappa^2 + \omega\varkappa + 1)(\varkappa^2 + \omega^2\varkappa + 1)(\varkappa^2 + \omega^4\varkappa + 1)\\ =&f_1 f_2 f_3 f_4 f_5.\end{aligned}$$

Then f_1, f_2, f_3, f_4 and f_5 are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over $\mathbf{R_8} = \mathbb{F}_8 + u\mathbb{F}_8 + u^2\mathbb{F}_8 + u^3\mathbb{F}_8$ listed in table 7.

Table 7 *Reversible cyclic codes of length* 9 *over* $\mathbf{R_8}$

Non zero generator polynomial	*Parameter*	*Remark*
f_1	$[9, 8, 2]$	*Optimal*
f_3	$[9, 7, 3]$	*Optimal*
f_1f_3	$[9, 6, 4]$	*Optimal*
f_2f_3	$[9, 5, 5]$	*Optimal*
$f_1f_3f_5$	$[9, 4, 6]$	*Optimal*
$f_2f_4f_5$	$[9, 3, 7]$	*Optimal*
$f_1f_2f_4f_5$	$[9, 2, 8]$	*Optimal*
$f_2f_3f_4f_5$	$[9, 1, 9]$	*Optimal*
f_1f_5, u^3f_5	$[9, 7, 3]$	*Optimal*
$f_1f_2f_3, u^3f_2f_3$	$[9, 5, 5]$	*Optimal*

Example 7.7 *let* $n = 14$, *consider the factorization over* $\mathbb{F}_3$

$$\begin{aligned}\varkappa^{14} - 1 =&(\varkappa + 1)(\varkappa + 2)(\varkappa^6 + \varkappa^5 + \varkappa^4 + \varkappa^3 + \varkappa^2 + \varkappa + 1)(\varkappa^6 + 2\varkappa^5 + \varkappa^4 + 2\varkappa^3 + \varkappa^2 + 2\varkappa + 1)\\ =&f_1f_2f_3f_4.\end{aligned}$$

Then f_1, f_3, f_4 *are self-reciprocal polynomials. Then using Theorem 4.3, we constructed reversible cyclic codes over* $\mathbf{R_3} = \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3 + u^3\mathbb{F}_3$ *listed in table 8.*

Table 8 *Reversible cyclic codes of length* 14 *over* $\mathbf{R_3}$

Non zero generator polynomial	*Parameter*	*Remark*
f_1	$[14, 13, 2]$	*Optimal*
f_4	$[14, 8, 2]$	
f_1f_3	$[14, 7, 4]$	
f_1f_4	$[14, 7, 2]$	
f_3f_4	$[14, 2, 7]$	
$f_1f_3f_4$,	$[14, 1, 14]$	*Optimal*

Table 2 DNA Code of length 28

AAAAAAAAAAAAAAAAAAAAAAAAAAAA	AAATAAATAAATAAATAAATAAATAAAT
AAAGAAAGAAAGAAAGAAAGAAAGAAAG	AAACAAACAAACAAACAAACAAACAAAC
ATCAATCAATCAATCAATCAATCAATCA	ATCTATCTATCTATCTATCTATCTATCT
ATCGATCGATCGATCGATCGATCGATCG	ACGGACGGACGGACGGACGGACGGACGG
ATCCATCCATCCATCCATCCATCCATCC	ACGAACGAACGAACGAACGAACGAACGA
ACGCACGCACGCACGCACGCACGCACGC	AATAAATAAATAAATAAATAAATAAATA
AATTAATTAATTAATTAATTAATTAATT	AATGAATGAATGAATGAATGAATGAATG
AATCAATCAATCAATCAATCAATCAATC	AGAAAGAAAGAAAGAAAGAAAGAAAGAA
ATTAATTAATTAATTAATTAATTAATTA	TAAATAAATAAATAAATAAATAAATAAA
AGAGAGAGAGAGAGAGAGAGAGAGAGAG	AGACAGACAGACAGACAGACAGACAGAC
ACCTACCTACCTACCTACCTACCTACCT	ACCGACCGACCGACCGACCGACCGACCG
ACCCACCCACCCACCCACCCACCCACCC	TTGATTGATTGATTGATTGATTGATTGA
TTGTTTGTTTGTTTGTTTGTTTGTTTGT	TTGGTTGGTTGGTTGGTTGGTTGGTTGG
TTGCTTGCTTGCTTGCTTGCTTGCTTGC	AAGTAAGTAAGTAAGTAAGTAAGTAAGT
ATATATATATATATATATATATATATAT	CATTCATTCATTCATTCATTCATTCATT
AAGGAAGGAAGGAAGGAAGGAAGGAAGG	CAGGCAGGCAGGCAGGCAGGCAGGCAGG
ACCAACCAACCAACCAACCAACCAACCA	CAGCCAGCCAGCCAGCCAGCCAGCCAGC
GAAAGAAAGAAAGAAAGAAAGAAAGAAA	CAGACAGACAGACAGACAGACAGACAGA
GTGCGTGCGTGCGTGCGTGCGTGCGTGC	CACCCACCCACCCACCCACCCACCCACC
GGTCGGTCGGTCGGTCGGTCGGTCGGTC	GAACGAACGAACGAACGAACGAACGAAC
TACATACATACATACATACATACATACA	GGTAGGTAGGTAGGTAGGTAGGTAGGTA

CACTCACTCACTCACTCACTCACTCACT	CTCACTCACTCACTCACTCACTCACTCA
ACGTACGTACGTACGTACGTACGTACGT	GCTCGCTCGCTCGCTCGCTCGCTCGCTC
GATTGATTGATTGATTGATTGATTGATT	CCTGCCTGCCTGCCTGCCTGCCTGCCTG
GGGGGGGGGGGGGGGGGGGGGGGGGGGG	CGACCGACCGACCGACCGACCGACCGAC
GAGAGAGAGAGAGAGAGAGAGAGAGAGA	CAAACAAACAAACAAACAAACAAACAAA
GAGGGAGGGAGGGAGGGAGGGAGGGAGG	CGCCCGCCCGCCCGCCCGCCCGCCCGCC
GATGGATGGATGGATGGATGGATGGATG	GCCCGCCCGCCCGCCCGCCCGCCCGCCC
CGTACGTACGTACGTACGTACGTACGTA	CACACACACACACACACACACACACACA
GATAGATAGATAGATAGATAGATAGATA	GACTGACTGACTGACTGACTGACTGACT
AGCTAGCTAGCTAGCTAGCTAGCTAGCT	GAGCGAGCGAGCGAGCGAGCGAGCGAGC
CCTTCCTTCCTTCCTTCCTTCCTTCCTT	TTCTTTCTTTCTTTCTTTCTTTCTTTCT
AGGAAGGAAGGAAGGAAGGAAGGAAGGA	CTGCCTGCCTGCCTGCCTGCCTGCCTGC
AAGCAAGCAAGCAAGCAAGCAAGCAAGC	AGTAAGTAAGTAAGTAAGTAAGTAAGTA
ACTAACTAACTAACTAACTAACTAACTA	GCCAGCCAGCCAGCCAGCCAGCCAGCCA
CTGGCTGGCTGGCTGGCTGGCTGGCTGG	CCCGCCCGCCCGCCCGCCCGCCCGCCCG
GGATGGATGGATGGATGGATGGATGGAT	CCAACCAACCAACCAACCAACCAACCAA
AGTGAGTGAGTGAGTGAGTGAGTGAGTG	AGTCAGTCAGTCAGTCAGTCAGTCAGTC
CATGCATGCATGCATGCATGCATGCATG	TAAGTAAGTAAGTAAGTAAGTAAGTAAG
TAACTAACTAACTAACTAACTAACTAAC	AGGGAGGGAGGGAGGGAGGGAGGGAGGG
CAACCAACCAACCAACCAACCAACCAAC	CGTCCGTCCGTCCGTCCGTCCGTCCGTC
AGGCAGGCAGGCAGGCAGGCAGGCAGGC	TATGTATGTATGTATGTATGTATGTATG
TCCATCCATCCATCCATCCATCCATCCA	TCTATCTATCTATCTATCTATCTATCTA
GTAGGTAGGTAGGTAGGTAGGTAGGTAG	CAAGCAAGCAAGCAAGCAAGCAAGCAAG
ACAAACAAACAAACAAACAAACAAACAA	TCGTTCGTTCGTTCGTTCGTTCGTTCGT
TATCTATCTATCTATCTATCTATCTATC	TGAGTGAGTGAGTGAGTGAGTGAGTGAG
CTAGCTAGCTAGCTAGCTAGCTAGCTAG	CTTACTTACTTACTTACTTACTTACTTA
TGACTGACTGACTGACTGACTGACTGAC	AGCGAGCGAGCGAGCGAGCGAGCGAGCG
CGGCCGGCCGGCCGGCCGGCCGGCCGGC	AGCCAGCCAGCCAGCCAGCCAGCCAGCC
CGAACGAACGAACGAACGAACGAACGAA	GTATGTATGTATGTATGTATGTATGTAT
CGTGCGTGCGTGCGTGCGTGCGTGCGTG	CCGCCCGCCCGCCCGCCCGCCCGCCCGC
GCTAGCTAGCTAGCTAGCTAGCTAGCTA	AGTTAGTTAGTTAGTTAGTTAGTTAGTT
TGTATGTATGTATGTATGTATGTATGTA	TGTGTGTGTGTGTGTGTGTGTGTGTGTG
GCGTGCGTGCGTGCGTGCGTGCGTGCGT	ACTTACTTACTTACTTACTTACTTACTT
CCGGCCGGCCGGCCGGCCGGCCGGCCGG	TGTCTGTCTGTCTGTCTGTCTGTCTGTC
GTTAGTTAGTTAGTTAGTTAGTTAGTTA	CGCTCGCTCGCTCGCTCGCTCGCTCGCT
GAAGGAAGGAAGGAAGGAAGGAAGGAAG	CCGACCGACCGACCGACCGACCGACCGA
CGGACGGACGGACGGACGGACGGACGGA	ATGAATGAATGAATGAATGAATGAATGA
GGAGGGAGGGAGGGAGGGAGGGAGGGAG	CCACCCACCCACCCACCCACCCACCCAC
CCCACCCACCCACCCACCCACCCACCCA	GGCAGGCAGGCAGGCAGGCAGGCAGGCA
TACGTACGTACGTACGTACGTACGTACG	TACCTACCTACCTACCTACCTACCTACC
GCAAGCAAGCAAGCAAGCAAGCAAGCAA	TATTTATTTATTTATTTATTTATTTATT
CGCGCGCGCGCGCGCGCGCGCGCGCGCG	GACAGACAGACAGACAGACAGACAGACA
GTCAGTCAGTCAGTCAGTCAGTCAGTCA	CCCTCCCTCCCTCCCTCCCTCCCTCCCT
TCCTTCCTTCCTTCCTTCCTTCCTTCCT	GACGGACGGACGGACGGACGGACGGACG
TTCGTTCGTTCGTTCGTTCGTTCGTTCG	TCATTCATTCATTCATTCATTCATTCAT
TGATTGATTGATTGATTGATTGATTGAT	CGGTCGGTCGGTCGGTCGGTCGGTCGGT
GACCGACCGACCGACCGACCGACCGACC	GGGCGGGCGGGCGGGCGGGCGGGCGGGC
CCTACCTACCTACCTACCTACCTACCTA	GGTTGGTTGGTTGGTTGGTTGGTTGGTT
TCACTCACTCACTCACTCACTCACTCAC	TCAGTCAGTCAGTCAGTCAGTCAGTCAG
GTACGTACGTACGTACGTACGTACGTAC	ATTCATTCATTCATTCATTCATTCATTC

ATTGATTGATTGATTGATTGATTGATTG	TCCCTCCCTCCCTCCCTCCCTCCCTCCC
GTTGGTTGGTTGGTTGGTTGGTTGGTTG	GCAGGCAGGCAGGCAGGCAGGCAGGCAG
TCCGTCCGTCCGTCCGTCCGTCCGTCCG	ATACATACATACATACATACATACATAC
AGGTAGGTAGGTAGGTAGGTAGGTAGGT	AGATAGATAGATAGATAGATAGATAGAT
CATCCATCCATCCATCCATCCATCCATC	GTTCGTTCGTTCGTTCGTTCGTTCGTTC
TGTTTGTTTGTTTGTTTGTTTGTTTGTT	AGCAAGCAAGCAAGCAAGCAAGCAAGCA
ATAGATAGATAGATAGATAGATAGATAG	ACTCACTCACTCACTCACTCACTCACTC
GATCGATCGATCGATCGATCGATCGATC	GAATGAATGAATGAATGAATGAATGAAT
ACTGACTGACTGACTGACTGACTGACTG	TCGCTCGCTCGCTCGCTCGCTCGCTCGC
GCCGGCCGGCCGGCCGGCCGGCCGGCCG	TCGGTCGGTCGGTCGGTCGGTCGGTCGG
GCTTGCTTGCTTGCTTGCTTGCTTGCTT	CATACATACATACATACATACATACATA
GCACGCACGCACGCACGCACGCACGCAC	GGCGGGCGGGCGGGCGGGCGGGCGGGCG
CGATCGATCGATCGATCGATCGATCGAT	TCAATCAATCAATCAATCAATCAATCAA
ACATACATACATACATACATACATACAT	ACACACACACACACACACACACACACAC
CGCACGCACGCACGCACGCACGCACGCA	TGAATGAATGAATGAATGAATGAATGAA
GGCCGGCCGGCCGGCCGGCCGGCCGGCC	ACAGACAGACAGACAGACAGACAGACAG
CAATCAATCAATCAATCAATCAATCAAT	GCGAGCGAGCGAGCGAGCGAGCGAGCGA
CTTCCTTCCTTCCTTCCTTCCTTCCTTC	GGCTGGCTGGCTGGCTGGCTGGCTGGCT
GCCTGCCTGCCTGCCTGCCTGCCTGCCT	TACTTACTTACTTACTTACTTACTTACT
CCTCCCTCCCTCCCTCCCTCCCTCCCTC	GGTGGGTGGGTGGGTGGGTGGGTGGGTG
GGGTGGGTGGGTGGGTGGGTGGGTGGGT	CCGTCCGTCCGTCCGTCCGTCCGTCCGT
ATGCATGCATGCATGCATGCATGCATGC	ATGGATGGATGGATGGATGGATGGATGG
CGTTCGTTCGTTCGTTCGTTCGTTCGTT	ATAAATAAATAAATAAATAAATAAATAA
GCGCGCGCGCGCGCGCGCGCGCGCGCGC	CTGTCTGTCTGTCTGTCTGTCTGTCTGT
CAGTCAGTCAGTCAGTCAGTCAGTCAGT	GGGAGGGAGGGAGGGAGGGAGGGAGGGA
TTTTTTTTTTTTTTTTTTTTTTTTTTTT	TTTATTTATTTATTTATTTATTTATTTA
TTTCTTTCTTTCTTTCTTTCTTTCTTTC	TTTGTTTGTTTGTTTGTTTGTTTGTTTG
TAGTTAGTTAGTTAGTTAGTTAGTTAGT	TAGATAGATAGATAGATAGATAGATAGA
TAGCTAGCTAGCTAGCTAGCTAGCTAGC	TGCCTGCCTGCCTGCCTGCCTGCCTGCC
TAGGTAGGTAGGTAGGTAGGTAGGTAGG	TGCTTGCTTGCTTGCTTGCTTGCTTGCT
TGCGTGCGTGCGTGCGTGCGTGCGTGCG	TTATTTATTTATTTATTTATTTATTTAT
TTAATTAATTAATTAATTAATTAATTAA	TTACTTACTTACTTACTTACTTACTTAC
TTAGTTAGTTAGTTAGTTAGTTAGTTAG	TCTTTCTTTCTTTCTTTCTTTCTTTCTT
TAATTAATTAATTAATTAATTAATTAAT	ATTTATTTATTTATTTATTTATTTATTT
TCTCTCTCTCTCTCTCTCTCTCTCTCTC	TCTGTCTGTCTGTCTGTCTGTCTGTCTG
TGGATGGATGGATGGATGGATGGATGGA	TGGCTGGCTGGCTGGCTGGCTGGCTGGC
TGGGTGGGTGGGTGGGTGGGTGGGTGGG	AACTAACTAACTAACTAACTAACTAACT
AACAAACAAACAAACAAACAAACAAACA	AACCAACCAACCAACCAACCAACCAACC
AACGAACGAACGAACGAACGAACGAACG	TTCATTCATTCATTCATTCATTCATTCA
TATATATATATATATATATATATATATA	GTAAGTAAGTAAGTAAGTAAGTAAGTAA
TTCCTTCCTTCCTTCCTTCCTTCCTTCC	GTCCGTCCGTCCGTCCGTCCGTCCGTCC
TGGTTGGTTGGTTGGTTGGTTGGTTGGT	GTCGGTCGGTCGGTCGGTCGGTCGGTCG
CTTTCTTTCTTTCTTTCTTTCTTTCTTT	GTCTGTCTGTCTGTCTGTCTGTCTGTCT
CACGCACGCACGCACGCACGCACGCACG	GTGGGTGGGTGGGTGGGTGGGTGGGTGG
CCAGCCAGCCAGCCAGCCAGCCAGCCAG	CTTGCTTGCTTGCTTGCTTGCTTGCTTG
ATGTATGTATGTATGTATGTATGTATGT	CCATCCATCCATCCATCCATCCATCCAT
GTGAGTGAGTGAGTGAGTGAGTGAGTGA	GAGTGAGTGAGTGAGTGAGTGAGTGAGT
TGCATGCATGCATGCATGCATGCATGCA	CGAGCGAGCGAGCGAGCGAGCGAGCGAG
CTAACTAACTAACTAACTAACTAACTAA	GGACGGACGGACGGACGGACGGACGGAC
CCCCCCCCCCCCCCCCCCCCCCCCCCCC	GCTGGCTGGCTGGCTGGCTGGCTGGCTG
CTCTCTCTCTCTCTCTCTCTCTCTCTCT	GTTTGTTTGTTTGTTTGTTTGTTTGTTT

CTCCCTCCCTCCCTCCCTCCCTCCCTCC	GCGGGCGGGCGGGCGGGCGGGCGGGCGG
CTACCTACCTACCTACCTACCTACCTAC	CGGGCGGGCGGGCGGGCGGGCGGGCGGG
GCATGCATGCATGCATGCATGCATGCAT	GTGTGTGTGTGTGTGTGTGTGTGTGTGT
CTATCTATCTATCTATCTATCTATCTAT	CTGACTGACTGACTGACTGACTGACTGA
TCGATCGATCGATCGATCGATCGATCGA	CTCGCTCGCTCGCTCGCTCGCTCGCTCG
GGAAGGAAGGAAGGAAGGAAGGAAGGAA	AAGAAAGAAAGAAAGAAAGAAAGAAAGA

8. Conclusion

We have studied cyclic code $\mathbf{C}$ of length n over the ring $\mathbf{R} = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + u^3\mathbb{F}_q, u^4 = 0$, and obtained the necessary and sufficient conditions for $\mathbf{C}$ to be reversible and reversible-complement cyclic code when $(n, q) = 1$. We also studied the dual $\mathbf{C}^\perp$ of the cyclic code $\mathbf{C}$ and obtained the generator polynomials of $\mathbf{C}^\perp$ in terms of the generator polynomials of $\mathbf{C}$, when the length n is coprime with q. Further we have given sufficient condition of $\mathbf{C}^\perp$ to be reversible. Moreover, cyclic DNA code are constructed as the images of reversible-complement codes over $\mathbf{R_1}$ by providing a bijection between $\mathbf{R_1}$ and the set of all DNA 4-mers. The reversibility problem is also addressed with the help of this bijection. Finally, some reversible optimal codes are also constructed, provided $(n, q) = 1$. However, for future study, obtaining the reversibility conditions for cyclic codes over $\mathbf{R}$ of arbitrary length is of interest.

References

1. Abualrub, T., Oehmke R., *On the generators of Z_4 cyclic codes of length 2^e*. IEEE Trans. Inform. Theory 49, 2126-2133 (2003).
2. Abualrub, T., Ghrayeb, A., Zeng, X., *Construction of cyclic codes over $GF(4)$ for DNA computing*. J. Franklin Inst. 343, 448-457 (2006).
3. Abualrub, T., Siap, I., *Cyclic codes over the rings $Z_2 + uZ_2$ and $Z_2 + uZ_2 + u^2Z_2$*. Des.Codes Cryptogr. 42, 273-287 (2007).
4. Adleman, L. M., *Molecular Computation of Solutions to Combinatorial Problems*. Science 266, 1021-1024 (1994).
5. Adleman, L. M., Rothmund, P. W. K., Roweis, S., Winfree, E., *On applying molecular Computation to the Data Encryption Standard*. J. Comput. Biol 6, 53-63 (1999).
6. Alali, A. S., Ashraf, M., Rehman, W., Mohammad G., Asim, M., *On reversibility problem in DNA bases over a class of rings*. Appl. Math. Sci. Eng. (2024). https://doi.org/10.1080/27690911.2024.2358213
7. Al-Ashker, M. M., Chen, J., *Cyclic codes of arbitrary length over $F_q + uF_q + u^2F_q + ... + u^{k-1}F_q$*. Palest. J. Math. 2, 72-80(2013).
8. Ashraf, M., Rehman, W., Mohammad G., Asim, M., *On reversible codes over a non-chain ring*. Comput. Appl. Math. 42, 1-26 (2023).
9. Dinh, H. Q., *Constacyclic codes of length p^s over $F_{p^m} + uF_{p^m}$*. J. Algebra 324, 940-950 (2010).
10. Dinh, H. Q., Ashraf, M., Rehman, W., Mohammad G., Asim, M., *On reversible DNA codes over the ring $\mathbf{Z_4}[u, v]/\langle u^2 - 2, uv - 2, v^2, 2u, 2v\rangle$ based on deletion distance*. Appl. Algebra Engrg. Comm. Comput. (2024). https://doi.org/10.1007/s00200-024-00661-7
11. Gaborit, P., King, O. D., *Linear constructions for DNA codes*. Theoret. Comput. Sci. 334, 99-113 (2005).
12. Liang, J., Wang, L., *On cyclic DNA codes over $Z_2 + uZ_2$*. J. Appl. Math. Comput. 51, 81-91 (2016).
13. Mansuripur, M., Khulbe, P. K., Kuebler, S. M., Perry, J. W., Giridhar, M. S., Erwin, J. K., Seong, K., Marder, S., Peyghambarian, N., *Information storage and retrieval using macromolecules as storage media*. In: Proc. SPIE, 5069, optical data storage, (2003).
14. Mostafanasab, H., Darani, A. Y., *On cyclic DNA codes over $F_2 + uF_2 + u^2F_2$*. Commun. Math. Stat. 9, 39-52 (2021).
15. Massey, J. L., *Reversible codes*. Inf. Control 7(3), 369–380 (1964).
16. Ouyang, Q., Kaplan, P. D., Liu, S., Libchaber, A., *DNA solution of the maximal clique problem*. Science 278, 446-449 (1997).
17. Oztas, E. S., Siap I., Yildiz, B., *Reversible codes and application to DNA*. In: Mathematical software ICMS: 4th international congress, seoul, South Korea, proceeding 4, August 5-9, Springer, Berlin, 124-128.
18. Pless, V. S., Qian, Z., *Cyclic codes and quadratic residue codes over Z_4*. IEEE Trans. Inform. Theory 42, 1594-1600 (1996).

19. Prakash, O., Patel, S., Yadav, S., *Reversible cyclic codes over some finite rings and their application to DNA codes.* Comput. Appl. Math. 40 1–17 (2021).

20. Rehman, N., Fareed, M. F., Azmi, M., *Construction of reversible cyclic codes over* $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 31(2), 155-176 (2023).

21. Siap, I., Abualrub, T., Ghrayeb, A., *Cyclic DNA codes over the ring* $F_2[u]/\langle u^2 - 1\rangle$ *based on the deletion distance.* J. Franklin Inst. 346, 731-740 (2009).

22. Yildiz, B., Aydin, N., *On cyclic codes over* $Z_4 + uZ_4$ *and their* Z_4*-images.* Int. J. Inf. Coding Theory 2, 226-237 (2014).

23. Yildiz, B., Siap, I., *Cyclic codes over* $F_2[u]/\langle u^4 - 1\rangle$ *and applications to DNA codes.* Comput. Math. Appl. 63, 1169-1176 (2012).

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