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Exploring Graph Energies of k-Copies of Complete Graph K_n with a Common Vertex

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ABSTRACT: In this study, we have calculated the energy, Seidel energy, and distance energy for k-copies of the complete graph K_n, where the graphs share a single common vertex. Additionally, we derived expressions for the Laplacian energy, Laplacian distance energy, and Laplacian Seidel energy for these graphs. To facilitate these computations, we also developed a Python code that generates the corresponding energy values.

Key Words: Complete graph, Energy, Seidel energy, distance energy, Laplacian energy, Laplacian Seidal energy, Laplacian distance energy.

Contents

1. Introduction

A graph is said to be simple if it does not possess directed, weighted, or multiple edges, and self-loops [11]. In this article, we consider only simple graphs. The concept of energy of a graph was introduced by I.Gutman [3] in the year 1978. Let G be a simple graph of order n, with vertex set V= $\{v_1, v_2, \cdots, v_n\}$ and size m. Its adjacency matrix $A(G) = (a_{ij})$ is a square symmetric matrix of order n whose $(i,j)^{th}$ element is defined as

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent.} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)$ be the eigenvalues of $A(G)$. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. that is,

$$E(G)=\sum_{i=1}^{n} |\lambda_i(G)| \ .$$

As $A(G)$ is real symmetric, the eigenvalues of G are real with sum equal to zero. The set of eigenvalues with their multiplicities is known as spectrum of a graph and it is denoted by $Spec(G)$. The theory of graph energy is nowadays a well elaborated field of applied mathematics and mathematical chemistry [14]. One should recall the concept of graph energy has a chemical origin and a chemical interpretation [12]. There are more than a thousand papers on graph energy and its variants [13]. Practically all these papers are concerned with simple graphs. For details on the mathematical aspects of the theory of graph energy see the papers [1,2,4,5] and the references cited there in.

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1.1. Distance energy

On addressing the loop-switching problem, R. L. Graham and H. O. Pollak [6] defined the distance matrix of a graph. The concept of distance energy was later introduced by Prof. G. Indulal and colleagues [8] in 2008.
Let d_{ij} represents the distance between the vertices v_i and v_j. The $n \times n$ matrix $D(G) = (d_{ij})$ is known as the distance matrix of G. The eigenvalues of $D(G)$ are referred to as the distance eigenvalues of G. Since $D(G)$ is a real symmetric matrix with a trace of zero, the distance eigenvalues are real, and their sum is also zero. The distance energy $E_D(G)$ of the graph G is defined as sum of the absolute value of these distance eigenvalues. i.e.,

$$E_D(G) = \sum_{i=1}^{n} |\mu_i|.$$

Bounds on distance energy can be seen in [7].

1.2. Seidel energy

The Seidel matrix of G is an $n \times n$ matrix denoted by $S(G) := (s_{ij})$, where the entries are defined as follows:

$$s_{ij} = \begin{cases} -1 & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } v_i v_j \notin E(G) \\ 0 & \text{if } v_i = v_j \end{cases}$$

The Seidel eigenvalues of the graph G are the eigenvalues of the Seidel matrix $S(G)$. Since $S(G)$ is real and symmetric, its eigenvalues are real numbers. The Seidel energy $SE(G)$ [9] of the graph G is defined as the sum of the absolute values of these eigenvalues:

$$\text{i,e.} \qquad SE(G) : = \sum_{i=1}^{n} |\lambda_i|.$$

1.3. Laplacian energy

In 2006, I. Gutman and B. Zhou [10] introduced the concept of Laplacian energy for a graph G. The Laplacian matrix of G, denoted by $L = (L_{ij})$, is an $n \times n$ square matrix, where the elements are defined as follows:

$$L_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i & \text{if } i = j \end{cases}$$

where d_i represents the degree of vertex v_i. The eigenvalues of the matrix L are called Laplacian eigenvalues. The Laplacian energy $LE(G)$ of G is defined as the sum of the absolute differences between each Laplacian eigenvalue and $\frac{2m}{n}$, where m is the number of edges in the graph:

$$\text{i,e.} \qquad \text{LE(G)} = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|.$$

2. Energy and Laplacian energy of k-copies of K_n sharing a single common vertex

Consider a graph G obtained by taking $k-$copies$(k \geq 2)$ of complete graph $K_n(n \geq 2)$ which shares a common vertex. Let $v_{11}, v_{12}, v_{13} \cdots, v_{1n}$ be the vertices of first copy of k_n, $v_{21}, v_{22}, v_{23}, \cdots, v_{2n}$ be the vertices of second copy of k_n and $v_{k1}, v_{k2}, v_{k3}, \cdots, v_{kn}$ be the vertices of k^{th} copy of k_n. i,e $v_{i1}, v_{i2}, v_{i3}, \cdots, v_{in}$ are the vertices of i^{th} copy of K_n where i=$1, 2, 3, \cdots k$. Let G be the graph obtained by sharing the common vertex v_{i1} and $\forall\ i = 1, 2, 3 \cdots, k$. i,e the vertices $v_{11} = v_{21} = v_{31} = \cdots v_{k1}$. Hence the total number of vertices is $k(n-1)+1$ and the total number of edges is $\frac{nk(n-1)+1}{2}$.

Theorem 2.1 *The Energy of a graph G obtained by taking k-copies($k \geq 2$) of complete graph $K_n(n \geq 2)$ sharing a single common vertex is $2nk - 4k - n + 2 + \sqrt{n^2 + 4nk - 4n - 4k + 4}$.*

Proof: The Adjacency matrix of G is given by

$$A(G)=\left(\begin{array}{c|cccc|cccc|c|cccc}
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
1 & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
1 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 0
\end{array}\right)_{[k(n-1)+1]\times[k(n-1)+1]}$$

$$= \begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where, $A=\begin{pmatrix} 0 \end{pmatrix}_{1\times 1}$, $B=\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times(n-1)}$, $B^T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(n-1)\times 1}$,

$C=\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$ and $D=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is, $(-1)^{nk-k+1}(\lambda+1)^{nk-2k}[\lambda-(n-2)]^{k-1}[\lambda^2-(n-2)\lambda-(n-1)k] = 0$
Spectrum of G is,

$$\text{Spec}(G)=\begin{pmatrix} -1 & n-2 & \dfrac{(n-2) \pm \sqrt{n^2+4nk-4n-4k+4}}{2} \\ nk-2k & k-1 & 1 \end{pmatrix}$$

Energy of G is,

$$E(G)=|-1|\,(nk-2k)+|(n-2)|\,(k-1)+\left|\frac{(n-2)+\sqrt{n^2+4nk-4n-4k+4}}{2}\right|(1)$$
$$+\left|\frac{(n-2)-\sqrt{n^2+4nk-4n-4k+4}}{2}\right|(1)$$
$$= 2nk - 4k - n + 2 + \sqrt{n^2+4nk-4n-4k+4}.$$

∴ The energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is $2nk - 4k - n + 2 + \sqrt{n^2 + 4nk - 4n - 4k + 4}$.

□

Python code to generate the energy of the graph G obtained by taking $k-$copies of the complete graph K_n by sharing a common vertex.

```
import numpy as np
from numpy.polynomial.polynomial import Polynomial
def construct_adjacency_matrix(n, k):
# Submatrices defining the block structure
A = np.array([[0]])  # Single-element matrix for the top-left block
B = np.ones((1, n - 1))  # Row matrix of ones (1 x (n-1))
BT = np.ones((n - 1, 1))  # Column matrix of ones ((n-1) x 1)
C = np.ones((n - 1, n - 1)) - np.eye(n - 1)  # Complete graph adjacency
D = np.zeros((n - 1, n - 1))  # Zero matrix for inter-block connections
# Constructing the first row: [A, B, B, ..., B]
first_row = np.hstack([A] + [B] * k)
# Constructing the remaining rows using submatrices BT, C, and D
block_rows = []
for block_row in range(k):
row_blocks = [BT]  # Each row starts with BT
for block_col in range(k):
if block_row == block_col:
row_blocks.append(C)  # Diagonal blocks are C
else:
row_blocks.append(D)  # Off-diagonal blocks are D
block_rows.append(np.hstack(row_blocks))  # Combine blocks for the current row
# Combine the first row and the block rows to form the complete matrix
adjacency_matrix = np.vstack([first_row] + block_rows)
return adjacency_matrix
def compute_characteristic_polynomial(matrix):
Returns the characteristic polynomial of 'matrix' with leading coefficient 1.
Coefficients are in descending powers of \lambda :
p(\lambda) = c[0] \lambda^n + c[1] \lambda^(n-1) + ... + c[n-1] \lambda + c[n]
# np.poly gives the coefficients of the polynomial whose roots
# are the eigenvalues of 'matrix'. Leading entry is always 1.
coeffs = np.poly(matrix)  # leading term is 1
# Round to nearest integer (assuming an integral adjacency matrix)
coeffs = [int(round(c)) for c in coeffs]
return coeffs
def format_polynomial(coefficients):
terms = []
degree = len(coefficients) - 1
for i, coef in enumerate(coefficients):
power = degree - i
if coef == 0:
continue
sign_str = f"+ {coef}" if coef > 0 and i > 0 else str(coef)
if power == 0:
# Constant term
terms.append(sign_str)
elif power == 1:
# Linear term (\lambda)
terms.append(f"{sign_str}\lambda")
else:
# Higher powers
terms.append(f"{sign_str}\lambda^{power}")
# Join everything together, then handle any '+ -' => '- ' replacements for tidiness
poly_str = " ".join(terms).replace("+ -", "- ")
# Remove leading plus sign if present
if poly_str.startswith("+ "):
poly_str = poly_str[2:]
return poly_str
```

```
# User input for graph parameters
n = int(input("Enter the number of vertices per complete graph (K_n): "))
k = int(input("Enter the number of disjoint copies of K_n: "))
# Generate the adjacency matrix for (K_n)^k
adj_matrix = construct_adjacency_matrix(n, k)
# Display the adjacency matrix and its order
print(f"\nAdjacency Matrix of (K_{n})^{k}:")
print(adj_matrix)
print(f"\nOrder of the matrix: {adj_matrix.shape}")
# Compute the characteristic polynomial of the adjacency matrix
char_poly_coefficients = compute_characteristic_polynomial(adj_matrix)
# Format the characteristic polynomial as a readable equation
formatted_char_poly = format_polynomial(char_poly_coefficients)
print(f"\nCharacteristic Polynomial of (K_{n})^{k}:")
print(f"P(\lambda) = {formatted_char_poly}")
# Display the eigenvalues
eigenvalues = np.linalg.eigvals(adj_matrix)
print(f"\nEigenvalues of the adjacency matrix (K_{n})^{k}:")
print(eigenvalues)
# Calculate and display the energy (sum of absolute eigenvalues)
energy = np.sum(np.abs(eigenvalues))
print(f"\nGraph Energy of (K_{n})^{k}:")
print(energy)
```

Example 2.1 *:The energy of the graph G obtained by taking two copies of complete graph K_4 sharing a common vertex is* $6+2\sqrt{7}$.

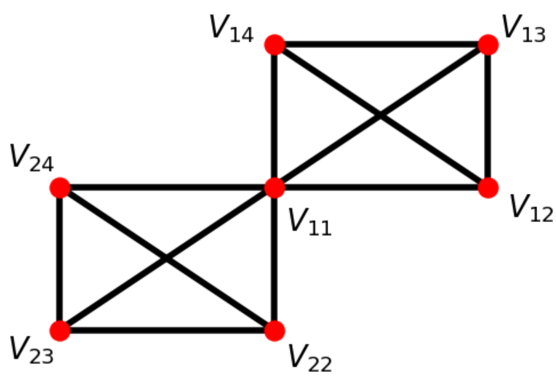

Figure 1: Two copies of complete graph K_4 sharing a common vertex

The Adjancency matrix is given by

$$A(K_4^2)=\left(\begin{array}{c|ccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)_{(7\times 7)}$$

Characteristic equation is, $-(\lambda+1)^4(\lambda-2)[\lambda^2-2\lambda-6]=0$
Spectrum of K_4^2 is,

$$\text{Spec}(K_4^2\ )=\begin{pmatrix} -1 & 2 & 1\pm\sqrt{7} \\ 4 & 1 & 1 \end{pmatrix}$$

Energy of K_4^2 is,
$E(K_4^2)=|-1|\,(4)+|2|\,(1)+\left|1+\sqrt{7}\right|(1)+\left|1-\sqrt{7}\right|(1)=6+2\sqrt{7}.$
$\therefore$ The energy of the graph G obtained by taking 2 copies of complete graph K_4 sharing a common vertex is $6+2\sqrt{7}$.

Theorem 2.2 *The Lapalacian energy of a graph G obtained by taking k-copies $(k\geq 2)$ of complete graph $K_n(n\geq 2)$ sharing a single common vertex is*

$$\frac{2n^2k^2-4nk^2+2k^2+2nk-4k+2}{nk-k+1}.$$

Proof: The Laplacian adjacency matrix is given by

$$A(G)=$$

$$\left(\begin{array}{c|cccc|cccc|c|cccc}
(n-1)k & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\ \hline
-1 & (n-1) & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & -1 & (n-1) & \cdots & -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & \cdots & (n-1) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & (n-1) & -1 & \cdots & -1 & \cdots & 0 & 0 & \cdots & 0 \\
-1 & 0 & 0 & \cdots & 0 & -1 & (n-1) & \cdots & -1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & (n-1) & \cdots & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & (n-1) & -1 & \cdots & -1 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & (n-1) & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & -1 & \cdots & (n-1)
\end{array}\right)$$

$$=\begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where, $A=\begin{pmatrix} (n-1)k \end{pmatrix}_{1\times 1}$, $B=\begin{pmatrix} -1 & -1 & \cdots & -1 \end{pmatrix}_{1\times(n-1)}$, $B^T=\begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}_{(n-1)\times 1}$,

$C=\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}_{(n-1)\times(n-1)}$ and $D=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is, $(-1)^{nk-k+1}\lambda(\lambda-n)^{nk-2k}(\lambda-1)^{k-1}[\lambda-(nk-k+1)]=0$
Laplacian spectrum of G is,
Number of edges m is $\frac{nk(n-1)}{2}$ and number of vertices n is $k(n-1)+1$

Then, $\frac{2m}{n}=\frac{n^2k-nk}{nk-k+1}$
$\text{LSpec(G)}=\begin{pmatrix} 0 & n & 1 & nk-k+1 \\ 1 & nk-2k & k-1 & 1 \end{pmatrix}$

Laplacian energy of G is,
$LE(G)=\left|0-\frac{n^2k-nk}{nk-k+1}\right|(1)+\left|n-\frac{n^2k-nk}{nk-k+1}\right|(nk-2k)+\left|1-\frac{n^2k-nk}{nk-k+1}\right|(k-1)+\left|(nk-k+1)-\frac{n^2k-nk}{nk-k+1}\right|(1)$
$=\left|\frac{-(n^2k-nk)}{nk-k+1}\right|(1)+\left|\frac{n}{nk-k+1}\right|(nk-2k)+\left|\frac{2nk-n^2k-k+1}{nk-k+1}\right|(k-1)+\left|\frac{n^2k^2+k^2-2nk^2-n^2k-2k+1}{nk-k+1}\right|(1)$
$=\frac{2n^2k^2+2k^2-4nk^2+2nk-4k+2}{nk-k+1}$
$\therefore$ The Laplacian energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is $\frac{2n^2k^2+2k^2-4nk^2+2nk-4k+2}{nk-k+1}$. □

Example 2.2 *:The Laplacian energy of the graph G obtained by taking two copies of complete graph K_4 sharing a common vertex is $\frac{82}{7}$.*

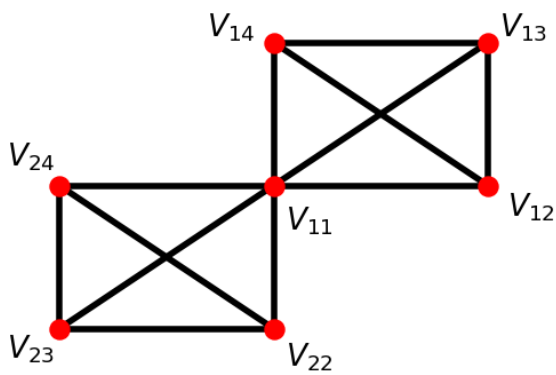

Figure 2: Two copies of complete graph K_4 sharing a common vertex

The Laplacian adjancency matrix of K_4^2 given by

$$A(K_4^2)=\left(\begin{array}{c|ccc|ccc} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 3 \end{array}\right)_{(7\times 7)}$$

Characteristic equation of K_4^2 is, $-\lambda(\lambda-1)(\lambda-7)(\lambda-4)^4=0$.
Laplacian spectrum of K_4^2 is,

$$\text{Spec}(K_4^2)=\begin{pmatrix} 0 & 1 & 4 & 7 \\ 1 & 1 & 4 & 1 \end{pmatrix}$$

Laplacian energy of K_4^2 is,
Number of edges m = 12 and number of vertices n = 7 then $\frac{2m}{n}=\frac{24}{7}$
$LE(K_4^2)=\left|0-\frac{24}{7}\right|(1)+\left|1-\frac{24}{7}\right|(1))+\left|7-\frac{24}{7}\right|(1)+\left|4-\frac{24}{7}\right|(4)$
$=\left|\frac{-24}{7}\right|(1)+\left|\frac{-17}{7}\right|(1))+\left|\frac{4}{7}\right|(4)+\left|\frac{25}{7}\right|(1)=\frac{82}{7}$.
$\therefore$ The Laplacian energy of the graph G obtained by taking two copies of complete graph K_4 sharing a common vertex is $\frac{82}{7}$.

3. Seidel energy and Laplacian Seidel energy of k-copies of K_n sharing a single common vertex

Theorem 3.1 *The Seidel energy of a graph G obtained by taking k-copies($k \geq 2$) of complete graph $K_n(n \geq 2)$ sharing a single common vertex is*
$3nk - 5k - 2n + 3 + \sqrt{n^2k^2 - 4n^2k - 2nk^2 + 4n^2 + k^2 + 14nk - 12n - 10k + 9}$.

Proof:

The Seidel adjacency matrix is given by

$$A(G)=\left(\begin{array}{c|cccc|cccc|c|cccc}
0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\ \hline
-1 & 0 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
-1 & -1 & 0 & \cdots & -1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
-1 & 1 & 1 & \cdots & 1 & 0 & -1 & \cdots & -1 & \cdots & 1 & 1 & \cdots & 1 \\
-1 & 1 & 1 & \cdots & 1 & -1 & 0 & \cdots & -1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
-1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 0 & -1 & \cdots & -1 \\
-1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & -1 & -1 & \cdots & 0
\end{array}\right)_{[k(n-1)+1]\times[k(n-1)+1]}$$

$$= \begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where $A=\begin{pmatrix} 0 \end{pmatrix}_{1\times 1}$, $B=\begin{pmatrix} -1 & -1 & \cdots & -1 \end{pmatrix}_{1\times(n-1)}$, $B^T = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}_{(n-1)\times 1}$,

$C=\begin{pmatrix} 0 & -1 & \cdots & -1 \\ -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$ and $D=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is,

$(-1)^{nk-k+1}(\lambda - 1)^{nk-2k}[\lambda + 2n - 3]^{k-1}\lambda^2 - [(k-2)n - (k-3)]\lambda - (nk - k) = 0$

Seidel spectrum of G is,

$$Spec(G)=\begin{pmatrix} 1 & -(2n-3) & \frac{nk-2n-k+3\pm\sqrt{n^2k^2-4n^2k-2nk^2+4n^2+k^2+14nk-12n-10k+9}}{2} \\ nk-2k & k-1 & 1 \end{pmatrix}$$

Seidel energy of G is,

$SE(G)=|1|\,(nk-2k)+|-(2n-3)|\,(k-1)$

$$+\left|\frac{nk-2n-k+3+\sqrt{n^2k^2-4n^2k-2nk^2+4n^2+k^2+14nk-12n-10k+9}}{2}\right|(1)$$

$$+\left|\frac{nk-2n-k+3-\sqrt{n^2k^2-4n^2k-2nk^2+4n^2+k^2+14nk-12n-10k+9}}{2}\right|(1)$$

$= 3nk-5k-2n+3+\sqrt{n^2k^2-4n^2k-2nk^2+4n^2+k^2+14nk-12n-10k+9}.$

$\therefore$ The Seidele energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is

$SE(G) = 3nk-5k-2n+3+\sqrt{n^2k^2-4n^2k-2nk^2+4n^2+k^2+14nk-12n-10k+9}.$

□

Python code to generate the Siedel energy of the graph G obtained by taking $k-$copies of the complete graph K_n by sharing a common vertex.

```
import numpy as np
from numpy.polynomial.polynomial import Polynomial
def construct_adjacency_matrix(n, k):
# Submatrices defining the block structure
A = np.array([[0]])  # Single-element matrix for the top-left block
B = -1 * np.ones((1, n - 1))  # Row matrix of -1s (1 x (n-1))
BT = -1 * np.ones((n - 1, 1))  # Column matrix of -1s ((n-1) x 1)
C = -1 * (np.ones((n - 1, n - 1)) - np.eye(n - 1))  # 0 diagonal and -1 elsewhere
D = np.ones((n - 1, n - 1))  # Matrix with all entries as 1
# Constructing the first row: [A, B, B, ..., B]
first_row = np.hstack([A] + [B] * k)
# Constructing the remaining rows using submatrices BT, C, and D
block_rows = []
for block_row in range(k):
row_blocks = [BT]  # Each row starts with BT
for block_col in range(k):
if block_row == block_col:
row_blocks.append(C)  # Diagonal blocks are C
else:
row_blocks.append(D)  # Off-diagonal blocks are D
block_rows.append(np.hstack(row_blocks))  # Combine blocks for the current row
# Combine the first row and the block rows to form the complete matrix
adjacency_matrix = np.vstack([first_row] + block_rows)
# Explicitly set diagonal elements to 0 to avoid negative zero
np.fill_diagonal(adjacency_matrix, 0)
return adjacency_matrix

def format_polynomial(coefficients):
terms = []
degree = len(coefficients) - 1
for i, coef in enumerate(coefficients):
if np.isclose(coef, 0):  # Skip terms with negligible coefficients
continue
coef = int(round(coef))  # Round to the nearest integer
power = degree - i
if power == 0:
terms.append(f"{coef}")  # Constant term
elif power == 1:
terms.append(f"{coef}\lambda")  # Linear term
else:
terms.append(f"{coef}\lambda^{power}")  # Higher degree terms
return " + ".join(terms).replace("+ -", "- ")  # Clean up formatting

# User input for graph parameters
```

```
n = int(input("Enter the number of vertices per complete graph (K_n): "))
k = int(input("Enter the number of disjoint copies of K_n: "))
# Generate the adjacency matrix for (K_n)^k
adj_matrix = construct_adjacency_matrix(n, k)
# Display the adjacency matrix and its order
print(f"\nSeidel adjacency Matrix of (K_{n})^{k}:")
print(adj_matrix)
print(f"\nOrder of the matrix: {adj_matrix.shape}")
# Compute eigenvalues of the adjacency matrix
eigenvalues = np.linalg.eigvals(adj_matrix)
# Compute the characteristic polynomial from the eigenvalues
char_poly = Polynomial.fromroots(eigenvalues)
# Ensure integral coefficients
integral_coefficients = np.round(char_poly.coef[::-1]).astype(int)
# Format the characteristic polynomial as a readable equation
formatted_char_poly = format_polynomial(integral_coefficients)
print(f"\nSeidel characteristic Polynomial of (K_{n})^{k}:")
print(f"P(\lambda) = {formatted_char_poly}")
# Display the eigenvalues
print(f"\nSeidel eigenvalues of the adjacency matrix (K_{n})^{k}:")
print(eigenvalues)
# Calculate and display the energy (sum of absolute eigenvalues)
energy = np.sum(np.abs(eigenvalues))
print(f"\nSeidel Energy of (K_{n})^{k}:")
print(energy)
```

Example 3.1 *:The Seidel energy of the graph G obtained by taking three copies of complete graph K_3 sharing a common vertex is $9+\sqrt{33}$.*

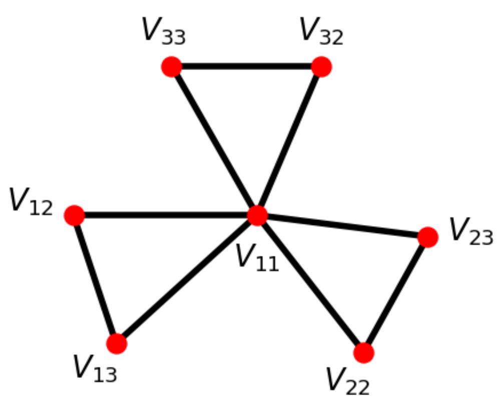

Figure 3: Three copies of complete graph K_3 sharing a common vertex

The Seidel adjancency matrix of K_3^3 is given by

$$A(K_3^3)=\left(\begin{array}{c|cc|cc|cc} 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 1 \\ \hline -1 & 1 & 1 & 0 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 1 & 1 \\ \hline -1 & 1 & 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 0 \end{array}\right)_{(7\times 7)}$$

Characteristic equation of K_3^3 is, $-(\lambda-1)^3(\lambda+3)^2[\lambda^2-3\lambda-6]=0$

Seidel spectrum of K_3^3 is,

$$\text{Spec}(K_3^3\ )=\begin{pmatrix} 1 & -3 & \frac{3\pm\sqrt{33}}{2} \\ 3 & 2 & 1 \end{pmatrix}$$

Seidel energy of K_3^3 is,

$SE(K_3^3)\ =|1|\,(3)+|-3|\,(2)+\left|\dfrac{3+\sqrt{33}}{2}\right|(1)+\left|\dfrac{3-\sqrt{33}}{2}\right|(1)= 9+\sqrt{33}.$

$\therefore$ The Seidel energy of the graph G obtained by taking three copies of complete graph K_3 sharing a common vertex is $9+\sqrt{33}$.

Theorem 3.2 *The Lapalcian Seidel energy of a graph G obtained by taking k-copies($k\geq 2$) of complete graph $K_n(n\geq 2)$ sharing a single common vertex is* $\frac{4n^2k^2-12nk^2-3n^2k+8k^2+13nk-3n-12k+4}{nk-k+1}+\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}$.

Proof:

The Laplacian Seidel adjacency matrix is given by

A(G)=

$$\left(\begin{array}{c|cccc|cccc|c|cccc}
(n-1)k & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
1 & n-1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\
1 & 1 & n-1 & \cdots & 1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & n-1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\ \hline
1 & -1 & -1 & \cdots & -1 & n-1 & 1 & \cdots & 1 & \cdots & -1 & -1 & \cdots & -1 \\
1 & -1 & -1 & \cdots & -1 & 1 & n-1 & \cdots & 1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & n-1 & \cdots & -1 & -1 & \cdots & -1 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & n-1 & 1 & \cdots & 1 \\
1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & 1 & n-1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & 1 & 1 & \cdots & n-1
\end{array}\right)_{[k(n-1)+1]\times[k(n-1)+1]}$$

$$=\begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where A=$\begin{pmatrix} (n-1)k \end{pmatrix}_{1\times 1}$, B=$\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times(n-1)}$, $B^T=\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(n-1)\times 1}$,

C=$\begin{pmatrix} n-1 & 1 & \cdots & 1 \\ 1 & n-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & n-1 \end{pmatrix}_{(n-1)\times(n-1)}$ D=$\begin{pmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is

, $(-1)^{nk-k+1}\ [\lambda-(n-2)]^{nk-2k}[\lambda-(3n-4)]^{k-1}\lambda^2-(3n-4)\lambda-(n-1)[(n-1)k^2-(3n-5)k]=0$

Lapalcian Seidel spectrum of G is,

Spec(G)=$\begin{pmatrix} n-2 & 3n-4 & \frac{3n-4\pm\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}}{2} \\ nk-2k & k-1 & 1 \end{pmatrix}$

Lapalcian Seidel energy of G is,

Number of edges m is $\frac{nk(n-1)}{2}$ and number of vertices n is $k(n-1)+1$ Then $\frac{2m}{n}=\frac{n^2k-nk}{nk-k+1}$

$E(G)=\left|(n-2)-\frac{n^2k-nk}{nk-k+1}\right|(nk-2k)+\left|(3n-4)-\frac{n^2k-nk}{nk-k+1}\right|(k-1)$

$+\left|\frac{3n-4+\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}}{2}-\frac{n^2k-nk}{nk-k+1}\right|(1)$
$+\left|\frac{3n-4-\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}}{2}-\frac{n^2k-nk}{nk-k+1}\right|(1)$
$=\left|\frac{-2nk+2k+n-2}{nk-k+1}\right|(nk-2k)+\left|\frac{2n^2k-6nk+3n+4k-4}{nk-k+1}\right|(k-1)$
$+\left|\frac{n^2k-5nk+3n+4k-4+(nk-k+1)\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}}{2(nk-k+1)}\right|(1)$
$+\left|\frac{n^2k-5nk+3n+4k-4-(nk-k+1)\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}}{2(nk-k+1)}\right|(1)$
$\therefore$ The Laplacian Seidel Energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is $LSE(G)=\frac{4n^2k^2-12nk^2-3n^2k+8k^2+13nk-3n-12k+4}{nk-k+1}$
$+\sqrt{4n^2k^2-12n^2k-8nk^2+9n^2+4k^2+32nk-24n-20k+16}$. □

Example 3.2 *:The Laplacian Seidel energy of the graph G obtained by taking three copies of complete graph K_3 sharing a common vertex is* $\frac{67+7\sqrt{73}}{7}$.

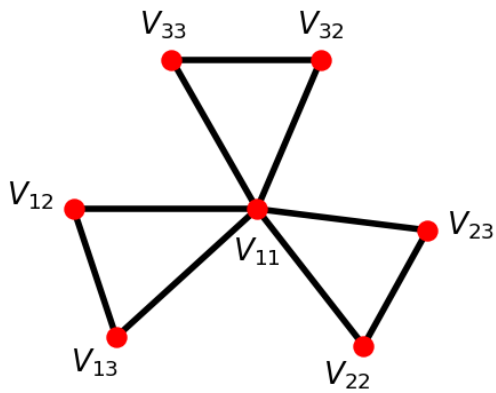

Figure 4: Three copies of complete graph K_3 sharing a common vertex

The Laplacian Seidel Adjancency matrix of K_3^3 is given by

$$A(K_3^3)=\left(\begin{array}{c|cc|cc|cc} 6 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 2 & -1 & -1 & -1 & -1 \\ \hline 1 & -1 & -1 & 2 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 2 & -1 & -1 \\ \hline 1 & -1 & -1 & -1 & -1 & 2 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 2 \end{array}\right)_{(7\times 7)}$$

Characteristic equation of K_3^3 is, $-(\lambda-5)^2(\lambda-1)^3(\lambda^2-5\lambda-12)=0$
Lapalcian Seidel spectrum of K_3^3 is,

$$\text{Spec}(K_3^3)=\begin{pmatrix} 5 & 1 & \frac{5\pm\sqrt{73}}{2} \\ 2 & 3 & 1 \end{pmatrix}$$

Lapalcian Seidel energy of K_3^3 is,
Number of edges m = 9 and number of vertices n = 7 then $\frac{2m}{n}=\frac{18}{7}$

$LSE(K_3^3) = \left|5 - \frac{18}{7}\right|(2) + \left|1 - \frac{18}{7}\right|(3) + \left|\frac{5+\sqrt{73}}{2} - \frac{18}{7}\right|(1) + \left|\frac{5-\sqrt{73}}{2} - \frac{18}{7}\right|(1)$

$= \left|\frac{17}{7}\right|(2) + \left|\frac{-11}{7}\right|(3) + \left|\frac{-1+7\sqrt{73}}{14}\right|(1) + \left|\frac{-1-7\sqrt{73}}{14}\right|(1) = \frac{67+7\sqrt{73}}{7}.$

$\therefore$ The Laplacian Seidel energy of the graph G obtained by taking three copies of complete graph K_3 sharing a common vertex is $\frac{67+7\sqrt{73}}{7}$.

4. Distance energy and Laplacian distance energy of k-copies of K_n sharing a single common vertex

Theorem 4.1 *The Distance energy of a graph G obtained by taking k-copies($k \geq 2$) of complete graph $K_n(n \geq 2)$ sharing a single common vertex is* $2nk - 2k - n + \sqrt{4n^2k^2 - 4n^2k - 8k^2n + n^2 + 4k^2 + 8nk - 4k}$.

Proof: The distance adjacency matrix is given by

$$A(G)=\left(\begin{array}{c|cccc|cccc|c|cccc}
0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
1 & 0 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\
1 & 1 & 0 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 2 & 2 & \cdots & 2 & \cdots & 2 & 2 & \cdots & 2 \\ \hline
1 & 2 & 2 & \cdots & 2 & 0 & 1 & \cdots & 1 & \cdots & 2 & 2 & \cdots & 2 \\
1 & 2 & 2 & \cdots & 2 & 1 & 0 & \cdots & 1 & \cdots & 2 & 2 & \cdots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & 2 & \cdots & 2 & 1 & 1 & \cdots & 0 & \cdots & 2 & 2 & \cdots & 2 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & \cdots & 1 & 1 & \cdots & 0
\end{array}\right)_{[k(n-1)+1]\times[k(n-1)+1]}$$

$$= \begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where A= $\begin{pmatrix} 0 \end{pmatrix}_{1\times1}$, B= $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times(n-1)}$, $B^T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(n-1)\times1}$,

C= $\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}_{(n-1)\times(n-1)}$ and D= $\begin{pmatrix} 2 & 2 & \cdots & 2 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is
$(-1)^{nk-k+1}\ (\lambda+1)^{nk-2k}(\lambda+n)^{k-1}\lambda^2 - [(2kn-n)-2k]\lambda - (nk-k) = 0$
Distance spectrum of G is,

$$\text{Spec(G)}=\begin{pmatrix} -1 & -n & \frac{2nk-2k-n\pm\sqrt{4n^2k^2-4n^2k-8k^2n+n^2+4k^2+8nk-4k}}{2} \\ nk-2k & k-1 & 1 \end{pmatrix}$$

Distance energy of G is,

$E(G)=|-1|\,(nk-2k)+|-n|\,(k-1)$

$+\left|\frac{2nk-2k-n+\sqrt{4n^2k^2-4n^2k-8k^2n+n^2+4k^2+8nk-4k}}{2}\right|(1)$

$+\left|\frac{2nk-2k-n-\sqrt{4n^2k^2-4n^2k-8k^2n+n^2+4k^2+8nk-4k}}{2}\right|(1)$

$=2nk-2k-n+\sqrt{4n^2k^2-4n^2k-8k^2n+n^2+4k^2+8nk-4k}$.

$\therefore$ The Distance energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is

$DE(G)=2nk-2k-n+\sqrt{4n^2k^2-4n^2k-8k^2n+n^2+4k^2+8nk-4k}$.

□

Python code to find the distance energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex

```
import numpy as np
def construct_distance_adjacency_matrix(n, k):
# Define the core submatrices
A = np.array([[0]])
B = np.ones((1, n - 1))
BT = np.ones((n - 1, 1))
C = np.ones((n - 1, n - 1)) - np.eye(n - 1)
D = 2 * np.ones((n - 1, n - 1))
# First row block: [ A | B | B | ... | B ] with k copies of B
first_row = np.hstack([A] + [B] * k)
# Remaining rows: each has [ B^T | (C or D) | ... | (C or D) ]
block_rows = []
for i in range(k):
row_blocks = [BT]  # Start each row with B^T
for j in range(k):
if i == j:
row_blocks.append(C)  # Diagonal block C
else:
row_blocks.append(D)  # Off-diagonal block D
block_rows.append(np.hstack(row_blocks))
# Stack all rows into the final matrix
adjacency_matrix = np.vstack([first_row] + block_rows)
return adjacency_matrix
def compute_monic_characteristic_polynomial(matrix):
eigenvals = np.linalg.eigvals(matrix)
# np.poly(...) with a 1D array of roots returns a monic polynomial.
# Coefficients are in descending order: [1, a_{n-1}, ..., a_0].
polynomial = np.poly(eigenvals)
# Round to nearest integer to clean up numerical noise
polynomial = np.round(polynomial).astype(int)
return polynomial.tolist()
def format_polynomial_descending(coefficients):
degree = len(coefficients) - 1
terms = []
for i, coef in enumerate(coefficients):
current_power = degree - i
if coef == 0:
continue
# Sign and absolute value
sign_str = " - " if coef < 0 else (" + " if i > 0 else "")
```

```
abs_coef = abs(coef)
if current_power == 0:
term_str = f"{sign_str}{abs_coef}"
elif current_power == 1:
if abs_coef == 1:
term_str = f"{sign_str}\lambda"
else:
term_str = f"{sign_str}{abs_coef}\lambda"
else:
if abs_coef == 1:
term_str = f"{sign_str}\lambda^{current_power}"
else:
term_str = f"{sign_str}{abs_coef}\lambda^{current_power}"
terms.append(term_str)
# Combine terms and clean up any leading plus sign
polynomial_str = "".join(terms).strip()
if polynomial_str.startswith("+ "):
polynomial_str = polynomial_str[2:]
return polynomial_str
def compute_graph_energy(matrix):
eigenvalues = np.linalg.eigvals(matrix)
energy = np.sum(np.abs(eigenvalues))
return eigenvalues, energy
if __name__ == "__main__":
# User input
n = int(input("Enter the number of vertices per complete graph (K_n): "))
k = int(input("Enter the number of disjoint copies of K_n: "))
# Construct the distance adjacency matrix
dist_adj_matrix = construct_distance_adjacency_matrix(n, k)
print(f"\nDistance Adjacency Matrix of (K_{n})^{k}:")
print(dist_adj_matrix)
print(f"\nOrder of the matrix: {dist_adj_matrix.shape}")
# Compute and display the monic characteristic polynomial
char_poly_coeffs = compute_monic_characteristic_polynomial(dist_adj_matrix)
poly_str = format_polynomial_descending(char_poly_coeffs)
print(f"\nDistance Characteristic Polynomial of (K_{n})^{k}:")
print(f"P(\lambda) = {poly_str}")
# Compute and display eigenvalues and energy
eigenvalues, energy = compute_graph_energy(dist_adj_matrix)
print(f"\nDistance Eigenvalues of the adjacency matrix (K_{n})^{k}:")
print(eigenvalues)
print(f"\nDistance Energy of (K_{n})^{k}: {energy:.4f}")
```

Example 4.1 : *The Distance energy of the graph G obtained by taking two copies of a complete graph K_5 sharing a common vertex is* $11 + \sqrt{153}$.

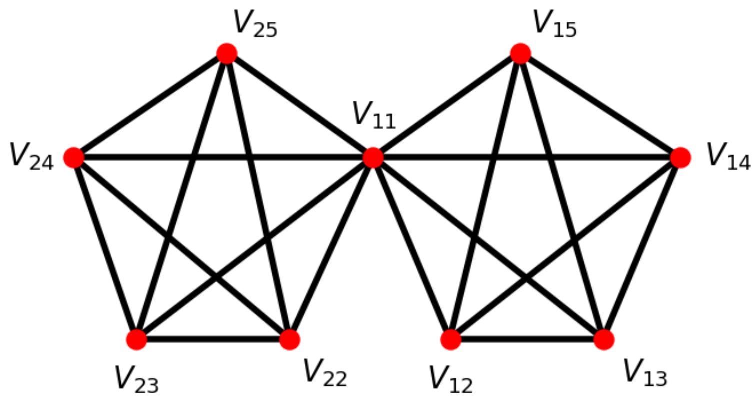

Figure 5: Two copies of complete graph K_5 sharing a common vertex

The distance adjacency matrix of K_5^2 is given by

$$A(K_5^2)=\left(\begin{array}{c|cccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{array}\right)_{9\times 9}$$

Characteristic equation of K_5^2 is, $-(\lambda+1)^6(\lambda+5)(\lambda^2-11\lambda-8)=0$

Distance spectrum of K_5^2 is,

$$\text{Spec}(K_5^2)=\begin{pmatrix} -1 & -5 & \frac{11\pm\sqrt{153}}{2} \\ 6 & 1 & 1 \end{pmatrix}$$

Distance energy of K_5^2 is,

$DE(K_5^2) = |-1|\,(6)+|-5|\,(1)+\left|\frac{11+\sqrt{153}}{2}\right|(1)+\left|\frac{11-\sqrt{153}}{2}\right|(1) = 11+\sqrt{153}.$

$\therefore$ The Distance energy of the graph G obtained by taking 2 copies of complete graph K_5 sharing a common vertex is $11+\sqrt{153}$.

Theorem 4.2 *The Laplacian distance energy of a graph G obtained by taking k-copies($k \geq 2$) of complete graph $K_n(n \geq 2)$ sharing a single common vertex is*
$\frac{n^2k^2-2nk^2+2nk+k^2-2k-2n+1}{nk-k+1}+\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}.$

Proof:

The Laplacian distance adjacency matrix is given by

A(G)=

$$\left(\begin{array}{c|cccc|cccc|c|cccc} (n-1)k & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & -1 & \cdots & -1 \\ \hline -1 & n-1 & -1 & \cdots & -1 & -2 & -2 & \cdots & -2 & \cdots & -2 & -2 & \cdots & -2 \\ -1 & -1 & n-1 & \cdots & -1 & -2 & -2 & \cdots & -2 & \cdots & -2 & -2 & \cdots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 & -2 & -2 & \cdots & -2 & \cdots & -2 & -2 & \cdots & -2 \\ \hline -1 & -2 & -2 & \cdots & -2 & n-1 & -1 & \cdots & -1 & \cdots & -2 & -2 & \cdots & -2 \\ -1 & -2 & -2 & \cdots & -2 & -1 & n-1 & \cdots & -1 & \cdots & -2 & -2 & \cdots & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -2 & -2 & \cdots & -2 & -1 & -1 & \cdots & n-1 & \cdots & -2 & -2 & \cdots & -2 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline -1 & -2 & -2 & \cdots & -2 & -2 & -2 & \cdots & -2 & \cdots & n-1 & -1 & \cdots & -1 \\ -1 & -2 & -2 & \cdots & -2 & -2 & -2 & \cdots & -2 & \cdots & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -2 & \cdots & -2 & -2 & -2 & \cdots & -2 & \cdots & -1 & -1 & \cdots & n-1 \end{array}\right)_{[k(n-1)+1]\times[k(n-1)+1]}$$

$$= \begin{pmatrix} A & B & B & \cdots & B \\ B^T & C & D & \cdots & D \\ B^T & D & C & \cdots & D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B^T & D & D & \cdots & C \end{pmatrix}$$

where,

A=$\begin{pmatrix} (n-1)k \end{pmatrix}_{1\times 1}$, B=$\begin{pmatrix} -1 & -1 & \cdots & -1 \end{pmatrix}_{1\times(n-1)}$, $B^T = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}_{(n-1)\times 1}$,

C=$\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}_{(n-1)\times(n-1)}$ D=$\begin{pmatrix} -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 \\ \vdots & \vdots & \vdots & \vdots \\ -2 & -2 & \cdots & -2 \end{pmatrix}_{(n-1)\times(n-1)}$

Characteristic equation of G is,
$(-1)^{nk-k+1}\ (\lambda-n)^{nk-2k}[\lambda-(2n-1)]^{k-1}\ \lambda^2+[(nk-2n)-(k-1)]\lambda-[(2k^2-2k)(n^2-2n+1)]=0$
Laplacian distance spectrum of G is,
Spec(G)=$\begin{pmatrix} n & 2n-1 & \frac{2n-nk+k-1\pm\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}}{2} \\ nk-2k & k-1 & 1 \end{pmatrix}$

Laplacian distanceenergy of G is,
Number of edges m is $\frac{nk(n-1)}{2}$ and number of vertices n is $k(n-1)+1$
Then, $\frac{2m}{n}=\frac{n^2k-nk}{nk-k+1}$
$LDE(G)=\left|n-\frac{n^2k-nk}{nk-k+1}\right|(nk-2k)+\left|(2n-1)-\frac{n^2k-nk}{nk-k+1}\right|(k-1)$
$+\left|\frac{2n-nk+k-1+\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}}{2}-\frac{n^2k-nk}{nk-k+1}\right|(1)$
$+\left|\frac{2n-nk+k-1-\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}}{2}-\frac{n^2k-nk}{nk-k+1}\right|(1)$
$=\left|\frac{n}{nk-k+1}\right|(nk-2k)+\left|\frac{n^2k-2nk+2n+k-1}{nk-k+1}\right|(k-1)$
$+\left|\frac{-n^2k^2+2nk^2-k^2-2nk+2n+2k-1+(nk-k+1)\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}}{2(nk-k+1)}\right|(1)$
$+\left|\frac{-n^2k^2+2nk^2-k^2-2nk+2n+2k-1-(nk-k+1)\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}}{2(nk-k+1)}\right|(1)$
$=\frac{n^2k^2-2nk^2+2nk+k^2-2k-2n+1}{nk-k+1}+\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}.$
$\therefore$ The Laplacian distance energy of the graph G obtained by taking k-copies of the complete graph K_n by sharing a common vertex is $\frac{n^2k^2-2nk^2+2nk+k^2-2k-2n+1}{nk-k+1}+$
$\sqrt{9n^2k^2-12n^2k-18k^2n+4n^2+9k^2+22nk-10k-4n+1}$ □

Example 4.2 *:The Laplacian distance energy of the graph G obtained by taking two copies of complete graph* K_5 *sharing a common vetex is* $\frac{71+9\sqrt{257}}{9}$.

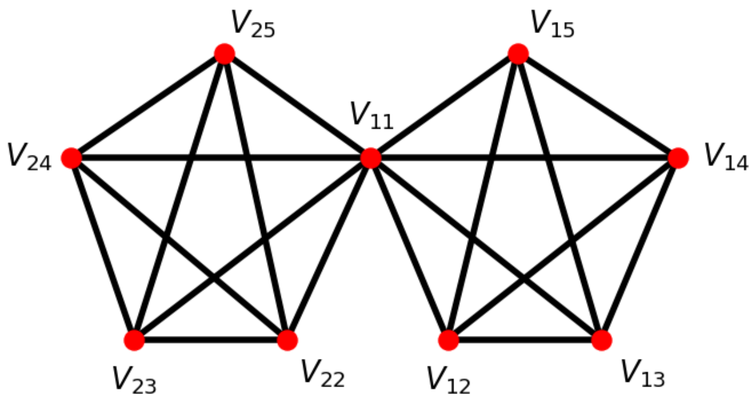

Figure 6: Two copies of complete graph K_5 sharing a common vertex.

The Laplacian distance adjacency matrix of K_5^2 is given by

$$A(K_5^2)=\left(\begin{array}{c|cccc|cccc}
8 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\hline
-1 & 4 & -1 & -1 & -1 & -2 & -2 & -2 & -2 \\
-1 & -1 & 4 & -1 & -1 & -2 & -2 & -2 & -2 \\
-1 & -1 & -1 & 4 & -1 & -2 & -2 & -2 & -2 \\
-1 & -1 & -1 & -1 & 4 & -2 & -2 & -2 & -2 \\
\hline
-1 & -2 & -2 & -2 & -2 & 4 & -1 & -1 & -1 \\
-1 & -2 & -2 & -2 & -2 & -1 & 4 & -1 & -1 \\
-1 & -2 & -2 & -2 & -2 & -1 & -1 & 4 & -1 \\
-1 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & 4
\end{array}\right)_{9\times 9}$$

Characteristic equation of K_5^2 is, $-(\lambda-5)^6(\lambda-9)(\lambda^2-\lambda-64)=0$

Laplacian distance spectrum of K_5^2 is, Spec(K_5^2)= $\begin{pmatrix} 5 & 9 & \frac{1\pm\sqrt{257}}{2} \\ 6 & 1 & 1 \end{pmatrix}$

Laplacian distance energy of K_5^2 is,
Number of edges m = 20 and number of vertices n = 9 then $\frac{2m}{n}=\frac{40}{9}$

$$LDE(K_5^2)=\left|5-\frac{40}{9}\right|(6)+\left|9-\frac{40}{9}\right|(1)+\left|\frac{1+\sqrt{257}}{2}-\frac{40}{9}\right|(1)+\left|\frac{1-\sqrt{257}}{2}-\frac{40}{9}\right|(1)$$
$$=\left|\frac{5}{9}\right|(6)+\left|\frac{41}{9}\right|(1)+\left|\frac{9+9\sqrt{257}-80}{18}\right|(1)+\left|\frac{9-9\sqrt{257}-80}{18}\right|(1)=\frac{71+9\sqrt{257}}{9}.$$

$\therefore$ The Laplacian distance energy of the graph G obtained by taking 2 copies of complete graph K_5 sharing a common vetex is $LDE(K_5^2)=\frac{71+9\sqrt{257}}{9}$.

5. Conclusion

In this work, we systematically computed the energy, Seidel energy, and distance energy for the class of graphs formed by taking k copies of the complete graph K_n joined at a single common vertex. Further, we derived explicit expressions for the Laplacian energy, Laplacian distance energy, and Laplacian Seidel energy of these graphs, thereby providing a comprehensive spectral analysis of this interesting family of composite graphs. To aid in computational verification and potential future exploration, we also developed a Python code capable of generating the corresponding energy values efficiently.

Future work could extend these calculations to other types of graph energies (such as the signless Laplacian energy or Randić energy), or consider different graph operations, such as identifying multiple vertices or introducing weighted edges. Moreover, studying the asymptotic behavior of these energies as n or k grows large could offer deeper insights into the spectral dynamics of complex graph structures.

Overall, this study contributes to the growing body of knowledge connecting algebraic graph theory, energy computations, and algorithmic techniques — opening doors for further mathematical exploration and computational experimentation.

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