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Comparison of Some Numerical and Analytical Methods for Solving Nonlinear Volterra Integral Equations with Applications

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ABSTRACT: This paper presents an in-depth investigation into the numerical solutions of nonlinear Volterra integral equations of the second kind (NVIE-II). Two distinct methods are applied for approximating the solutions: combining the Laplace transform and the Adomian Decomposition Method (ADM) using He's polynomials, and the Variational Iteration Method (VIM). The first method involves transforming the integral equation using the Laplace transform, followed by solving it through the Adomian Decomposition Method to produce a series of approximations, which are then compared with the exact solution. The second method, based on VIM, converts the Volterra integral equation into an integro-differential equation and demonstrates the rapid convergence of successive approximations to the exact solution. A thorough comparison is conducted between the exact solution, the approximate results obtained from both methods, and the associated error estimations. The findings offer a detailed evaluation of the effectiveness and accuracy of these methods, providing insight into their practical application for solving nonlinear (NVIE-II).

Key Words: Volterra integral equation, Adomian decomposition method, Laplace transform, Lagrange multiplier, variational iteration method.

Contents

1. Introduction

Volterra integral equations are fundamental in the mathematical modeling of numerous biological, physical, and engineering systems, especially those characterized by memory-dependent behavior, where the cumulative history of the system influences the present state. Vito Volterra initiated the study of these equations in 1884, and they were formally named in his honor in 1908. For a deeper insight into Volterra and other integral equations, comprehensive treatments are available in the works of Wazwaz [1,2] Given such equations' intrinsic nonlinearity and structural complexity, obtaining exact analytical solutions is often infeasible. This challenge has led to the development of various analytical and numerical techniques aimed at generating reliable approximations. Among the widely recognized methods are the Laplace–Adomian Decomposition Method (LADM) and the Variational Iteration Method (VIM), which have demonstrated notable success in addressing nonlinear Volterra integral equations. Ahmad and Singh [3] have provided solutions using LADM for nonlinear formulations of these equations. LADM effectively combines the Adomian Decomposition Method with the Laplace transform, enabling the resolution of nonlinear problems without resorting to linearization or imposing restrictive assumptions. The foundation of ADM was laid by Adomian [7] , and its integration with the Laplace transform is elaborated by Wazwaz [8], with enhancements proposed by Rani and Mishra [10]. This approach has been successfully

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employed in solving problems such as heat transfer equations, Burgers' equations, and weakly singular integral equations. Further advancements and practical applications of LADM have been presented by Ullah et al. [4].

The Variational Iteration Method (VIM) introduced by Ji-Huan He in the late 1990s [9] , employs variational principles to iteratively solve nonlinear integral and differential equations. This method involves transforming the original Volterra equation into a differential form, followed by constructing iterative corrections. VIM is widely recognized for its efficiency, rapid convergence, and versatility, as discussed in the works of Al-Saar and Ghadle [5] and Mirzaei [6].

This paper aims to compare the effectiveness of LADM and VIM in solving (NVIE-II). By applying both techniques to a selected test equation, we analyze and contrast their accuracy, rate of convergence, and computational efficiency. LADM proves to be particularly adept at addressing nonlinear terms without simplifications, while VIM demonstrates strong performance in rapidly converging toward accurate solutions. Additional insights and alternative techniques are presented in references such as Atkinson [11] and Saha Ray & Sahu [12].

Our investigation into nonlinear Volterra integral equations of the second kind starts with the following expression:

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))\,dt. \tag{1.1}$$

Where $F(u(x))$ is a nonlinear function of $u(x)$, such as $u^3(x)$, $\cos(u(x))$, and $e^{u(x)}$, and the kernel $K(x,t)$, and the function $f(x)$, are given real-valued functions. It will be found that the unknown function $u(x)$ occurs both inside and outside the integral sign.

2. Application of the Laplace–Adomian decomposition method for solving nonlinear Volterra integral equations of the second kind

In this section, we introduce the Laplace–Adomian Decomposition Method (LADM). This hybrid analytical technique combines the strengths of the classical Adomian Decomposition Method (ADM) with the operational power of the Laplace transform. This method is particularly effective in addressing nonlinear Volterra integral equations of the second kind, especially when dealing with convolution-type kernels and nonlinear terms that are difficult to handle directly. The idea of LADM originates from the need to simplify the integration process and avoid iterative convolution, which often arises in classical methods. By applying the Laplace transform to the entire equation, the integral operator is transformed into an algebraic form, significantly easing the manipulation of nonlinearities. The ADM component then decomposes the nonlinear term into a rapidly converging series of Adomian polynomials. This structured approach allows for the recursive calculation of each term in the solution series, making the method both systematic and efficient.

Over the past decades, LADM has been successfully applied in various scientific and engineering problems involving integro-differential and nonlinear integral equations. Its ability to yield approximate analytical solutions without linearization or discretization makes it a powerful tool for both theoretical analysis and numerical approximation.

2.1. Adomian decomposition method (ADM)

: The Adomian decomposition method was developed by George Adomian.
The ADM decomposes the solution $u(x)$ into an infinite series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

Substituting this expansion into the nonlinear Volterra integral equation, we get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt,$$

By matching terms, the recursive relation becomes:

$$u_0(x) = f(x),$$
$$u_{n+1}(x) = \lambda \int_0^x k(x,t) u_n(t)\, dt, \quad (n \geq 0),$$

When we apply the Laplace transform to both sides of equation (1.1), we obtain:

$$\mathcal{L}[u(x)] = \mathcal{L}[f(x)] + \lambda\, \mathcal{L}[k(x-t)] * \mathcal{L}[F(u(x))] \tag{2.1}$$

The approximate solution can still be represented as:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.2}$$

Additionally, the nonlinear term is decomposed as:

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{2.3}$$

where A_n are the Adomian polynomials computed by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{k=0}^{\infty} \lambda^k u_k(x) \right) \right] \Bigg|_{\lambda=0}, \tag{2.4}$$

Substituting (2.2) and (2.3) into (2.1), we get

$$\sum_{n=0}^{\infty} \mathcal{L}\{u_n(x)\} = \mathcal{L}\{f(x)\} + \lambda\, \mathcal{L}\{K(x-t)\} \sum_{n=0}^{\infty} \mathcal{L}\{A_n(x)\}, \tag{2.5}$$

This leads to the iterative scheme:

$$\mathcal{L}\{u_0(x)\} = \mathcal{L}\{f(x)\}, \tag{2.6}$$
$$\mathcal{L}\{u_{n+1}(x)\} = \lambda\, \mathcal{L}\{K(x-t)\} \mathcal{L}\{A_n(x)\}, \quad (n \geq 0), \tag{2.7}$$

Ultimately, upon applying the inverse Laplace transform to equations (2.6) and (2.7), we obtain:

$$u_0(x) = \mathcal{L}^{-1}\left[\mathcal{L}[f(x)]\right]$$

$$u_{n+1}(x) = \mathcal{L}^{-1}\left[\mathcal{L}[k(x-t)] \cdot \mathcal{L}[A_n(x)]\right], \quad (n \geq 0).$$

Example 2.1. Solve the following Volterra nonlinear integral equation

$$u(x) = f(x) + \int_0^x (x-t) u^2(t)\, dt, \tag{2.8}$$

where

$$f(x) = \sin x + \cos x + \frac{\sin 2x}{4} - \frac{x}{2} - \frac{x^2}{2}$$

This has exact solution $u(x) = \sin x + \cos x$,
Solution. Utilizing the linearity property of the Laplace transform on both sides of (2.8):

$$\mathcal{L}[u(x)] = \mathcal{L}\left[\sin x + \cos x + \frac{\sin 2x}{4} - \frac{x}{2} - \frac{x^2}{2}\right] + \mathcal{L}\left[\int_0^x (x-t) u^2(t)\, dt\right] \tag{2.9}$$

Before applying analytical iterative techniques like the Laplace Adomian Decomposition Method (LADM) to solve the nonlinear Volterra integral equation, we first write the known portion of the problem as follows:

$$f(x) = \sin x + \cos x + \frac{\sin 2x}{4} - \frac{x}{2} - \frac{x^2}{2}$$

as a Taylor series expansion about $x = 0$:

$$f(x) = 1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots$$

With the help of this expansion, we can present a power series representation of the given function, which makes it easier to compute successive terms under iterativity.

Using recursive relations in LADM, the $f(x)$ series form is then utilized to calculate the consecutive correction terms $u_n(x)$ and identify the beginning term $u_0(x)$, Next, using the series solution approach, the function $u(x)$ is

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(x)\right] = \mathcal{L}\left[1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots\right] + \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{2.10}$$

After decomposing the nonlinear term $u^2(x)$, we apply the formula (2.4), When using modified Adomian polynomials, the following happens:

$$\begin{cases} A_0 = u_0^2 \\ A_1 = 2u_0u_1 \\ A_2 = (2u_0u_2 + u_1^2) \\ A_3 = (2u_0u_3 + 2u_1u_2) \\ A_4 = (2u_0u_4 + 2u_1u_3 + u_2^2) \\ \vdots \end{cases}$$

Comparing both sides of Equation (2.10), we get:

$$\mathcal{L}[u_0(x)] = \mathcal{L}\left[1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots\right],$$

$$\mathcal{L}[u_0(x)] = \frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^3} - \frac{3}{s^4} + \frac{1}{s^5} + \frac{9}{s^6} - \frac{1}{s^7} + \cdots$$

On both sides, we use the inverse Laplace transform to obtain,

$$u_0(x) = 1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots$$

In general,

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[A_n(x)]\right\}, \quad n \geq 0$$

$$u_1(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{3}{20}x^5 + \frac{1}{360}x^6 + \frac{37}{1260}x^7 + \frac{113}{20160}x^8 - \frac{7}{2592}x^9 + \cdots$$

$$u_2(x) = \frac{x^4}{12} + \frac{x^5}{12} - \frac{x^6}{60} - \frac{7}{180}x^7 - \frac{151}{20160}x^8 + \frac{1387}{181440}x^9 + \frac{2581}{907200}x^{10} - \frac{1481}{1995840}x^{11} + \cdots$$

$$u_3(x) = \frac{x^6}{72} + \frac{x^7}{63} - \frac{x^8}{10080} - \frac{17}{2160}x^9 - \frac{2279}{907200}x^{10} + \frac{773}{498960}x^{11} + \frac{1489}{1478400}x^{12} - \frac{4271}{57657600}x^{13} + \cdots$$

Then the approximate solution becomes,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{2}{315}x^7 - \frac{1}{504}x^8 - \frac{59}{20160}x^9 - \frac{817}{1814400}x^{10} + \frac{27}{30800}x^{11} + \frac{5207}{9979200}x^{12} - \frac{721}{11119680}x^{13} + \cdots.$$

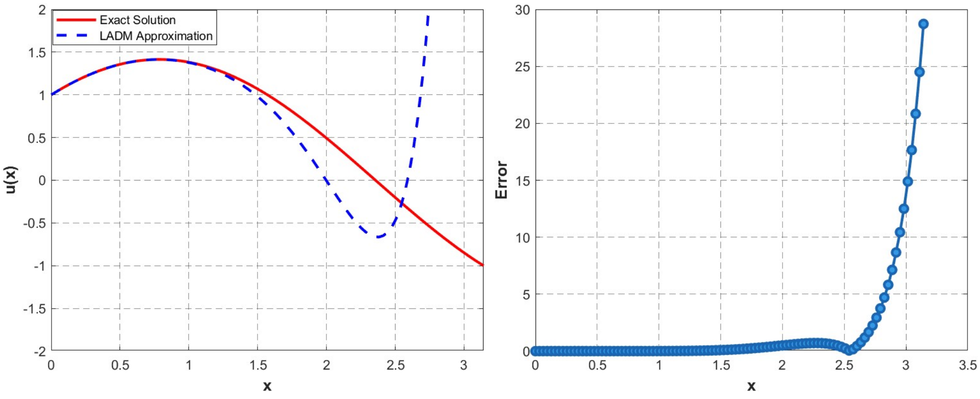

Figure 1: Comparison of exact vs. approximate solutions and absolute error by using (LADM), for Example (2.1).

Figure 1 shows a close agreement between the LADM approximation (blue dashed line) and the actual solution $u(x) = \sin(x)+\cos(x)$ (red solid line) in $0 \le x \le 2.5$, but a discernible divergence beyond $x = 2.5$. Because of series truncation, the absolute error plot shows slight error in this range but significant rise thereafter, peaking close to $x = 3.2$. For broader domains, better refining is required, even if LADM is very accurate locally.

Example 2.2. Solve the following Volterra nonlinear integral equation:

$$u(x) = f(x) - \frac{1}{18}\int_0^x (x-t)u^2(t)\,dt, \tag{2.11}$$

where

$$f(x) = 3x + \frac{1}{24}x^4,$$

The exact solution of (2.11) is $u(x) = 3x$,

Solution. Using the Laplace transform's linearity feature and applying it to both sides of (2.11), we obtain:

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left[3x + \tfrac{1}{24}x^4\right] - \tfrac{1}{18}\mathcal{L}\left\{\int_0^x (x-t)u^2(t)\,dt\right\}, \tag{2.12}$$

The function $u(x)$, thus, is obtained by applying the series solution approach,

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(x)\right] = \mathcal{L}\left[3x + \frac{1}{24}x^4\right] - \frac{1}{18s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(x)\right], \tag{2.13}$$

$$\mathcal{L}[u_0(x)] = \mathcal{L}\left[3x + \frac{1}{24}x^4\right],$$

$$\mathcal{L}[u_0(x)] = \frac{3}{s^2} + \frac{1}{s^5},$$

$$u_0(x) = \mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{1}{s^5}\right],$$

$$u_0(x) = 3x + \frac{x^4}{24},$$

In general,

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \mathcal{L}^{-1}\left[-\frac{1}{18s^2}\mathcal{L}[A_n(x)]\right], \quad n \geq 0,$$

$$u_1(x) = -\frac{1}{24}x^4 - \frac{1}{3024}x^7 - \frac{1}{933120}x^{10},$$

$$u_2(x) = \frac{1}{3024}x^7 + \frac{11}{3265920}x^{10} + \frac{37}{3056901120}x^{13} + \frac{1}{48372940800}x^{16},$$

$$u_3(x) = -\frac{1}{435456}x^{10} - \frac{41}{1528450560}x^{13} - \frac{3937}{30813563289600}x^{16} - \frac{193}{645198284390400}x^{19} - \frac{1}{2896339504988160}x^{22},$$

The approximate solution then becomes,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 3x - \frac{1}{67931136}x^{13} - \frac{11}{102711877632}x^{16} - \frac{193}{645198284390400}x^{19} - \frac{1}{2896339504988160}x^{22} + \cdots$$

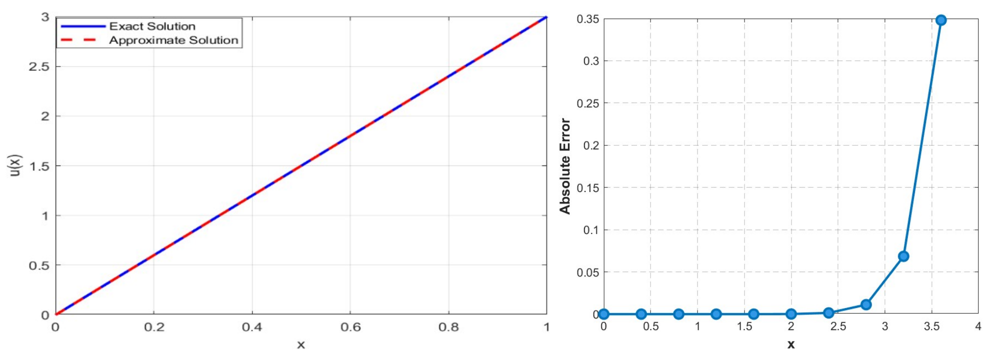

Figure 2: Comparison of exact vs. approximate solutions and absolute error by using **LADM**: for example (2.2).

The accuracy of LADM is confirmed by Figure 2, which displays excellent agreement over $[0, 1]$ between the LADM approximation (blue dashed line) and the exact solution $u(x) = 3x$ (red solid line). Because of series truncation, the error subplot is insignificant until $x \approx 2.5$, at which point it slightly increases and peaks close to $x = 4$. The resilience of LADM is demonstrated by the inaccuracy remaining tiny in spite of this.

Example 2.3. Solve the following Volterra nonlinear integral equation

$$u(x) = f(x) + \int_0^x (x-t)u^3(t)\,dt, \tag{2.14}$$

$$f(x) = e^x - \frac{1}{9}e^{3x} + \frac{1}{9} + \frac{1}{3}x,$$

The exact solution of (2.14) is given as $u(x) = e^x$,

Solution: Using the linearity property of the Laplace transform on both sides of (2.14), we get

$$\mathcal{L}[u(x)] = \mathcal{L}\left[e^x - \tfrac{1}{9}e^{3x} + \tfrac{1}{9} + \tfrac{1}{3}x\right] + \mathcal{L}\left[\int_0^x (x-t)u^3(t)dt\right], \tag{2.15}$$

To simplify the application of the iterative method, we expand the known function

$$f(x) = e^x - \tfrac{1}{9}e^{3x} + \tfrac{1}{9} + \tfrac{1}{3}x,$$

into a Taylor series about $x = 0$, yielding:

$$f(x) = 1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 - \tfrac{13}{60}x^5 - \tfrac{1}{9}x^6 + \cdots.$$

Thus, the series solution strategy generates the function $u(x)$:

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(x)\right] = \mathcal{L}\left[1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 - \tfrac{13}{60}x^5 - \tfrac{1}{9}x^6 + \cdots\right] + \tfrac{1}{s^2}\mathcal{L}\left(\sum_{n=0}^{\infty} A_n(x)\right), \tag{2.16}$$

and decompose the nonlinear term:

$$u^3(x) = \sum_{n=0}^{\infty} A_n(x),$$

where $A_n(x)$ are Adomian polynomials defined recursively:

$$\begin{cases} A_0 = u_0^3, \\ A_1 = 3u_0^2u_1, \\ A_2 = 3u_0^2u_2 + 3u_0u_1^2, \\ A_3 = 3u_0^2u_3 + 6u_0u_1u_2 + u_1^3, \\ \quad\vdots \end{cases}$$

$$\mathcal{L}[u_0(x)] = \mathcal{L}[f(x)],$$

$$\mathcal{L}[u_0(x)] = \mathcal{L}\left[1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 - \tfrac{13}{60}x^5 - \tfrac{1}{9}x^6 + \cdots\right],$$

$$u_0(x) = \mathcal{L}^{-1}\left[1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 - \tfrac{13}{60}x^5 - \tfrac{1}{9}x^6 + \cdots\right],$$

In general,

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}[A_n(x)]\right), \quad n \geq 0,$$

$$u_1(x) = \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 - \frac{1}{10}x^6 - \frac{73}{840}x^7 - \frac{23}{560}x^8 - \frac{19}{4320}x^9 + \frac{11}{900}x^{10} + \cdots$$

$$u_2(x) = \frac{1}{8}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6 + \frac{1}{21}x^7 - \frac{51}{1120}x^8 - \frac{1403}{20160}x^9 - \frac{7411}{151200}x^{10} - \frac{19777}{1108800}x^{11} + \cdots$$

$$u_3(x) = \frac{3}{80}x^6 + \frac{7}{80}x^7 + \frac{3}{32}x^8 + \frac{1013}{20160}x^9 - \frac{29}{6720}x^{10} - \frac{25787}{739200}x^{11} - \frac{236683}{6652800}x^{12} + \cdots$$

Then the approximate solution becomes,

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{27}{560}x^7 + \frac{1}{140}x^8 + \cdots$$

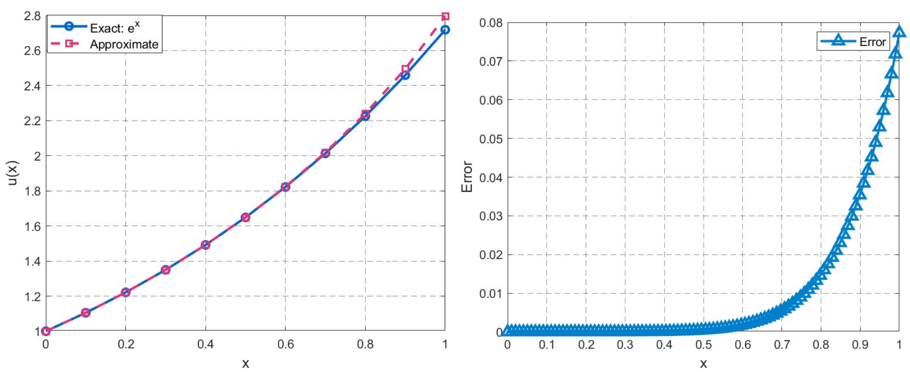

Figure 3: Using LADM, compare exact versus approximate answers and absolute error for example (2.3). Figure 3 shows a significant agreement over $[0, 1]$, despite a little variation near $x = 1$, between the exact solution $u(x) = e^x$ (solid blue line with circles) and the LADM approximation (dashed pink line with squares). The efficacy of LADM for such issues is validated by the error subplot, which shows great accuracy with the absolute error staying below 0.08 even for the cubic nonlinear $u^3(x)$.

3. Application of the Variational Iteration method for solving (NVIEs-II)

In this section, we discuss the Variational Iteration Method (VIM), a semi-analytical technique introduced to efficiently solve various types of linear and nonlinear problems, particularly differential and integral equations. This method is especially effective in handling nonlinearities without the need for linearization or perturbation techniques, which often limit the applicability of traditional analytical methods. The main strength of VIM lies in its flexibility and simplicity in constructing correction functionals based on variational theory. Unlike decomposition methods that require additional constructs such as special polynomials, VIM relies on Lagrange multipliers that are optimally identified via variational principles. These multipliers are embedded within an iterative scheme that refines the solution through successive corrections, leading to rapid convergence toward the exact solution when it exists.

The VIM has been widely applied to integro-differential problems and nonlinear Volterra integral equations of the second kind (NVIEs-II), as it provides an elegant framework for converting the problem into a form suitable for iteration. By applying a combination of transformation techniques and integral correction structures, VIM not only simplifies the computational process but also retains the nonlinear structure of the original equation throughout the solution steps.

The general form of the correction functional employed in VIM is given by:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\big(Lu_n(\xi) + NF(u_n(\xi)) - g(\xi)\big)\,d\xi, \quad n \geq 0, \tag{3.1}$$

where:

- $\lambda(\xi)$ denotes the Lagrange multiplier,
- L is a linear operator,
- $NF(u)$ represents the nonlinear component of the equation,
- $g(\xi)$ is a known function.

For first-order Volterra differential equations, the Lagrange multiplier is typically selected as $\lambda(\xi) = -1$, following the formulation introduced by He (1999). The exact solution is obtained in the limit:

$$u(x) = \lim_{n\to\infty} u_n(x).$$

To apply VIM in the context of nonlinear Volterra integral equations, the following procedural steps are carried out:

1. **Transformation:** Differentiate both sides of the Volterra integral equation using Leibniz's rule to convert it into a corresponding integro-differential equation.
2. **Identification of Multiplier:** Analyze the structure and order of the transformed equation to determine the appropriate Lagrange multiplier $\lambda(\xi)$.
3. **Initial Approximation:** Select an initial approximation, typically derived from the initial conditions of the problem.
4. **Iterative Computation:** Generate successive approximations using the correction function defined in Equation (3.1), refining the solution at each step. This step significantly enhances the accurate treatment of nonlinearities and offers a practical framework for solving a broad class of Volterra integral equations.

Example 3.1. Solve the following Volterra nonlinear integral equation by using (VIM).

$$u(x) = f(x) + \int_0^x (x-t)u^2(t)\,dt, \tag{3.2}$$

$$f(x) = \sin x + \cos x + \frac{\sin 2x}{4} - \frac{x}{2} - \frac{x^2}{2},$$

The exact solution of (3.2) is given as $u(x) = \sin x + \cos x$.

Solution. We first determine the known portion of the problem as follows before starting to solve the provided nonlinear Volterra integral equation using the Variational Iteration Method (VIM):

$$f(x) = \sin x + \cos x + \frac{\sin 2x}{4} - \frac{x}{2} - \frac{x^2}{2},$$

Using a Taylor series about $x = 0$, yielding

$$f(x) = 1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots,$$

This yields a power series representation suitable for computing the initial term $u_0(x)$ and constructing the recursive approximation sequence.

$$u(x) = 1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots$$
$$+ \int_0^x (x-t)u^2(t)\,dt,$$

Using Leibniz's rule, we differentiate both sides of equation (3.2) for x, yielding the corresponding integro-differential equation:

$$u'(x) = 1 - 2x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{8}x^4 - \frac{1}{120}x^5 + \cdots + \int_0^x u^2(t)\,dt, \tag{3.3}$$

$$u(0) = 1,$$

By substituting $x = 0$ into equation (3.2), we obtain the initial condition: $u(0) = 1$. Thus, we can choose the initial approximation as $u_0(x) = 1$,

The correction functional corresponding to equation (3.2), using the first-order integro-differential form and choosing the Lagrange multiplier $\lambda = -1$, is given by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 + 2\xi + \frac{3}{2}\xi^2 - \frac{1}{6}\xi^3 - \frac{3}{8}\xi^4 + \frac{1}{120}\xi^5 - \int_0^\xi u_n^2(r)\,dr \right) d\xi, \tag{3.4}$$

Using the formula (3.4), we obtain the following approximations:

$$u_0(x) = 1,$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - 1 + 2\xi + \frac{3}{2}\xi^2 - \frac{1}{6}\xi^3 - \frac{3}{8}\xi^4 + \frac{1}{120}\xi^5 + \cdots \right.$$
$$\left. - \int_0^\xi u_0^2(r)\,dr \right) d\xi.$$

$$u_1(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{3}{40}x^5 - \frac{1}{720}x^6 + \cdots$$

$$u_2(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{40}x^5 - \frac{17}{720}x^6 + \frac{11}{630}x^7 + \frac{2}{315}x^8 - \frac{43}{25920}x^9 + \cdots$$

$$u_3(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{210}x^7$$
$$- \frac{1}{504}x^8 - \frac{1}{60480}x^9 + \frac{1639}{1814400}x^{10} - \frac{13}{623700}x^{11} - \frac{16717}{119750400}x^{12} + \cdots.$$

Therefore, the approximate solution converges to the exact solution:

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x + \cos x.$$

Example 3.2. Solve the (NVIE-II), by using the Variational Iteration Method.

$$u(x) = 3x + \frac{1}{24}x^4 - \frac{1}{18}\int_0^x (x-t)u^2(t)\,dt, \tag{3.5}$$

Has exact solution as $u(x) = 3x$,

Solution. To find the integro-differential equation, differentiate both sides of equation (3.5), around x, using Leibniz's rule,

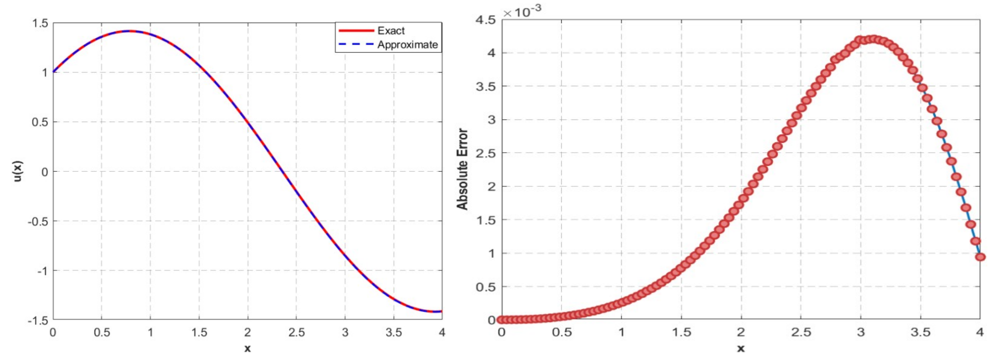

Figure 4: Using VIM, compare exact versus approximate answers and absolute error for Example (3.1). Figure 4 displays nearly comparable curves for Example (3.1) when comparing the VIM approximation to the actual answer, $u(x) = \sin(x) + \cos(x)$. The accuracy of VIM in solving nonlinear Volterra equations is confirmed by the absolute error being below 0.004 across $[0, 4]$.

$$u'(x) = 3 + \frac{1}{6}x^3 - \frac{1}{18}\int_0^x u^2(t)\,dt, \quad u(0) = 0, \tag{3.6}$$

We can select $u_0(x) = 0$ by entering $x = 0$ into equation (3.5), which yields the initial condition $u(0) = 0$. For equation (3.5), the correction functional is,

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 3 - \frac{1}{6}\xi^3 + \frac{1}{18}\int_0^\xi u_n^2(r)\,dr \right) d\xi, \tag{3.7}$$

where we selected the integro-differential equation of first order with $\lambda = -1$. As mentioned before, we can use the initial condition to select $u_0(x) = 0$, which will lead to the following approximations.

$$u_0(x) = 0,$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - 3 - \frac{1}{6}\xi^3 + \frac{1}{18}\int_0^\xi u_0^2(r)\,dr \right) d\xi,$$

$$u_1(x) = 3x + \frac{1}{24}x^4,$$

$$u_2(x) = 3x - \frac{1}{3024}x^7 - \frac{1}{933120}x^{10}.$$

$$u_3(x) = 3x + \frac{1}{816480}x^{10} + \frac{1}{436700160}x^{13} - \frac{1}{39504568320}x^{16} - \frac{1}{7240848762470400}x^{22},$$

As a result, the approximate and exact solutions converge:

$$u(x) = \lim_{n\to\infty} u_n(x) = 3x.$$

The VIM approximation and the actual solution $u(x) = 3x$ for Example (3.2) are compared in Figure 5, which demonstrates the close alignment of the two curves. The stability and dependability of VIM are confirmed by the error profile, which shows a modest increase close to $x = 4$, although the absolute error stays below 0.018.

Example 3.3. Solve the (NVIE-II), by using the Variational Iteration Method.

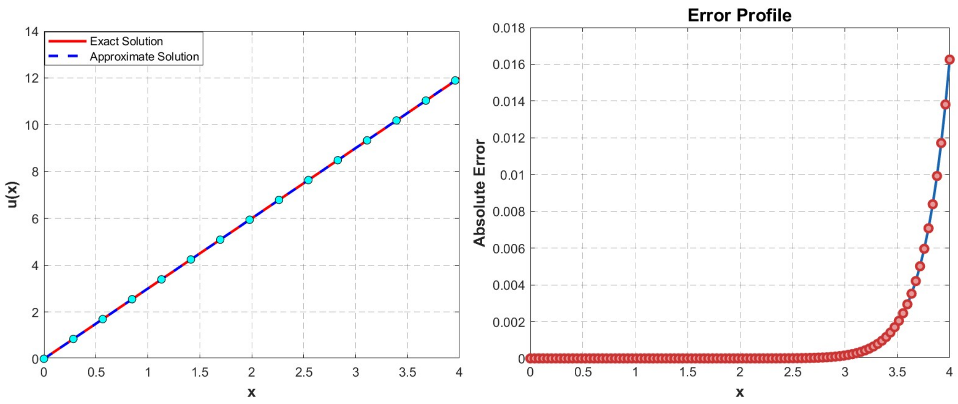

Figure 5: Comparison of exact vs. approximate solutions and absolute error by using (VIM), for example (3.2).

$$u(x) = f(x) + \int_0^x (x-t)u^3(t)\,dt, \tag{3.8}$$

$$f(x) = e^x - \frac{1}{9}e^{3x} + \frac{1}{9} + \frac{1}{3}x,$$

Has exact solution as $u(x) = e^x$,

Solution. To simplify the application of the iterative method, we expand the known function

$$f(x) = e^x - \frac{1}{9}e^{3x} + \frac{1}{9} + \frac{1}{3}x,$$

We expand $f(x)$ in a Taylor series about $x = 0$, to facilitate iterative computations:

$$f(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{13}{60}x^5 - \frac{1}{9}x^6 + \cdots,$$

$$u(x) = 1 + x - \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{13}{60}x^5 - \frac{1}{9}x^6 + \cdots + \int_0^x (x-t)u^3(t)\,dt,$$

Differentiate both sides of equation (3.8), concerning x, using Leibniz's rule. Thus, we get:

$$u'(x) = 1 - x^2 - \frac{4}{3}x^3 - \frac{13}{12}x^4 - \frac{2}{3}x^5 + \cdots + \int_0^x u^3(t)\,dt, \tag{3.9}$$

From equation (3.8), set $x = 0$ to find:

$$u(0) = 1,$$

So we choose:

$$u_0(x) = 1,$$

We construct the correction functional using Lagrange multiplier $\lambda = -1$, as follows:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 1 + \xi^2 + \frac{4}{3}\xi^3 + \frac{13}{12}\xi^4 + \frac{2}{3}\xi^5 + \cdots - \int_0^\xi u_n^3(r)\,dr \right) d\xi, \tag{3.10}$$

Let's compute the first few iterations:

$$u_0(x) = 1,$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - 1 + \xi^2 + \frac{4}{3}\xi^3 + \frac{13}{12}\xi^4 + \frac{2}{3}\xi^5 + \cdots - \int_0^\xi u_0^3(r)\,dr \right) d\xi,$$

$$u_1(x) = 1 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{13}{60}x^5 - \frac{1}{9}x^6,$$

$$u_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{15}x^5 - \frac{49}{360}x^6 - \frac{13}{140}x^7 + \cdots .$$

$$u_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{3}{70}x^7 + \frac{67}{6720}x^8 + \cdots .$$

Therefore, the approximate solution converges to the exact solution:

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x.$$

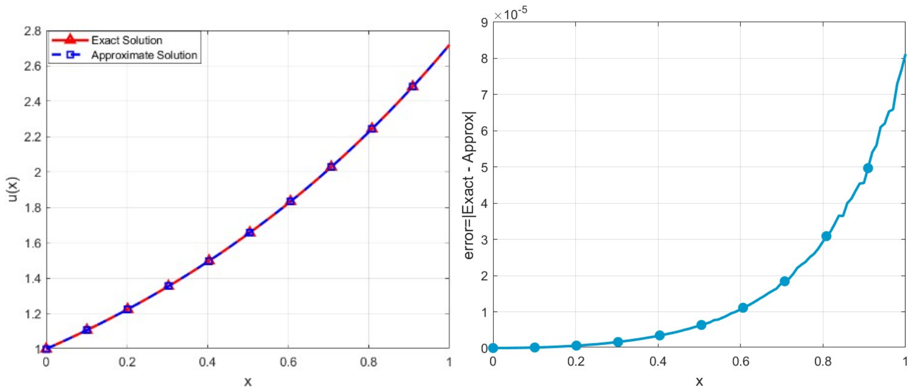

Figure 6: Comparison of exact and approximate solutions and absolute error by using (VIM) for Example (3.3).
Strong agreement over $[0, 1]$ is seen when comparing the VIM approximation for Example (3.3) with the exact solution, $u(x) = e^x$, in Figure 6. The efficacy of VIM for nonlinear integral equations is validated by the error profile, which shows great accuracy with an absolute error of less than 0.018.

4. A Comparative Performance Analysis and Accuracy Assessment of VIM and LADM Methods

This section provides a comparative analysis of the numerical performance of the Variational Iteration Method (VIM) and the Laplace–Adomian Decomposition Method (LADM) by comparing their results with the exact solutions of the integral equations. The absolute errors for each method were computed to assess their accuracy and efficiency in handling the given examples.

The comparative study includes three test cases (Examples 2.1, 2.3, and 3.3) to demonstrate the effectiveness of both methods across different problem configurations. The following tables summarize the numerical values and errors for each example.

Table 1: "Exact and Numerical Solutions with Error Analysis for Examples 2.1 and 3.1 Using VIM and LADM

x	Exact	Approximate (LADM)	Error (LADM)	Approximate (VIM)	Error (VIM)
0	1.0000	1.0000	0.000000	0.0000	0.0000
0.1	1.09484	1.09477	7.0252e^{-05}	1.09461	0.00023
0.2	1.17874	1.17861	1.2146e^{-04}	1.17868	6e^{-05}
0.3	1.25086	1.25071	1.4625e^{-04}	1.25059	0.00027
0.4	1.31048	1.31034	1.4003e^{-04}	1.31036	0.00012
0.5	1.35701	1.35690	1.0590e^{-04}	1.3567	0.00031
0.6	1.38998	1.38991	6.78704e^{-05}	1.38976	0.00022
0.7	1.40906	1.40888	1.8377e^{-04}	1.4087	0.00036
0.8	1.41406	1.41338	6.8118e^{-04}	1.41373	0.00033
0.9	1.40494	1.40316	1.7801e^{-03}	1.40451	0.00043
1.0	1.38177	1.37766	4.1140e^{-03}	1.38131	0.00046

Table 1 presents the absolute errors for the Laplace Adomian Decomposition Method (LADM) and the Variational Iteration Method (VIM) across the domain of the problem. Initially, both methods start with zero error, but differences emerge as x increases. LADM shows a fluctuating error behavior that grows steadily and reaches a maximum of approximately 0.0041. In contrast, VIM demonstrates more stable and relatively lower errors throughout the interval, with a maximum error of around 0.00046. This comparison highlights that VIM outperforms LADM in terms of numerical accuracy and error stability, particularly for larger values of x. The consistently lower error values in VIM suggest it is more reliable for solving this class of nonlinear Volterra integral equations.

Table 2: "Exact and Numerical Solutions with Error Analysis for Examples 2.2 and 3.2 Using VIM and LADM

x	Exact	Approximate (LADM)	Error (LADM)	Approximate (VIM)	Error (VIM)
0.00000	0.00000	0.00000	0.000000	0.00000	0.000000
0.12121	0.36364	0.39364	0.000000	0.36364	0.000000
0.20202	0.60606	0.60606	0.000000	0.60606	0.000000
0.32323	0.96970	0.96970	6.2172e^{-15}	0.96970	0.000000
0.40404	1.21210	1.21210	1.1258e^{-13}	1.45450	2.2204e^{-16}
0.52525	1.57580	1.57580	3.4133e^{-12}	1.69700	4.4409e^{-16}
0.60606	1.81820	1.81820	2.1945e^{-11}	1.81820	1.5543e^{-15}
0.72727	2.18180	2.18180	2.3508e^{-10}	2.18180	2.5757e^{-14}
0.80808	2.42420	2.42420	9.2578e^{-10}	2.42420	1.3545e^{-13}
0.92929	2.78790	2.78790	5.7075e^{-09}	2.78790	1.2315e^{-12}
1.00000	3.00000	3.00000	1.4828e^{-08}	3.00000	4.6114e^{-12}

From the above error comparison, it is evident that the Variational Iteration Method (VIM) provides a solution that is significantly closer to the exact solution compared to the Laplace Adomian Decomposition Method (LADM). The error values in VIM remain extremely small throughout the interval, especially in the first several steps, showing near-zero deviation from the exact solution. This indicates that VIM achieves higher accuracy and better convergence properties for this specific problem, as shown in Table 2.

Table 3: "Exact and Numerical Solutions with Error Analysis for Examples 2.3 and 3.3 Using VIM and LADM

x	Exact	Approximate (LADM)	Error (LADM)	Approximate (VIM)	Error (VIM)
0	1.0000	1.0000	0.000000	1.0000	$0.0000e^{+00}$
0.1	1.1052	1.1052	$4.9908e^{-09}$	1.10629	$1.4493e^{-07}$
0.2	1.2214	1.2214	$6.656e^{-07}$	1.22387	$6.5931e^{-07}$
0.3	1.3499	1.3499	$1.1885e^{-05}$	1.35396	$1.6602e^{-06}$
0.4	1.4918	1.4919	$9.337e^{-05}$	1.49787	$3.4814e^{-06}$
0.5	1.6487	1.6492	0.00046874	1.64042	$5.9812e^{-06}$
0.6	1.8221	1.8239	0.0017742	1.81478	$1.0459e^{-05}$
0.7	2.0138	2.0193	0.0055249	2.00768	$1.7647e^{-05}$
0.8	2.2255	2.2404	0.014878	2.22108	$2.9082e^{-05}$
0.9	2.4596	2.4953	0.035665	2.45717	$4.5580e^{-05}$
1	2.7183	2.7955	0.077266	2.71836	$8.1190e^{-05}$

Table 3 illustrates the absolute error values obtained by the Laplace-Adomian Decomposition Method (LADM) and the Variational Iteration Method (VIM). It is observed that both methods yield very small errors at the beginning of the interval. However, as x increases, the error of LADM grows significantly faster than that of VIM. The maximum error in LADM reaches approximately 0.0773, while VIM maintains a much lower maximum error of about 8.12×10^{-5}. These results indicate that VIM provides better numerical accuracy and stability compared to LADM in this case, especially for larger values of x.

5. Conclusion

This work presents a detailed investigation into the use of two numerical schemes, the Laplace Adomian Decomposition Method (LADM) and the Variational Iteration Method (VIM), for solving nonlinear Volterra integral equations of the second kind. The numerical results obtained from several test examples confirmed the ability of both methods to produce accurate approximations. However, a closer examination revealed that VIM generally achieves better accuracy and faster convergence, particularly as the complexity or domain of the problem increases. While LADM remains effective, especially over narrow intervals, its precision tends to decline more significantly in extended cases. VIM, on the other hand, demonstrated enhanced stability and flexibility, making it more suitable for a wider variety of nonlinear systems.

Overall, the study supports the application of both techniques but highlights VIM's stronger performance across multiple scenarios. Future work may consider combining these methods or adapting them to hybrid frameworks to further improve accuracy, computational speed, and applicability to complex integral models.

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