Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.78036

On Ideal-Statistical Convergence in Partial Metric Space

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ABSTRACT: This paper explores the foundational aspects of $\mathcal{I}$-statistical convergence in the partial metric space. We also investigate key properties, provide illustrative examples, and establish several fundamental theorems characterizing the behavior of $\mathcal{I}$-statistically convergent sequences in a partial metric space.

Key Words: Partial metric space, $\mathcal{I}$- statistical convergence, $\mathcal{I}$- statistical limit points, $\mathcal{I}$- statistical cluster points.

Contents

1. Introduction

The natural density of a set $K \subset \mathbb{N}$ (the set of natural numbers) is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|.$$

Fast [6] and Steinhaus [17] independently proposed the idea of statistical convergence in 1951, which is based on the natural density of subsets of $\mathbb{N}$. $\mathcal{I}$-convergence [12] and $\mathcal{I}$-statistical convergence [5] are prominent extensions of statistical convergence that incorporate the concept of ideals. Later, this topic drew the attention of many scholars, and several studies have been conducted on it (see J.A Fridy [7], E. Savas & P. Das [16], T. Salat et al. [18], M. Mursaleen et al. [14]). For a more comprehensive overview, we refer the reader to [2,10,9,11,3,4].

In 1994, S. G. Matthews [13] first introduced the concept of partial metric spaces as an expansion to typical metric spaces. A partial metric permits a non-zero self-distance, in contrast to classical metrics space, where the distance between a point and itself is always zero. The concept of statistical convergence in a partial metric space [15] emerged from combining the flexibility of statistical convergence with the structure of partial metric spaces. A sequence $\{x_k\}$ in a partial metric space (X, p) is said to be statistically convergent to $a_0 \in X$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| = 0.$$

In this expanded framework, various types of sequence convergence have been studied in recent times. For a more comprehensive perspective, we refer to [1,8]. This work explores $\mathcal{I}$-statistical convergence and its features in partial metric spaces.

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2010 *Mathematics Subject Classification:* 40A05, 40A35, 30L99.
Submitted July 24, 2025. Published October 29, 2025

2. Preliminaries

Definition 2.1 [12] A non-empty family of subsets $\mathcal{I} \subset P(\mathbb{N})$ is called an *ideal* of $\mathbb{N}$ if and only if the following conditions are satisfied,

(i) $\emptyset \in \mathcal{I}$;

(ii) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(iii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.2 [12] A family of subsets $\mathcal{I} \subset P(\mathbb{N})$ is called a *filter* of $\mathbb{N}$ if and only if the following assertions are satisfied:

(i) $\emptyset \notin \mathcal{F}$;

(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

Definition 2.3 [12] An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$

In this paper, $\mathcal{I}$ will denote a proper admissible ideal of $\mathbb{N}$.

Definition 2.4 [13] Let X be a nonempty set and a function $p : X \times X \to \mathbb{R}$ such that if p is satisfies the following conditions for all $x, y, z \in X$
$PM_1 : 0 \leq p(x, x) \leq p(x, y)$ (non-negativity and small self-distances),
PM_2 : if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (indistancy implies equality),
$PM_3 : p(x, y) = p(y, x)$ (symmetry), and
$PM_4 : p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ (tri-angularity).

then p is called the partial metric on X and (X, p) is called the partial metric space.

Definition 2.5 [8] Let (X, p) be a partial metric space and $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$.Then a sequence $\{x_k\}$ is said to be $\mathcal{I}$-convergent to a_0 if for each $\epsilon > 0$

$$\{k \in \mathbb{N} : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\} \in \mathcal{I}$$

where a_0 is called the $\mathcal{I}$-limit of the sequence $\{x_k\}$ and we write $\mathcal{I}_p\text{-}\lim p(x_k, a_0) = p(a_0, a_0)$

3. $\mathcal{I}_p$-statistical convergence

Definition 3.1 Let $\{x_k\}$ be a sequence in a partial metric space (X, p). Then $\{x_k\}$ is said to be $\mathcal{I}_p$-statistical convergent to $a_0 \in X$ if for each $\epsilon > 0$ and $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

We can also write $\mathcal{I}_p\text{-st}\lim p(x_k, a_0) = p(a_0, a_0)$.

Example 3.1 *Let (X, p) be a partial metric space where $X = [0, \infty)$ and $p(x, y) = max\{x, y\}$. Also, let a sequence $\{x_n : n \in \mathbb{N}\}$ in (X, p) such that*

$$x_n = \begin{cases} 4, & if\ n = k^2,\ for\ some\ k \in \mathbb{N} \\ 2, & otherwise. \end{cases}$$

Consider an admissible ideal $\mathcal{I}$.
Now, $\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, 4) - p(4, 4)| \geq \epsilon\}| \geq \delta\} = \emptyset \in \mathcal{I}$.
Therefore, $\{x_n\}$ is $\mathcal{I}_p$-statistical convergent to 4.
Again, $\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, 2) - p(2, 2)| \geq \epsilon\}| \geq \delta\} = \{finite\ set\} \in \mathcal{I}$.
Therefore, $\{x_n\}$ is $\mathcal{I}_p$-statistical convergent to 2.
Thus, the limit of $\mathcal{I}$-statistical convergent sequence in a partial metric space may not be unique.

Corollary 3.1 *The collection of all the limits of a $\mathcal{I}$-statistical convergent sequence in a partial metric space is called the $\mathcal{I}_p$-statistical limit set of the sequence and is denoted by $\mathcal{I}_p$-st(L_x). Also, if the set $\mathcal{I}_p$-st$(L_x) \neq \emptyset$, then the sequence is considered $\mathcal{I}_p$-statistical convergent.*

Theorem 3.1 *Let $\{x_k\}$ be a sequence in a partial metric space (X, p). If st-$\lim p(x_k, a_0) = p(a_0, a_0)$ then $\mathcal{I}_p$-st $\lim p(x_k, a_0) = p(a_0, a_0)$.*

Proof: Let the sequence $\{x_k\}$ statistically convergent to a_0. Then for each $\epsilon > 0$ there exists

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| = 0.$$

So for each $\epsilon > 0$ and each $\delta > 0$, the set

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \text{ must be finite.}$$

Therefore, this set belongs to $\mathcal{I}$.
Hence $\mathcal{I}_p$-st $\lim p(x_k, a_0) = p(a_0, a_0)$.

Example 3.2 *The following example illustrates how the linearity property does not hold for partial metric spaces, in contrast to the metric space.*

Let $(\mathbb{R}^-, p)$ be a partial metric space where $p(x, y) = -\min\{x, y\}$. Also, let $\{x_k\}$ and $\{y_k\}$ be two sequences in the partial metric space $(\mathbb{R}^-, p)$ such that

$$x_k = \begin{cases} -2, & \text{if } k \text{ is odd,} \\ -1, & \text{if } k \text{ is even.} \end{cases}$$

and

$$y_k = \begin{cases} -3, & \text{if } k \text{ is odd,} \\ -5, & \text{if } k \text{ is even.} \end{cases}$$

Let us take, $\mathcal{I} = \mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$.
Now for any $\epsilon > 0$ and $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, -2) - p(-2, -2)| \geq \epsilon\}| \geq \delta\} = \emptyset \in \mathcal{I}.$$

Therefore, $\{x_k\}$ is $\mathcal{I}_p$-statistically convergent to -2.
Similarly, for any $\epsilon > 0$ and $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(y_k, -5) - p(-5, -5)| \geq \epsilon\}| \geq \delta\} = \emptyset \in \mathcal{I}.$$

Therefore $\{y_k\}$ is $\mathcal{I}_p$-statistically convergent to -5.
But

$$x_k + y_k = \begin{cases} -5, & \text{if } k \text{ is odd,} \\ -6, & \text{if } k \text{ is even.} \end{cases}$$

doesn't $\mathcal{I}_p$-statistically convergent to $(-2) + (-5) = -7$.

In this example, the underlying space is $\mathbb{R}^- = \{x \in \mathbb{R} : x \leq 0\} \subset \mathbb{R}$ preserves the conventional addition operation. Therefore, the addition used in this example is valid because of the structure that was inherited from $\mathbb{R}$, but it cannot be generalized to arbitrary partial metric spaces.

Theorem 3.2 *In a partial metric space (X, p), if each subsequence of $\{x_k\}$ is $\mathcal{I}_p$-statistically convergent to a_0, then the sequence $\{x_k\}$ is also $\mathcal{I}_p$-statistically convergent to a_0.*

Proof: Suppose that $\{x_k\}$ is not $\mathcal{I}_p$-statistically convergent to a_0 in a partial metric space (X,p). Then there exist $\epsilon, \delta > 0$ such that,

$$A = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \notin \mathcal{I}$$

Since $\mathcal{I}$ is an admissible ideal, A must be an infinite set.

Let $A = \{n_1 < n_2 < \cdots < n_m < \dots\}$.

Let us define $y_m = x_{k_m}$ for $m \in \mathbb{N}$. Here, $\{y_m\}$ is a subsequence of $\{x_k\}$ that does not statistically converge to a_0, leading to a contradiction.

Hence, the theorem is proved.

Theorem 3.3 *Let (X,p) be a partial metric space and let $\{x_k\}, \{y_k\}$ and $\{z_k\}$ be three sequences in X. If the following conditions*

1. *$\mathcal{I}_p$-st $\lim p(x_k, a_0) = p(a_0, a_0)$ and $\mathcal{I}_p$-st $\lim p(z_k, a_0) = p(a_0, a_0)$*
2. *$p(x_k, a_0) \leq p(y_k, a_0) \leq p(z_k, a_0)$ for every $k \in H$ where $H \in \mathcal{F}(\mathcal{I})$*

are provided, then $\mathcal{I}_p$-st $\lim p(y_k, a_0) = p(a_0, a_0)$.

Proof: Assume that $\mathcal{I}_p$-$st \lim p(x_k, a_0) = p(a_0, a_0)$ and $\mathcal{I}_p$-$st \lim p(z_k, a_0) = p(a_0, a_0)$. Then for every $\epsilon, \delta > 0$, we have

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}$$

$$\text{and} \quad \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(z_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

This implies that the sets

$$G = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta\} \in \mathcal{F}(\mathcal{I})$$

$$\text{and} \quad K = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(z_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta\} \in \mathcal{F}(\mathcal{I})$$

Also, $p(x_k, a_0) \leq p(y_k, a_0) \leq p(z_k, a_0)$ for every $k \in H \in \mathcal{F}(\mathcal{I})$

So, $G \cap K \cap H \neq \emptyset$ and $G \cap K \cap H \in \mathcal{F}(\mathcal{I})$.

Now for all $n \in G \cap K \cap H$ and $\epsilon, \delta > 0$, $\frac{1}{n} |\{k \leq n : |p(y_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta$

i.e., $\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(y_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta\} \supseteq G \cap K \cap H \in \mathcal{F}(\mathcal{I})$.

Therefore,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(y_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

Hence Proved.

4. $\mathcal{I}_p$-statistical limit point and $\mathcal{I}_p$-statistical cluster point

Definition 4.1 Let (X,p) be a partial metric space. An element $a_0 \in X$ is said to be $\mathcal{I}_p$-statistical limit point of a sequence $\{x_k\}$ if for every $\epsilon > 0$, there exists a set $M = \{m_1 < m_2 < .....\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and st-lim $p(x_{m_i}, a_0) = p(a_0, a_0)$.

The set of all $\mathcal{I}_p$ statistical limit points of the sequence $\{x_k\}$ is denoted by $\mathcal{I}_p$-st(Λ_x).

Definition 4.2 Let (X,p) be a partial metric space. An element $a_0 \in X$ is said to be $\mathcal{I}_p$-statistical cluster point of a sequence $\{x_k\}$ if for every $\epsilon > 0$ and $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta\} \notin \mathcal{I}$$

The set of all $\mathcal{I}_p$-statistical cluster points of the sequence $\{x_k\}$ is denoted by $\mathcal{I}_p$-st(Γ_x).

Example 4.1 *Let $\mathcal{I}$ be an ideal that contains sets with natural density 0. Let us take the partial metric on $X = \mathbb{R}$ as $p(x,y) = \max\{x,y\}$. Let us define a sequence $\{x_k\}$ such that*

$$x_k = \begin{cases} \frac{1}{\sqrt{k}}, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases}$$

Now, for any $\epsilon > 0$, $\{k \leq n : |p(x_k,1) - p(1,1)| \geq \epsilon\}$ is an empty set.
Therefore for any $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k,1) - p(1,1)| \geq \epsilon\}| < \delta\} \notin \mathcal{I}$$

i.e., 1 is a $\mathcal{I}_p$-statistical cluster point of $\{x_k\}$, whereas 0 is not.
However, 0 and 1 both are the $\mathcal{I}_p$-statistical limit point of $\{x_k\}$.

Theorem 4.1 *For any sequence $\{x_k\}$ in a partial metric space (X,p), $\mathcal{I}_p$-$st(\Gamma_x) \subseteq \mathcal{I}_p$-$st(\Lambda_x)$.*

Proof: Consider a sequence $\{x_k\}$ in a partial metric space (X,p). Let $a_0 \in \mathcal{I}_p$-$st(\Gamma_x)$. Then for every $\epsilon, \delta > 0$ there exists a set

$$M = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k,a_0) - p(a_0,a_0)| \geq \epsilon\}| < \delta\} \notin \mathcal{I}$$

Let us consider a subsequence $\{x_{m_n}\}$ of $\{x_k\}$ such that $m_n \in M$. Since $\delta > 0$ is arbitrary,
Therefore,

$$\lim_{m_n \to \infty} \frac{1}{m_n} |\{m_n \in M : |p(x_{m_n},a_0) - p(a_0,a_0)| \geq \epsilon\}| = 0.$$

i.e., st-lim $p(x_{m_n},a_0) = p(a_0,a_0)$.
Hence, $a_0 \in \mathcal{I}_p$-$st(\Lambda_x) \Rightarrow \mathcal{I}_p$-$st(\Gamma_x) \subseteq \mathcal{I}_p$-$st(\Lambda_x)$.
However, the example 4.1 clearly shows that the converse is not true.

Theorem 4.2 *If $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences such that $\{k \in \mathbb{N} : x_k \neq y_k\} \in \mathcal{I}$ then*

1. *$\mathcal{I}_p$-$st(\Lambda_x) = \mathcal{I}_p$-$st(\Lambda_y)$*
2. *$\mathcal{I}_p$-$st(\Gamma_x) = \mathcal{I}_p$-$st(\Gamma_y)$*

Proof:

1. Consider $a_0 \in \mathcal{I}_p$-$st(\Lambda_x)$.

   So by definition there must exist a set $M = \{m_1 < m_2 < m_3 < ...\}$ of $\mathbb{N}$ such that $M \notin \mathcal{I}$ and st-lim $p(x_{m_k},a_0) = p(a_0,a_0)$.

   Since, $\{k \in \text{M} : x_n \neq y_n\} \subset \{k \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$.

   $M' = \{k \in M : x_n = y_n\} \notin \mathcal{I}$ and $M' \subseteq M$.

   So, st-lim $p(x_{m_k},a_0) = p(a_0,a_0) \Rightarrow$ st-lim $p(y_{m'_k},a_0) = p(a_0,a_0)$ where $m'_k \in M'$.

   i.e., $a_0 \in \mathcal{I}_p$-$st(\Lambda_y)$.

   So, $\mathcal{I}_p$-$st(\Lambda_x) \subseteq \mathcal{I}_p$-$st(\Lambda_y)$.

   Similarly, we can show that $\mathcal{I}_p$-$st(\Lambda_y) \subseteq \mathcal{I}_p$-$st(\Lambda_x)$.

   Hence, $\mathcal{I}_p$-$st(\Lambda_x) = \mathcal{I}_p$-$st(\Lambda_y)$.

2. Suppose that $a_0 \in \mathcal{I}_p$-$st(\Lambda_x)$.

   So according to definition, for each $\epsilon, \delta > 0$, the set

$$T = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |p(x_k,a_0) - p(a_0,a_0)| \geq \varepsilon\}| < \delta\} \notin \mathcal{I}$$

Let

$$V = \{n \in \mathbb{N} : \frac{1}{n}\,|\{k \leq n : |p(y_k, a_0) - p(a_0, a_0)| \geq \varepsilon\}| < \delta\}$$

We have to show that $V \notin \mathcal{I}$. If possible, suppose $V \in \mathcal{I}$, then

$$V^c = \{n \in \mathbb{N} : \frac{1}{n}\,|\{k \leq n : |p(y_k, a_0) - p(a_0, a_0)| \geq \varepsilon\}| \geq \delta\} \in \mathcal{F}(\mathcal{I}).$$

Also, by the hypothesis $C = \{k \in \mathbb{N} : x_k = y_k\} \in \mathcal{F}(\mathcal{I})$.

Thus, $V^c \cap C \in \mathcal{F}(\mathcal{I})$. Also, $V^c \cap C \subset T^c \in \mathcal{F}(\mathcal{I})$

i.e., $T \in \mathcal{I}$, which is a contradiction.

Therefore, $V \notin \mathcal{I}$.

i.e., $a_0 \in \mathcal{I}_p\text{-}st(\Gamma_y)$.

So, $\mathcal{I}_p\text{-}st(\Gamma_x) \subseteq \mathcal{I}_p\text{-}st(\Gamma_y)$.

Similarly, we can show that $\mathcal{I}_p\text{-}st(\Gamma_y) \subseteq \mathcal{I}_p\text{-}st(\Gamma_x)$.

Hence, $\mathcal{I}_p\text{-}st(\Gamma_x) = \mathcal{I}_p\text{-}st(\Gamma_y)$.

Theorem 4.3 *For any sequence $\{x_k\}$ in a partial metric space (X, p), the set $\mathcal{I}_p\text{-}st(\Gamma_x)$ is closed.*

Proof: Let (X, p) be a partial metric space and $\{x_k\}$ be a sequence in X. Let us assume y_0 is a limit point of the set $\mathcal{I}_p\text{-}st(\Gamma_x)$, then for any $\epsilon > 0$ we have,

$$\mathcal{I}_p - st(\Gamma_x) \cap B(y_0, \varepsilon) \neq \emptyset$$

where, $B(y_0, \epsilon) = \{z \in X : |p(z, y_0) - p(y_0, y_0)| < \varepsilon\}$.

Let $z_0 \in \mathcal{I}_p\text{-}st(\Gamma_x) \cap B(y_0, \epsilon)$, Let us choose $\epsilon_1 > 0$ such that $B(z_0, \epsilon_1) \subseteq B(y_0, \epsilon)$

Then we have

$$\{k \leq n : |p(x_k, z_0) - p(z_0, z_0)| \geq \epsilon_1\} \supseteq \{k \leq n : |p(x_k, y_0) - p(y_0, y_0)| \geq \epsilon\}$$

Now, for any $\delta > 0$

$$\{n \in \mathbb{N} : \frac{1}{n}\{k \leq n : |p(x_k, z_0) - p(z_0, z_0)| \geq \varepsilon_1\} < \delta\} \subseteq$$

$$\{n \in \mathbb{N} : \frac{1}{n}\{n \leq k : |p(x_k, y_0) - p(y_0, y_0)| \geq \varepsilon\} < \delta\}$$

Since $z_0 \in \mathcal{I}_p\text{-}st(\Gamma_x)$,

$$\therefore \{n \in \mathbb{N} : \frac{1}{n}\{n \leq k : |p(x_k, y_0) - p(y_0, y_0)| \geq \varepsilon\} < \delta\} \notin \mathcal{I}.$$

i.e., $y_0 \in \mathcal{I}_p\text{-}st(\Gamma_x)$.

$\therefore$ The set $\mathcal{I}_p\text{-}st(\Gamma_x)$ is closed.

Theorem 4.4 *Let $\{x_k\}$ be a sequence in a partial metric space (X, p). Then $\mathcal{I}_p\text{-}st(L_x) \subseteq \mathcal{I}_p\text{-}st(\Gamma_x) \subseteq \mathcal{I}_p\text{-}st(\Lambda_x)$.*

Proof: Let the sequence $\{x_n\}$ be $\mathcal{I}_p$-statistical convergent to a_0. Then for each $\epsilon > 0$ and each $\delta > 0$ there exists,

$$A = \{n \in \mathbb{N} : \frac{1}{n}\,|\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}$$

So

$$A^c = \{n \in \mathbb{N} : \frac{1}{n}\,|\{k \leq n : |p(x_k, a_0) - p(a_0, a_0)| \geq \epsilon\}| < \delta\} \notin \mathcal{I}$$

i.e., $a_0 \in \mathcal{I}_p\text{-}st(\Gamma_x)$.
Therefore, $\mathcal{I}_p\text{-}st(L_x) \subseteq \mathcal{I}_p\text{-}st(\Gamma_x)$. Again, Theorem 4.1 yields $\mathcal{I}_p\text{-}st(\Gamma_x) \subseteq \mathcal{I}_p\text{-}st(\Lambda_x)$.
So, $\mathcal{I}_p\text{-}st(L_x) \subseteq \mathcal{I}_p\text{-}st(\Gamma_x) \subseteq \mathcal{I}_p\text{-}st(\Lambda_x)$.

However, the converse of this theorem is not true. This can be easily shown with the help of the example 4.2 given below:

Example 4.2 *Let $\mathcal{I} = \{A \subset \mathbb{N} : \delta(A) = 0\}$ and the partial metric on $X = \mathbb{R}$ as $p(x, y) = \max\{x, y\}$. Let $\{x_k\}$ be a sequence on X such that*

$$x_k = \begin{cases} k, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases}$$

Then the subsequence $\{x_{k_i}\}$ where $k_i \in \{2n + 1 : n \in \mathbb{N}\} \notin \mathcal{I}$ of $\{x_k\}$ is statistically convergent to the limit 1.
So, $\mathcal{I}_p\text{-}st(\Lambda_x) = \{1\}$.
But $\{x_k\}$ is not $\mathcal{I}_p$-statistically convergent. So, $\mathcal{I}_p\text{-}st(L_x) = \emptyset$.

5. Conclusion

In this paper, we investigate the concept of ideal statistical convergence in the context of partial metric spaces, broadening traditional ideas of convergence by including both $\mathcal{I}$-statistical convergence and partial self-distances. Our results extended and enriched the previous convergence theorems by providing increased flexibility and application. This theoretical framework has potential applications in various fields such as computational mathematics, computer science, fuzzy systems, and data analysis. Future research could explore the integration of various generalized convergence methods inside partial metric spaces, as well as more specific applications in fixed point theory, optimization, and applied topology.

Authors Contribution: The authors contributed equally to the establishment of this work.

Conflict of Interest: The authors of this article do not have any conflicts of interest.

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