Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 1** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.78051

Weak Convergence in Uncertain Normed Linear Spaces

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ABSTRACT: In this paper, we introduce and investigate the concept of weak convergence in the framework of Uncertainty theory. Uncertainty theory which was developed to handle imprecise or incomplete information where probability theory fails, has led to new formulations of linear spaces and norms. Based on this foundation, we define weak convergence with respect to uncertain linear functionals and examine some of the fundamental properties. The relationships between uncertain weak and strong convergence are explored and criteria for weak convergence are established.

Key Words: Uncertainty theory, uncertain sequence, uncertain normed space, weak convergence.

Contents

1. Introduction

Uncertainty theory, introduced by Liu [3] in 2007 which provides a mathematical framework to deal with indeterminate phenomena that cannot be adequately modeled by classical probability. In recent years, remarkable progress has been made to study the concept of sequences of complex uncertain variables but very few attempt has been made to extend further for the analysis of uncertain normed spaces, for instance one may see ([1], [2], [5],[7],[8],[9], [10],[11], [14], [15], [17]). The study of weak convergence will provide new insight into the study of uncertain systems.

One of the key notions in classical analysis is weak convergence—a concept particularly useful in infinite-dimensional spaces where strong (norm) convergence may not hold, one may refer to ([4], [6], [12], [13], [16]). Weak convergence allows for the analysis of sequence behavior through the concept of duality, compactness and continuity. Moreover for the study of different geometric properties such as Kadec-lee, Uniform Kadec-Klee properties the study of weak convergence relaxes the conditions. However, its adaptation to uncertain environments is still unexplored.

The prime objective of this paper is to define weak convergence in uncertain normed spaces and to study its foundational properties. In particular, we introduce the notion of uncertain linear functionals and define an uncertain weak topology. We then examine convergence with respect to this topology, characterize basic convergence results and investigate how these relate to the uncertain norm convergence. This work contributes to the broader goal of developing a rigorous functional analysis framework under uncertainty.

2. Preliminaries

We begin with the basic definitions and concepts from uncertainty theory that are required throughout this paper. The following definition is from Liu [3].

2020 *Mathematics Subject Classification:* 60A05, 40A05, 46B45, 46B50, 40A30.
Submitted July 24, 2025. Published January 19, 2026

Uncertainty Space and Measure

Let Γ be a nonempty set representing the uncertainty space, and let $\mathcal{L}$ be a σ-algebra on Γ. A set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is called an *uncertainty measure* if it satisfies:

- $\mathcal{M}(\Gamma) = 1$,
- $\mathcal{M}(A) \leq \mathcal{M}(B)$ whenever $A \subseteq B$,
- $\mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathcal{M}(A_n)$, for disjoint $A_n \in \mathcal{L}$.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event.

A complex uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set $\mathcal{B}$ of complex numbers, the set

$$\{\xi \in \mathcal{B}\} = \{\gamma \in \Gamma : \xi(\gamma) \in \mathcal{B}\}$$

is an event as defined by Chen et al. [17]. When the range is the set of real numbers, we call it an uncertain variable, introduced and investigated by Liu [3].

3. Weak Convergence in Uncertain Normed Spaces

In this section, we introduce the concept of uncertain normed spaces and weak convergence in uncertain normed spaces and study its fundamental properties.

Definition 3.1 (Uncertain Normed Space) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and let $\mathcal{U}$ be a real or complex linear space of uncertain variables (or uncertain-valued functions) defined on Γ. A mapping

$$\|\cdot\|_{\mathcal{M}} : \mathcal{U} \to \mathcal{F}(\Gamma, \mathbb{R}^+),$$

where $\mathcal{F}(\Gamma, \mathbb{R}^+)$ denotes the set of all non-negative uncertain functions on Γ, is said to be an *uncertain norm* if for all $\xi, \eta \in \mathcal{U}$, $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), and for all $\gamma \in \Gamma$, the following properties are satisfied:

1. **Definiteness:** $\|\xi\|_{\mathcal{M}}(\gamma) = 0$ for all $\gamma \in \Gamma$ if and only if $\xi = 0$,
2. **Homogeneity:** $\|\alpha\xi\|_{\mathcal{M}}(\gamma) = |\alpha| \cdot \|\xi\|_{\mathcal{M}}(\gamma)$,
3. **Triangle inequality:** $\|\xi + \eta\|_{\mathcal{M}}(\gamma) \leq \|\xi\|_{\mathcal{M}}(\gamma) + \|\eta\|_{\mathcal{M}}(\gamma)$.

A pair $(\mathcal{U}, \|\cdot\|_{\mathcal{M}})$ satisfying the above conditions is called an *uncertain normed space.*

Definition 3.2 (Uncertain Dual Space) Let $(\mathcal{U}, \|\cdot\|_{\mathcal{M}})$ be an uncertain normed space. The *uncertain dual space*, denoted by $\mathcal{U}^*_{\mathcal{M}}$, is the collection of all *uncertain linear functionals* $\phi : \mathcal{U} \to \mathcal{F}(\Gamma, \mathbb{R})$ such that:

1. ϕ is linear: for all $\xi, \eta \in \mathcal{U}$ and scalars $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$ in the complex case),

$$\phi(\alpha\xi + \beta\eta) = \alpha\phi(\xi) + \beta\phi(\eta);$$

2. ϕ is continuous with respect to the uncertain norm $\|\cdot\|_{\mathcal{M}}$; that is, there exists a constant $C_\phi > 0$ such that for all $\xi \in \mathcal{U}$,

$$|\phi(\xi)|(\gamma) \leq C_\phi \, \|\xi\|_{\mathcal{M}}(\gamma), \quad \text{for almost all } \gamma \in \Gamma.$$

Definition 3.3 (Weak Convergence in Uncertainty Theory) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and let $\mathcal{U}$ be an uncertain normed space with dual space $\mathcal{U}^*_{\mathcal{M}}$. A sequence $\{\xi_n\} \subset \mathcal{U}$ is said to converge *weakly* to $\xi \in \mathcal{U}$, denoted $\xi_n \xrightarrow{w} \xi$, if for every uncertain linear functional $\phi \in \mathcal{U}^*_{\mathcal{M}}$, we have

$$\lim_{n\to\infty} \mathcal{M}\left(\{\gamma \in \Gamma : |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| \geq \varepsilon\}\right) = 0, \quad \text{for all } \varepsilon > 0.$$

Example 3.1 Let $\Gamma = [0,1]$ with $\mathcal{L}$ the Borel σ-algebra. We define the uncertain measure $\mathcal{M}$ by

$$\mathcal{M}(A)(\gamma) = \delta_\gamma(A) = \begin{cases} 1, & \text{if } \gamma \in A, \\ 0, & \text{if } \gamma \notin A, \end{cases} \qquad A \in \mathcal{L},\ \gamma \in \Gamma,$$

where δ_γ denotes the Dirac measure supported at γ.

Let $\mathcal{U}$ be the set of all real-valued measurable functions $\xi : \Gamma \to \mathbb{R}$.

For $\xi \in \mathcal{U}$, define

$$\|\xi\|_{\mathcal{M}}(\gamma) = \int_\Gamma |\xi(t)|\, d\big(\mathcal{M}(\cdot)(\gamma)\big)(t) = \int_\Gamma |\xi(t)|\, d\delta_\gamma(t) = |\xi(\gamma)|, \quad \gamma \in \Gamma.$$

We check the axioms of an uncertain norm:

1. **Definiteness:** If $\|\xi\|_{\mathcal{M}}(\gamma) = 0$ for all $\gamma \in \Gamma$, then $|\xi(\gamma)| = 0$ for all γ, hence $\xi = 0$.
2. **Homogeneity:** For $\alpha \in \mathbb{R}$ (or $\mathbb{C}$),
$$\|\alpha\xi\|_{\mathcal{M}}(\gamma) = |\alpha\xi(\gamma)| = |\alpha|\,|\xi(\gamma)| = |\alpha|\,\|\xi\|_{\mathcal{M}}(\gamma).$$
3. **Triangle inequality:** For $\xi, \eta \in \mathcal{U}$,
$$\|\xi + \eta\|_{\mathcal{M}}(\gamma) = |\xi(\gamma) + \eta(\gamma)| \le |\xi(\gamma)| + |\eta(\gamma)| = \|\xi\|_{\mathcal{M}}(\gamma) + \|\eta\|_{\mathcal{M}}(\gamma).$$

Thus, $(\mathcal{U}, \|\cdot\|_{\mathcal{M}})$ is an *uncertain normed space.*

Remark 3.1 *The convergence $\phi(\xi_n) \to \phi(\xi)$ is considered under the uncertainty measure $\mathcal{M}$, reflecting the uncertain environment in which the variables $\xi_n, \xi \in \mathcal{U}$ evolve.*

Remark 3.2 *Weak convergence in uncertainty generalizes classical weak convergence by incorporating uncertainty in the values of the functionals $\phi \in \mathcal{U}^*$, where $\mathcal{U}^*$ denotes the uncertain dual space.*

Remark 3.3 *As in classical analysis, weak convergence in uncertainty is weaker than strong convergence, i.e., strong convergence $\|\xi_n - \xi\|_{\mathcal{M}} \to 0$ implies weak convergence $\phi(\xi_n) \to \phi(\xi)$ under $\mathcal{M}$, but the converse may not hold.*

3.1. Main Results

Theorem 3.1 *If $\{\xi_n\} \subset \mathcal{U}$ converges strongly to $\xi \in \mathcal{U}$, then $\xi_n \to \xi$ weakly in uncertainty.*

Proof: By the definition of strong convergence in the uncertain normed space $\mathcal{U}$, we have

$$\|\xi_n - \xi\|_{\mathcal{M}}(\gamma) \to 0 \quad \text{for almost all } \gamma \in \Gamma,$$

that is, there exists a set

$$A = \left\{\gamma \in \Gamma : \lim_{n\to\infty} \|\xi_n - \xi\|_{\mathcal{M}}(\gamma) \neq 0\right\}$$

such that

$$\mathcal{M}(A) = 0.$$

Let $\phi \in \mathcal{U}^*$ be any uncertain linear functional. By the continuity of ϕ, there exists a constant $C_\phi > 0$ such that for all $n \in \mathbb{N}$ and all $\gamma \in \Gamma$,

$$|\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| = |\phi(\xi_n - \xi)(\gamma)| \le C_\phi \|\xi_n - \xi\|_{\mathcal{M}}(\gamma).$$

Now we define the set

$$B = \left\{\gamma \in \Gamma : \lim_{n\to\infty} |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| \neq 0\right\}.$$

For every $\gamma \in \Gamma \setminus A$, we have $\|\xi_n - \xi\|_{\mathcal{M}}(\gamma) \to 0$, and thus

$$|\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| \leq C_\phi \|\xi_n - \xi\|_{\mathcal{M}}(\gamma) \to 0.$$

This implies $\gamma \notin B$, so $B \subseteq A$. Hence, by monotonicity of the uncertainty measure,

$$\mathcal{M}(B) \leq \mathcal{M}(A) = 0.$$

Therefore, for every $\phi \in \mathcal{U}^*$, the convergence

$$\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma)$$

holds in uncertainty measure $\mathcal{M}$, i.e., $\xi_n \to \xi$ weakly in uncertainty. □

Remark 3.4 *The converse of the above theorem is generally not true; weak convergence in uncertainty does not necessarily imply strong convergence.The following example illustrates this.*

Example 3.2 Consider the uncertain normed space $\mathcal{U} = \ell_\infty^U$, the space of all bounded uncertain sequences

$$\xi = (\xi_1, \xi_2, \dots), \quad \xi_k : \Gamma \to \mathbb{R},$$

equipped with the uncertain supremum norm

$$\|\xi\|_{\mathcal{M}}(\gamma) = \sup_k |\xi_k(\gamma)|.$$

We define the sequence $\{\xi_n\} \subset \mathcal{U}$ by

$$\xi_n = (0, 0, \dots, \underbrace{1}_{n\text{-th position}}, 0, 0, \dots),$$

where the n-th coordinate is the constant uncertain function 1 (for all $\gamma \in \Gamma$) and zero elsewhere.

We now show that $\xi_n \to 0$ weakly in uncertainty.

Consider any uncertain linear functional $\phi \in \mathcal{U}^*$. By the Hahn-Banach-type extension results in uncertain normed spaces (or by analogy with the classical dual of ℓ_∞), such a functional ϕ acts as

$$\phi(\xi) = \sum_{k=1}^{\infty} a_k \xi_k,$$

where $a = (a_1, a_2, \dots)$ is an uncertain sequence satisfying boundedness with respect to the uncertainty norm.

Then,

$$\phi(\xi_n)(\gamma) = a_n(\gamma) \cdot 1 = a_n(\gamma).$$

Since $a = (a_1, a_2, \dots)$ is bounded in the uncertain norm, we have

$$a_n(\gamma) \to 0 \quad \text{for almost all } \gamma \in \Gamma,$$

because the boundedness of ϕ implies $a_n(\gamma) \to 0$ in the uncertainty sense.

Therefore,

$$\phi(\xi_n)(\gamma) \to 0 = \phi(0)(\gamma),$$

and hence

$$\xi_n \to 0 \quad \text{weakly in uncertainty.}$$

Now we show that $\xi_n \not\to 0$ strongly in uncertainty.

We observe that,

$$\|\xi_n - 0\|_{\mathcal{M}}(\gamma) = \|\xi_n\|_{\mathcal{M}}(\gamma) = \sup_k |\xi_{n,k}(\gamma)| = 1, \quad \text{for all } \gamma \in \Gamma,$$

since the n-th coordinate is constantly 1 for every γ.

Thus,

$$\|\xi_n - 0\|_{\mathcal{M}}(\gamma) \not\to 0,$$

and the sequence $\{\xi_n\}$ does not converge strongly to zero in uncertainty.

Remark 3.5 *This counterexample demonstrates that weak convergence in uncertainty does not imply strong convergence, mirroring the classical result in ℓ_∞, but now within the uncertain functional analytic framework.*

Theorem 3.2 (Characterization of Weak Convergence in Uncertainty) *Let $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$ be an uncertain normed space, and let $\{\xi_n\} \subset \mathcal{U}$, $\xi \in \mathcal{U}$. Then $\xi_n \to \xi$ weakly in uncertainty if and only if for every uncertain linear functional $\phi \in \mathcal{U}^*$, we have*

$$\mathcal{M}\left(\{\gamma \in \Gamma : |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| \geq \varepsilon\}\right) \to 0, \quad \textit{for all } \varepsilon > 0.$$

Proof: By definition, weak convergence in uncertainty means

$$\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma) \quad \text{for } \mu\text{-almost all } \gamma \in \Gamma,$$

which is equivalent to saying that the set

$$E_{n,\varepsilon} = \{\gamma \in \Gamma : |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| \geq \varepsilon\}$$

satisfies

$$\lim_{n\to\infty} \mathcal{M}(E_{n,\varepsilon}) = 0, \quad \text{for all } \varepsilon > 0.$$

Conversely, if this condition holds, then for each $\varepsilon > 0$, $\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma)$ in $\mathcal{M}$-measure. Therefore, by the standard subsequence principle, there exists a subsequence converging pointwise $\mathcal{M}$-almost everywhere. A diagonal argument ensures the whole sequence converges $\mathcal{M}$-almost everywhere, hence weak convergence in uncertainty follows. □

Remark 3.6 *This theorem provides a precise characterization of weak convergence in uncertain normed spaces via convergence in uncertain measure $\mathcal{M}$ for all continuous uncertain linear functionals.*

Remark 3.7 *The uncertain measure $\mathcal{M}$ quantifies the negligibly small subset of the uncertainty domain Γ where weak convergence may not occur, thereby distinguishing weak convergence from strong convergence in uncertain settings.*

3.2. Relation to Reflexivity in Uncertain Normed Spaces

Definition 3.4 An uncertain normed space $\mathcal{U}$ is said to be *reflexive under uncertainty* if the canonical embedding

$$J : \mathcal{U} \to \mathcal{U}^{**}_{\mathcal{M}}, \quad J(\xi)(\phi) = \phi(\xi), \quad \phi \in \mathcal{U}^*_{\mathcal{M}},$$

is surjective (onto) modulo $\mathcal{M}$-almost everywhere equality.

Theorem 3.3 (Reflexivity and Weak Compactness) *If $\mathcal{U}$ is reflexive under uncertainty, then every bounded sequence $\{\xi_n\} \subset \mathcal{U}$ has a subsequence $\{\xi_{n_k}\}$ that converges weakly in uncertainty. That is, there exists $\xi \in \mathcal{U}$ such that for all $\phi \in \mathcal{U}^*_{\mathcal{M}}$,*

$$\phi(\xi_{n_k})(\gamma) \to \phi(\xi)(\gamma) \quad \textit{for } \mathcal{M}\textit{-almost all } \gamma \in \Gamma.$$

Proof: Let $\{\xi_n\}$ be a bounded sequence in $\mathcal{U}$. That is, there exists $C > 0$ such that

$$\|\xi_n\|_{\mathcal{M}}(\gamma) \leq C \quad \text{for all } n \in \mathbb{N}, \text{ and for } \mathcal{M}\text{-almost all } \gamma \in \Gamma.$$

Since $\mathcal{U}$ is reflexive under uncertainty, the canonical embedding

$$J : \mathcal{U} \to \mathcal{U}^{**}_{\mathcal{M}}, \quad J(\xi)(\phi) = \phi(\xi),$$

is surjective up to $\mathcal{M}$-almost everywhere equality.

Reflexivity implies that the closed unit ball

$$B = \{\xi \in \mathcal{U} : \|\xi\|_{\mathcal{M}}(\gamma) \le 1 \text{ for } \mathcal{M}\text{-almost all } \gamma \in \Gamma\}$$

is weakly compact in the uncertain sense, i.e., compact with respect to the weak topology induced by all $\phi \in \mathcal{U}^*_{\mathcal{M}}$, where convergence is defined $\mathcal{M}$-almost everywhere in Γ.

Using the uncertain analogue of the Banach–Alaoglu theorem, which asserts that the unit ball in the dual space $\mathcal{U}^*_{\mathcal{M}}$ is weak* compact, we conclude that the closed unit ball $B \subset \mathcal{U}$ is compact under the weak topology defined by pointwise $\mathcal{M}$-almost everywhere convergence of functionals $\phi(\xi_n)$.

Since $\{\xi_n/C\} \subset B$, by weak compactness, there exists a subsequence $\{\xi_{n_k}\}$ and $\xi \in B$ such that

$$\phi(\xi_{n_k})(\gamma) \to \phi(\xi)(\gamma) \quad \text{for } \mathcal{M}\text{-almost all } \gamma \in \Gamma,$$

for every $\phi \in \mathcal{U}^*_{\mathcal{M}}$.

Rescaling by C preserves weak convergence. Hence, $\xi_{n_k} \to \xi$ weakly in uncertainty, with convergence holding outside a set of γ of $\mathcal{M}$-measure zero.

Therefore, every bounded sequence in $\mathcal{U}$ has a weakly convergent subsequence in the uncertain sense. □

Remark 3.8 *Reflexivity guarantees weak compactness in uncertain normed spaces, ensuring that bounded sequences possess weak limits with respect to convergence defined $\mathcal{M}$-almost everywhere in Γ.*

Compact Operators and Weak Convergence in Uncertainty

Definition 3.5 An operator $T : \mathcal{U} \to \mathcal{V}$ between uncertain normed spaces is called *compact under uncertainty* if for every bounded sequence $\{\xi_n\} \subset \mathcal{U}$, the image sequence $\{T\xi_n\} \subset \mathcal{V}$ has a subsequence $\{T\xi_{n_k}\}$ that converges *strongly in uncertainty*; that is, there exists $\eta \in \mathcal{V}$ such that

$$\|T\xi_{n_k} - \eta\|_{\mathcal{M}}(\gamma) \to 0 \quad \text{for } \mathcal{M}\text{-almost all } \gamma \in \Gamma.$$

Theorem 3.4 *Let $T : \mathcal{U} \to \mathcal{V}$ be a compact operator under uncertainty. If a sequence $\{\xi_n\} \subset \mathcal{U}$ converges weakly in uncertainty to $\xi \in \mathcal{U}$, i.e.,*

$$\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma) \quad \textit{for all } \phi \in \mathcal{U}^*_{\mathcal{M}} \textit{ and } \mathcal{M}\textit{-a.e. } \gamma \in \Gamma,$$

then

$$T\xi_n \to T\xi \quad \textit{strongly in uncertainty, i.e., } \|T\xi_n - T\xi\|_{\mathcal{M}}(\gamma) \to 0 \quad \textit{for } \mathcal{M}\textit{-a.e. } \gamma.$$

Proof: Since $\{\xi_n\}$ converges weakly in uncertainty to ξ, it is bounded; i.e., there exists $C > 0$ such that for all n,

$$\|\xi_n\|_{\mathcal{M}}(\gamma) \le C \quad \text{for } \mathcal{M}\text{-almost all } \gamma \in \Gamma.$$

By compactness of T, the bounded sequence $\{T\xi_n\} \subset \mathcal{V}$ admits a subsequence $\{T\xi_{n_k}\}$ converging strongly in uncertainty to some $\eta \in \mathcal{V}$, i.e.,

$$\|T\xi_{n_k} - \eta\|_{\mathcal{M}}(\gamma) \to 0 \quad \text{for } \mathcal{M}\text{-a.e. } \gamma.$$

Next, we show $\eta = T\xi$. Since T is linear and continuous, we have

$$\phi(T\xi_{n_k})(\gamma) = (\phi \circ T)(\xi_{n_k})(\gamma) \to (\phi \circ T)(\xi)(\gamma) = \phi(T\xi)(\gamma)$$

for all $\phi \in \mathcal{V}^*_{\mathcal{M}}$ and $\mathcal{M}$-a.e. γ, because $\phi \circ T \in \mathcal{U}^*_{\mathcal{M}}$.

But $T\xi_{n_k} \to \eta$ strongly in uncertainty implies

$$\phi(T\xi_{n_k})(\gamma) \to \phi(\eta)(\gamma) \quad \text{for all } \phi \in \mathcal{V}^*_{\mathcal{M}} \text{ and } \mathcal{M}\text{-a.e. } \gamma.$$

By uniqueness of weak limits in $\mathcal{V}$, we conclude that

$$\phi(\eta)(\gamma) = \phi(T\xi)(\gamma) \quad \text{for all } \phi \in \mathcal{V}^*_{\mathcal{M}} \text{ and } \mathcal{M}\text{-a.e. } \gamma,$$

hence $\eta = T\xi$ $\mathcal{M}$-a.e.

Finally, to show the entire sequence $\{T\xi_n\} \to T\xi$ strongly, suppose not. Then there exists $\varepsilon > 0$ and a subsequence $\{T\xi_{m_j}\}$ such that

$$\|T\xi_{m_j} - T\xi\|_{\mathcal{M}}(\gamma) > \varepsilon \quad \text{on a set of positive } \mathcal{M}\text{-measure.}$$

By compactness again, $\{T\xi_{m_j}\}$ has a strongly convergent subsequence. Its limit must be $T\xi$, contradicting the assumption.

Therefore, $T\xi_n \to T\xi$ strongly in uncertainty. □

Remark 3.9 *This theorem shows that compact operators under uncertainty "upgrade" weak convergence to strong convergence, respecting the uncertain measure $\mathcal{M}$ in the sense that convergence holds $\mathcal{M}$-almost everywhere on Γ. This extends classical compact operator theory to uncertain normed spaces.*

4. Measure-Theoretic Characterization of Weak Convergence in Uncertainty

In this section, we delve deeper into the role of the uncertain measure $\mathcal{M}$ in characterizing and analyzing weak convergence in uncertain normed spaces. By integrating measure-theoretic tools with functional analytic concepts, we develop a comprehensive framework that captures convergence behavior in the presence of uncertainty.

Almost Everywhere Convergence via $\mathcal{M}$

Recall that weak convergence in uncertainty relies on the convergence of the scalar-valued uncertain functionals $\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma)$ outside a $\mathcal{M}$-null set. This leads to the following definitions:

Definition 4.1 Let $\{\xi_n\} \subset \mathcal{U}$ and $\xi \in \mathcal{U}$. We say that:

1. $\xi_n \to \xi$ *weakly in uncertainty almost everywhere* if for every $\phi \in \mathcal{U}^*_{\mathcal{M}}$,

$$\mathcal{M}\left(\left\{\gamma \in \Gamma : \lim_{n\to\infty} \phi(\xi_n)(\gamma) \neq \phi(\xi)(\gamma)\right\}\right) = 0.$$

2. $\xi_n \to \xi$ *weakly in uncertainty in measure* if for every $\phi \in \mathcal{U}^*_{\mathcal{M}}$ and every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathcal{M}\left(\{\gamma \in \Gamma : |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| > \varepsilon\}\right) = 0.$$

Remark 4.1 *The distinction between almost everywhere convergence and convergence in measure with respect to $\mathcal{M}$ reflects subtle differences in how uncertainty impacts the mode of weak convergence. Convergence in measure generalizes almost everywhere convergence and plays a critical role in proving limit theorems under uncertainty.*

Uncertain Measure and Weak Compactness

The uncertain measure $\mathcal{M}$ also facilitates compactness arguments in the weak topology of uncertain normed spaces:

Theorem 4.1 (Uncertain Banach–Alaoglu Theorem) *Let $B = \{\xi \in \mathcal{U} : \|\xi\|_{\mathcal{M}}(\gamma) \leq 1\}$ be the uncertain unit ball. Then the polar set*

$$B^0 = \{\phi \in \mathcal{U}^*_{\mathcal{M}} : |\phi(\xi)(\gamma)| \leq 1 \textit{ for all } \xi \in B,\ \mathcal{M}\textit{-a.e. } \gamma \in \Gamma\}$$

is weak- compact in the uncertain weak-* topology induced by $\mathcal{M}$.*

Proof: Let $\mathcal{U}$ be an uncertain normed space with uncertainty parameter $\gamma \in \Gamma$, and let $\mathcal{M}$ be the uncertain measure on Γ.

In the uncertain setting, the dual space $\mathcal{U}^*_{\mathcal{M}}$ consists of uncertain linear functionals ϕ such that $\phi(\xi) : \Gamma \to \mathbb{C}$ is measurable with respect to $\mathcal{M}$ for every $\xi \in \mathcal{U}$.

Define the polar set:

$$B^0 = \{\phi \in \mathcal{U}^*_{\mathcal{M}} : |\phi(\xi)(\gamma)| \leq 1 \text{ for all } \xi \in B,\ \mathcal{M}\text{-a.e. } \gamma\}.$$

The uncertain weak-* topology is generated by seminorms of the form

$$p_\xi(\phi) = \|\phi(\xi)\|_{L^\infty(\Gamma,\mathcal{M})} := \inf\{M \geq 0 : \mathcal{M}(\{\gamma : |\phi(\xi)(\gamma)| > M\}) = 0\}.$$

By Tychonoff's theorem, the product space

$$\prod_{\xi \in B} \{g \in \mathcal{L}^\infty(\Gamma, \mathcal{M}) : \|g\|_\infty \leq 1\}$$

is compact in the product topology.

The polar set B^0 is a closed subset of this product, defined by the linearity and continuity conditions of ϕ, which are preserved under convergence modulo $\mathcal{M}$-null sets.

Hence B^0 is compact in the uncertain weak-* topology. □

Characterization via Uncertain Integration

Integration with respect to $\mathcal{M}$ provides another characterization of weak convergence in uncertainty:

Theorem 4.2 *A sequence $\{\xi_n\} \subset \mathcal{U}$ converges weakly in uncertainty to $\xi \in \mathcal{U}$ if and only if for every $\phi \in \mathcal{U}^*_{\mathcal{M}}$,*

$$\lim_{n\to\infty} \int_\Gamma |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)|\, d\mathcal{M}(\gamma) = 0.$$

Proof: Suppose $\xi_n \to \xi$ weakly in uncertainty. Then for all $\phi \in \mathcal{U}^*_{\mathcal{M}}$, we have

$$\phi(\xi_n)(\gamma) \to \phi(\xi)(\gamma) \quad \text{for } \mathcal{M}\text{-a.e. } \gamma \in \Gamma.$$

Let $g_n(\gamma) := |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)|$. Then $g_n(\gamma) \to 0$ almost everywhere. Since $\{\xi_n\}$ is bounded and ϕ is continuous, we have $g_n(\gamma) \leq 2\|\phi\| \cdot \sup_n \|\xi_n\|_{\mathcal{M}}(\gamma)$, which is integrable.

Thus, by the Dominated Convergence Theorem for uncertain measures,

$$\int_\Gamma g_n(\gamma)\, d\mathcal{M}(\gamma) \to 0.$$

Conversely, if the integrals tend to zero, then for every $\varepsilon > 0$, by Markov's inequality:

$$\mathcal{M}\left(\{\gamma : |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)| > \varepsilon\}\right) \leq \frac{1}{\varepsilon} \int_\Gamma |\phi(\xi_n)(\gamma) - \phi(\xi)(\gamma)|\, d\mathcal{M}(\gamma) \to 0.$$

Hence $\phi(\xi_n) \to \phi(\xi)$ in $\mathcal{M}$-measure, and a subsequence converges almost everywhere. Therefore, $\xi_n \to \xi$ weakly in uncertainty. □

5. Conclusion

In this paper, we developed the theory of weak convergence in the setting of uncertain normed spaces. By introducing a suitable notion of uncertainty-based duality, we defined weak convergence with respect to uncertain linear functionals and examined its foundational properties. We established the relation between weak and strong convergence in uncertainty theory and demonstrated the role of uncertain boundedness in the convergence behavior of sequences.

We further introduced the concept of reflexivity under uncertainty and proved that, analogous to classical analysis, reflexivity implies weak sequential compactness in the uncertain framework. Specifically, we showed that every bounded sequence in a reflexive uncertain normed space admits a weakly convergent subsequence, with convergence understood almost everywhere with respect to the uncertainty measure. Furthermore, we investigated compact operators within this uncertain framework and demonstrated that several classical results.

These findings lay a foundational bridge between classical functional analysis and uncertain analysis, opening avenues for future research in operator theory, optimization and variational problems under uncertainty.

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