Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.78011

Amplified eccentric connectivity energy of a graph

AJITHKUMARA*, PUTTASWAMY and PUNEETH S.

ABSTRACT: In this paper, we introduce and formalize the concept of amplified eccentric–connectivity energy (AECE), a novel spectral graph invariant that combines both local and global structural properties through a degree–eccentricity weighted adjacency matrix, termed the amplified eccentric–connectivity matrix. We investigate the spectral properties of this matrix and exact analytical expressions for AECE are derived for key graph classes.

Key Words: Energy, amplified eccentric-connectivity matrix, spectral radius.

Contents

1. Introduction

Let G be a simple graph and let $V(G) = \{v_1, v_2, \cdots, v_n\}$ be its vertex set. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by $d(v_i)$, is the number of the vertices adjacent to v_i.

In the last seven decades, topological graph indices increasingly attracted the researchers by their countless applications, especially in Chemistry [11,14,15]. One of the example is Prediction of Corrosion Inhibition Effectiveness by Molecular Descriptors of Weighted Chemical Graphs which has been studied by Niko Tratnik et al. [17]. The graph energy, Estrada index, Resolvent energy, and the Laplacian energy were tested accurately as parameters by Izudin Redžepović et al. [13] in the prediction of boiling points, heat formation, and octanol/water partition coefficients of alkanes.

Graph energies have been a central object of study in chemical graph theory and spectral graph theory. It provides a spectral measure that reflects the structural and connectivity features of a graph. Over the decades, this invariant has found applications beyond chemistry-including network theory, machine learning, and theoretical graph analysis [1,6,7,10]. Many generalizations of graph energy have since been proposed, involving alternative matrices such as the Laplacian, signless Laplacian, distance, and eccentricity matrices, each capturing distinct aspects of graph structure [2,9,12,16]. Inspired by the classical adjacency matrix and related topological indices, we define the amplified eccentric-connectivity matrix $AEC(G)$ of a graph G, and introduce its spectral invariant, the amplified eccentric-connectivity energy.

Veena Mathad et al. [8] introduced a new topological index, called amplified eccentric-connectivity (AEC) index of any connected graph G, which is defined as

$$M_{aec}(G) = \sum_{uv \in E(G)} \{d(u)e(u) + d(v)e(v)\},$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 05C50, 05C40, 15A18.
Submitted July 23, 2025. Published September 01, 2025

where $e(u)$ denotes the eccentricity of a vertex u, and it is defined as:

$$e(u) = max\left\{d(u,w)|w \in V(G)\right\}.$$

The concept of the amplified eccentric-connectivity index suggests that it is purposeful to associate a symmetric square matrix to the graph G, which we call as the amplified eccentricity-connectivity matrix denoted by $M_{aec}(G)$, and it is defined as $M_{aec}(G) = (M_{ij})_{n\times n}$, where

$$M_{ij} = \begin{cases} d(u)e(u) + d(v)e(v) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

2. The amplified eccentric - connectivity energy of graphs

Let G be a simple, finite, undirected graph. The energy $E(G)$ of G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, see [4,5]. Several authors defined several types of energy by taking a variety of matrices obtained from the given graph instead of the classical adjacency matrix. In this paper, we particularly study the amplified eccentric-connectivity energy of a graph.

Let $M_{aec}(G)$ be the amplified eccentric-connectivity matrix of the graph G. The characteristic polynomial of $M_{aec}(G)$ is denoted by $\phi_{AEC}(G,\zeta)$ and defined by

$$\phi_{AEC}(G,\zeta) = \det(\zeta I - M_{aec}(G)).$$

As the amplified eccentric-connectivity matrix is real and symmetric, its eigenvalues are all real numbers. Let us label them in non-increasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The amplified eccentric-connectivity energy is denoted by $\zeta_e(G)$ and is defined by

$$\zeta_e(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The remaining of this paper is structured as follows: In Section 3, we discuss several fundamental properties of the amplified eccentric-connectivity energy of a graph and in Section 4, we derive explicit expressions for the amplified eccentric-connectivity energy of various well-known classes of graphs and discuss few bounds.

3. Fundamental properties of $\zeta_e(G)$

Proposition 3.1 *The first three coefficients of the polynomial $\phi_{AEC}(G,\zeta)$ are given by*

(a) $a_0 = 1$

(b) $a_1 = 0$

(c) $a_2 = -Q^2$

where $Q = \sum_{uv\in E(G)} \{d(u)e(u) + d(v)e(v)\}$.

Proof: By the definition, we have $\phi_{AEC}(G,\zeta) = \det(\zeta I - M_{aec}(G))$. From this, after straightforward calculations it follows that $a_0 = 1$. The sum of the determinants of all 1×1 principal submatrices of $M_{aec}(G)$ is equal to the trace of $M_{aec}(G)$. Therefore

$$a_1 = (-1)\cdot \text{ trace of } (M_{aec}(G)) = 0.$$

Finally, we have

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= -Q^2,\end{aligned}$$

from which it follows that $a_2 = -Q^2$. This completes the proof. □

We now discuss an interesting and useful result for the sum of the squares of the amplified eccentric-connectivity eigenvalues.

Proposition 3.2 *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the amplified eccentric -connectivity eigenvalues of $M_{aec}(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2Q^2.$$

Proof: We know that

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2 \\ &= 2\sum_{i<j} a_{ij}^2 \\ &= 2Q^2.\end{aligned}$$

□

The next result gives an upper bound for the amplified eccentric -connectivity energy of a graph G in terms of the number of vertices and the number Q.

Theorem 3.1 *Let G be a graph with n vertices. Then $\zeta_e(G) \le \sqrt{2n}\, Q$.*

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix $M_{aec}(G)$. By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Choose $a_i = 1$ and $b_i = |\lambda_i|$ for $1 \le i \le n$. Then the inequality becomes

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right),$$

which simplifies to

$$(\zeta_e(G))^2 \le n \cdot \sum_{i=1}^{n} |\lambda_i|^2.$$

Since $\sum_{i=1}^{n} |\lambda_i|^2 = 2Q^2$, it follows that

$$(\zeta_e(G))^2 \leq 2nQ^2,$$

and hence

$$\zeta_e(G) \leq \sqrt{2n}\, Q.$$

□

The next result gives a lower bound for the amplified eccentric-connectivity energy of a graph G in terms of the number of vertices, the number Q and the determinant of the amplified eccentric -connectivity matrix of G.

Theorem 3.2 *Let G be a graph with n vertices. If $R = \det M_{aec}(G)$, then $\zeta_e(G) \geq \sqrt{2Q^2 + n(n-1)R^{\frac{2}{n}}}$.*

Proof: By definition,

$$\begin{aligned}\zeta_e(G)^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &= \sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\zeta_j| \\ &= \left(\sum_{i=1}^{n} |\lambda_i|^2\right) + \sum_{i \neq j} |\lambda_i| \, |\zeta_j| \, .\end{aligned}$$

Using the arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i| |\lambda_j|\right)^{\frac{1}{n(n-1)}}.$$

Therefore, we obtain the inequality

$$\begin{aligned}\zeta_e(G)^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i \neq j} |\lambda_i| \, |\lambda_j|\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)R^{\frac{2}{n}} \\ &= 2Q^2 + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Hence, we conclude that

$$\zeta_e(G) \geq \sqrt{2Q^2 + n(n-1)R^{\frac{2}{n}}}.$$

□

4. $\zeta_e(G)$ of certain standard graphs

Theorem 4.1 *For a complete graph K_n, we have $\zeta_e(K_n) = 4(n-1)^2$.*

Proof: Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The amplified eccentric-connectivity matrix of K_n is given by

$M_{aec}(K_n) =$

$$\begin{bmatrix} 0 & 2(n-1) & 2(n-1) & \ldots & 2(n-1) & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \ldots & 2(n-1) & 2(n-1) \\ 2(n-1) & 2(n-1) & 0 & \ldots & 2(n-1) & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(n-1) & 2(n-1) & \cdots & 2(n-1) & 0 & 2(n-1) \\ 2(n-1) & 2(n-1) & \cdots & 2(n-1) & 2(n-1) & 0 \end{bmatrix}.$$

The characteristic equation of this matrix is

$$(\lambda+1)^{n+1}(\lambda - \ (n-1))(2(n-1))^n = 0$$

from which the spectrum is given by

$$\mathrm{Spec}_{AEC}(K_n) = \begin{pmatrix} -2(n-1)^2 & 2(n-1)^2 \\ n-1 & 1 \end{pmatrix}.$$

Hence, the amplified eccentric-connectivity energy is

$$\zeta_e(K_n) = \ 4(n-1)^2.$$

□

Theorem 4.2 *For a cycle graph C_{2n}, we have $\zeta_e(C_{2n}) = 16n\left(\sum_{k=0,}^{n-1}\left|\cos\frac{\pi k}{n}\right|\right)$.*

Proof: The Amplified eccentric connectivity matrix corresponding to the cycle graph C_{2n} is

$$M_{aec}(C_{2n}) = \begin{bmatrix} 0 & 4n & 0 & 0 & 0 & \ldots & 0 & 4n \\ 4n & 0 & 4n & 0 & 0 & \ldots & 0 & 0 \\ 0 & 4n & 0 & 4n & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 4n \\ 4n & 0 & 0 & 0 & 0 & \ldots & 4n & 0 \end{bmatrix}.$$

This is a circullant matrix of order $2n$. The characteristic polynomial of the above matrix is:

$$\prod_{k=0}^{2n-1}\left(\lambda - 8n\cos\left(\frac{\pi k}{n}\right)\right) = 0$$

Therefore, the spectrum of amplified eccentric connectivity matrix is

$$\mathrm{Spec}_{AEC}(C_{2n}) = \left\{\left|8n\cos\left(\frac{\pi k}{n}\right)\right| k = 0, 1, 2, ..., 2n-1\right\}.$$

From the above, we finally obtain the desired result $\zeta_e(C_{2n}) = 16n\left(\sum_{k=0,}^{n-1}\left|\cos\frac{\pi k}{n}\right|\right)$. □

Theorem 4.3 *For a star graph $K_{1,n-1}$, we have $\zeta_e\left(K_{1,n-1}\right)=2(n+1)\sqrt{n-1}$.*

Proof: Let $K_{1,n-1}$ be the star graph with vertex set $V=\{v_1,v_2,\ldots,v_{n-1}\}$. The amplified eccentric connectivity matrix is given by

$$M_{aec}\left(K_{1,n-1}\right)=\begin{bmatrix} 0 & n+1 & n+1 & \cdots & n+1 & n+1 \\ n+1 & 0 & 0 & \cdots & 0 & 0 \\ n+1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n+1 & 0 & 0 & \cdots & 0 & 0 \\ n+1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then, the characteristic equation of $M_{aec}\left(K_{1,n-1}\right)$ is given by

$$\lambda^{n-2}(\lambda^2-\ (n+1)^2(n-1))=0$$

and therefore, the spectrum is given by

$$\mathrm{Spec}_{AEC}\left(K_{1,n-1}\right)=\begin{pmatrix} 0 & (n+1)\sqrt{n-1} & -(n+1)\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the amplified eccentric-connectivity energy of $K_{1,n-1}$ is

$$\zeta_e\left(K_{1,n-1}\right)=2(n+1)\sqrt{n-1}.$$

□

Theorem 4.4 *For a crown graph S_n^0, we have $\zeta_e\left(S_n^0\right)=16(n-1)^2$.*

Proof: Let S_n^0 denote the crown graph of order $2n$, with vertex set $\{u_1,u_2,\ldots,u_n,\ v_1,v_2,\ldots,v_n\}$. The amplified eccentric connectivity matrix is

$$M_{aec}\left(S_n^0\right)=\begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 4(n-1) & \ldots & 4(n-1) \\ 0 & 0 & \ldots & 0 & 4(n-1) & 0 & \ldots & 4(n-1) \\ 0 & 0 & \ldots & 0 & 4(n-1) & 4(n-1) & \ldots & 4(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 4(n-1) & 4(n-1) & \ldots & 0 \\ 0 & 4(n-1) & \ldots & 4(n-1) & 0 & 0 & \ldots & 0 \\ 4(n-1) & 0 & \ldots & 4(n-1) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4(n-1) & 4(n-1) & \ldots & 4(n-1) & 0 & 0 & \ldots & 0 \\ 4(n-1) & 4(n-1) & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

In this case, the characteristic equation is

$$\left(\lambda^2-\left[4(n-1)^2\right]^2\right)\left(\lambda^2+\left[4(n-1)\right]^2\right)^{n-1}=0$$

which implies that the spectrum is

$$\mathrm{Spec}_{AEC}\left(S_n^0\right)=\begin{pmatrix} 4(n-1)^2) & -4(n-1)^2 & 4(n-1) & -4(n-1) \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$$

Therefore, we arrive at

$$\zeta_e\left(S_n^0\right)=16\ (n-1)^2.$$

□

Theorem 4.5 *For a cocktail party graph $K_{n\times 2}$, we have $\zeta_e\left(K_{n\times 2}\right) = 32\ (n-1)^2$.*

Proof: Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The amplified eccentric connectivity matrix is given by

$$M_{aec}\left(K_{n\times 2}\right) = \begin{bmatrix} 0 & 8(n-1) & \dots & 8(n-1) & 0 & 8(n-1) & \dots & 8(n-1) \\ 8(n-1) & 0 & \dots & 8(n-1) & 8(n-1) & 0 & \dots & 8(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8(n-1) & 8(n-1) & \dots & 0 & 8(n-1) & 8(n-1) & \dots & 0 \\ 0 & 8(n-1) & \dots & 8(n-1) & 0 & 8(n-1) & \dots & 8(n-1) \\ 8(n-1) & 0 & \dots & 8(n-1) & 8(n-1) & 0 & \dots & 8(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8(n-1) & 8(n-1) & \dots & 0 & 8(n-1) & 8(n-1) & \dots & 0 \end{bmatrix}.$$

The characteristic equation of the above matrix is

$$\lambda^n\left(\lambda + 16n - 16\right)^{n-1}\left(\lambda - 16(n-1)^2\right) = 0,$$

which implies that the spectrum is

$$\text{Spec}_{AEC}\left(K_{n\times 2}\right) = \begin{pmatrix} 16(n-1)^2 & -16(n-1)^2 & 0 \\ 1 & n-1 & n-1 \end{pmatrix}.$$

From the above, it follows that

$$\zeta_e\left(K_{n\times 2}\right) = 32\ (n-1)^2.$$

□

Theorem 4.6 *The amplified eccentric connectivity energy of the complete bipartite graph $K_{m,n}$ is*

$$\zeta_e\left(K_{m,n}\right) = 4\left(m+n\right)\sqrt{mn}.$$

Proof: Let $K_{m,n}$ be the complete bipartite graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. The amplified eccentric connectivity energy is given by

$$AEC(K_{m,n}) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 2(m+n) & 2(m+n) & \cdots & 2(m+n) \\ 0 & 0 & \cdots & 0 & 2(m+n) & 2(m+n) & \cdots & 2(m+n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2(m+n) & 2(m+n) & \cdots & 2(m+n) \\ \hline 2(m+n) & 2(m+n) & \cdots & 2(m+n) & 0 & 0 & \cdots & 0 \\ 2(m+n) & 2(m+n) & \cdots & 2(m+n) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(m+n) & 2(m+n) & \cdots & 2(m+n) & 0 & 0 & \cdots & 0 \end{array}\right]$$

In this case, the characteristic equation is

$$\left(\lambda^{m+n-2}\right)\left[\lambda^2 - 4(m+n)^2 mn\right] = 0,$$

implying that the spectrum is

$$\text{Spec}_{AEC}\left(K_{m,n}\right) = \begin{pmatrix} 2mn\sqrt{mn} & -2mn\sqrt{mn} & 0 \\ 1 & 1 & m+n-2 \end{pmatrix}.$$

Therefore, we have

$$\zeta_e\left(K_{m,n}\right) = 4\left(m+n\right)\sqrt{mn}.$$

□

Theorem 4.7 *Let G be a connected k-regular graph on n vertices such that all vertices have the same eccentricity e. Then, the amplified eccentric-connectivity energy of G satisfies*

$$\zeta_e = 2ke \cdot E(G),$$

where $E(G)$ is the classical graph energy.

Proof: Since G is k-regular, each vertex has degree $d(v) = k$. Given that all vertices also share the same eccentricity e, it follows that for any adjacent pair of vertices $u \sim v$, the off-diagonal entry of the AEC matrix is

$$M_{uv} = d(u)e(u) + d(v)e(v) = ke + ke = 2ke.$$

Thus, every edge in the graph contributes a value of $2ke$ to the AEC matrix, and this structure exactly mirrors the adjacency matrix scaled by $2ke$. Therefore,

$$\zeta_e = 2ke \cdot A(G).$$

From spectral graph theory, scalar multiplication of a symmetric matrix scales all its eigenvalues. So, if $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigenvalues of $A(G)$, then the eigenvalues of AEC are $2ke \cdot \lambda_1, 2ke \cdot \lambda_2, \ldots, 2ke \cdot \lambda_n$. Hence,

$$\zeta_e = \sum_{i=1}^{n} \left|\lambda_i^{AEC}\right| = \sum_{i=1}^{n} |2ke \cdot \lambda_i| = 2ke \cdot \sum_{i=1}^{n} |\lambda_i| = 2ke \cdot E(G).$$

□

Theorem 4.8 *Let G and $\bar{G}$ be connected graphs of order $n \geq 4$. Then*

$$n(n-1) \leq \zeta_e(G) + \zeta_e(\bar{G}) \leq n(n-1)^3$$

and

$$\zeta_e(G) \cdot \zeta_e(\bar{G}) \leq \left(\frac{n(n-1)^2}{2}\right)^2.$$

Proof: The number of edges in G and $\bar{G}$ satisfies $|E(G)| + |E(\bar{G})| = \binom{n}{2} = \frac{n(n-1)}{2}$. For any edge uv in G or $\bar{G}$, we have $1 \leq d(u), e(u) \leq n-1$, hence

$$d(u)e(u) + d(v)e(v) \in [2, 2(n-1)^2].$$

Thus, each edge contributes at least 2 and at most $2(n-1)^2$ to AECE. Therefore,

$$\zeta_e(G) + \zeta_e(\bar{G}) \geq 2 \cdot \frac{n(n-1)}{2} = n(n-1),$$

$$\zeta_e(G) + \zeta_e(\bar{G}) \leq 2(n-1)^2 \cdot \frac{n(n-1)}{2} = n(n-1)^3.$$

Further, we use the following relation

$$\sqrt{\zeta_e(G) \cdot \zeta_e(\bar{G})} \leq \frac{\zeta_e(G) + \zeta_e(\bar{G})}{2},$$

to get

$$\zeta_e(G) \cdot \zeta_e(\bar{G}) \leq \left(\frac{n(n-1)^2}{2}\right)^2.$$

□

Theorem 4.9 *Let G be a connected graph of order n. Then*

$$\zeta_e(G) \leq n \cdot \rho(M_{aec}), \tag{1}$$
$$\zeta_e(G) \leq n \cdot \Delta_E, \tag{2}$$

where $\rho(M_{aec})$ be the spectral radius of $M_{aec}(G)$ and $\Delta_E = \max\limits_{uv \in E(G)} \{d(u)e(u) + d(v)e(v)\}$.

Proof: (1) Since all eigenvalues of a real symmetric matrix are real, the absolute values $|\lambda_i|$ are well-defined. We have $\rho(M_{aec}) = \max\limits_{1 \leq i \leq n} |\lambda_i|$, so

$$\zeta_e(G) = \sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} \rho(M_{aec}) = n \cdot \rho(M_{aec})$$

which proves inequality (1).

(2) By the Gershgorin Circle Theorem, each eigenvalue ζ of M_{aec} lies within a disc centered at $M_{aec_{ij}}$ with radius $R_i = \sum_{j \neq i} |M_{aec_{ij}}|$. Hence,

$$\rho(M_{aec}) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |M_{aec_{ij}}|.$$

Since each entry $M_{aec_{ij}}$ corresponds to $d(i)e(i) + d(j)e(j)$ for adjacent vertices $i \sim j$, we have:

$$\sum_{j=1}^{n} |M_{aec_{ij}}| \leq \Delta_E \quad \text{for all } i.$$

Therefore, $\rho(M_{aec}) \leq \Delta_E$. Combining this with inequality (1), we get:

$$\zeta_e(G) \leq n \cdot \rho(M_{aec}) \leq n \cdot \Delta_E,$$

proving inequality (2). □

5. Conclusion

In this paper, we introduced and studied the *amplified eccentric-connectivity energy* (AECE) of a graph, a new spectral invariant based on the amplified eccentric-connectivity matrix. We established fundamental properties and bounds for AECE and demonstrated its behavior through detailed computations for several standard graph families including complete graphs, cycles, stars, crown graphs, cocktail party graphs, and complete bipartite graphs. This study not only extends the growing family of graph energy concepts, but also opens up several avenues for further research. Potential directions include the investigation of AECE for random graphs, trees, and chemical graph models, as well as exploring its correlation with other topological indices in chemical graph theory and network science.

References

1. Aphale, R. K. and Pathan, S. H., *On Sombor Energy of Graphs with Self-Loops*, Commun. Appl. Nonlinear Anal., 32, 18–30, (2025).
2. Deidda, P., Segala, N. and Putti, M., *Graph p-Laplacian Eigenpairs as Saddle Points of a Family of Spectral Energy Functions*, arXiv preprint arXiv:2405.07056, (2024).
3. Gutman, I., Furtula, B. and Bozkurt, S. B., *On Randić Energy*, Linear Algebra Appl., 442, 50–57, (2014).
4. Gutman, I., *The Energy of a Graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz., 103, 1–22, (1978).
5. Gutman, I., *The Energy of a Graph: Old and New Results*, in: Combinatorics and Applications, Betten, A., Khoner, A., Laue, R., Wassermann, A. (eds.), Springer, Berlin, pp. 196–211, (2001).

6. Huang, Y. F. and Liu, H. C., *Bounds of Modified Sombor Index, Spectral Radius and Energy*, AIMS Math., 6(10), 11263–11274, (2021).

7. Kalaam, A. R. A., Greeni, A. B. and Arockiaraj, M., *Modified Reverse Degree Descriptors for Combined Topological and Entropy Characterizations of 2D Metal Organic Frameworks: Applications in Graph Energy Prediction*, Front. Chem., 12:1470231, (2024). DOI: 10.3389/fchem.2024.1470231

8. Mathad, V., Sujatha, H. N. and Puneeth, S., *Amplified Eccentric Connectivity Index of Graphs*, TWMS J. Appl. Eng. Math., 12(4), 1469–1479, (2022).

9. Nagalakshmi, A. R., Shrikanth, A. S., Kalavathi, G. K. and Sreekeshava, K. S., *The Degree Energy of a Graph*, Mathematics, 12(17), Article 2699, (2024).

10. Nikiforov, V., *Graphs and Matrices with Maximal Energy*, J. Math. Anal. Appl., 327(1), 735–738, (2007).

11. Puneeth, S., Mathad, V., Shalini, K. and Mahde, S., *Application of Eccentricity-Based Topological Indices to the Design and Optimization of Blood Cancer Drugs*, J. Chem., Article ID 2414974, (2024). DOI: 10.1155/joch/2414974.

12. Rather, B. A., Ganie, H. A., Das, K. C. and Shang, Y., *The General Extended Adjacency Eigenvalues of Chain Graphs*, Mathematics, 12(2), Article 192, (2024). DOI: 10.3390/math12020192

13. Redžepović, I. and Furtula, B., *Predictive Potential of Eigenvalue-Based Topological Molecular Descriptors*, J. Comput.-Aided Mol. Des., 34, 975–982, (2020).

14. Sarveshkumar, B. and Chaluvaraju, B., *Sharp Inequalities for the Generalized Steiner-Gutman Index*, J. Indian Acad. Phys. Sci., 10.61294/jiaps2025.2911, (2025).

15. Sharma, V., Goswami, R. and Madan, A. K., *Connectivity Index: A Novel Highly Discriminating Topological Descriptor for Structure–Eccentric Property and Structure–Activity Studies*, J. Chem. Inf. Comput. Sci., 37, 273–282, (1997).

16. Tang, Q. and Liu, Y., *On the r-th Power Energy of Connected Graphs*, arXiv preprint arXiv:2410.16604, (2024).

17. Tratnik, N., Radenković, S., Redžepović, I., Finšgar, M. and Pleteršek, P. Z., *Predicting Corrosion Inhibition Effectiveness by Molecular Descriptors of Weighted Chemical Graphs*, Croat. Chem. Acta., 94(3), 177–184, (2021).

AJITHKUMARA,
Department of Mathematics,
Vidyavardhaka College of Engineering,
Mysuru, India.
E-mail address: ajithkumar2191@gmail.com

and

PUTTASWAMY,
Department of Mathematics,
P E S College of Engineering,
Mandya, India.
E-mail address: prof.puttaswamy@gmail.com

and

PUNEETH S.,
Department of Advanced Computing
St. Joseph's University,
Bengaluru, Karnataka, India
E-mail address: puneethprajwal@gmail.com