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Critical Waves in a Nonlocal Dispersion Delayed SI_1I_2R Model with Generalized Nonlinear Incidence Function

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ABSTRACT: We investigate traveling wave solutions of a delayed nonlocal diffusion epidemic model divided into four compartments: susceptible (S), two infectious classes (I_1 and I_2), and recovered (R). The model integrates geographic dispersal using nonlocal integral operators, as well as a temporal delay in the transmission process to account for the latent period of infection. The incidence variables $L_1(S, I_1)$ and $L_2(S, I_2)$ allow for broad nonlinear interactions between susceptibles and infectious classes. We demonstrate the presence of traveling wave connecting the disease-free steady state to an endemic equilibrium using appropriate kernel function and incidence rate assumptions. The method is based on the development of higher and lower solutions. A detailed mathematical examination reveals that the fundamental reproduction number R_0 is critical in recognising the presence of traveling waves. We present a threshold condition for the least wave speed α^* and analyse the impact of delay and nonlocal dispersal on wave profile and propagation speed. The system permits nontrivial traveling wave solutions at all speeds $\alpha \geq \alpha^*$, but none exist for $0 < \alpha < \alpha^*$. In contrast, if $R_0 \leq 1$, no traveling wave solutions exist. Numerical simulations of sample nonlinear incidence functions are presented to validate and illustrate the analytical conclusions.

Key Words: Two infectious classes, nonlocal dispersion, minimal wave speed, delay, basic reproduction number R_0.

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1. Introduction

Spatial distribution of infectious diseases is a fundamental problem in epidemiology, particularly when it is regulated by heterogeneous mobility and time-lag transmission dynamics [23,24,25,26]. Traditional reaction-diffusion epidemic models have provided rudimentary understanding of infectious disease spread in space [1,2,18]. These models have a tendency to make local diffusion mechanisms, which might prove to be inadequate for long-range movements such as human travel or animal migration. To eliminate this deficiency, nonlocal diffusion models have been introduced, where population spread is described by integral operators with spatial kernels, Li et al. 2010 and Wang and Li 2012 are detailing this point [3,4,32].

Time lags in epidemiological models are based on biology, representing incubation periods, behavior response lags, and reporting delays [5,6]. Delayed transmission may qualitatively alter disease spreading dynamics, having effects on the stability of equilibria and the existence of traveling wave [7,36,37,42].

Here we study a delayed nonlocal SI_1I_2R model where the infected population is segregated into two different compartments: I_1, I_2 two class of infection. This is because in real epidemics like COVID-19,

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asymptomatic carriers play a very significant role in transmission dynamics see R. Li, S. Pei, B. Chen, et al. 2020 and He et al. 2020 [8,10].

We consider a partial integro-differential equation system with convolution-type spatial operators and discrete time delays in the incidence terms. The incidence functions $L_1(S, I_1)$ and $L_2(S, I_2)$ are considered to be general incidence functions. The primary goal is to find the existence of traveling wave solutions between the disease-free equilibrium and a positive endemic state [14,15,16,17,19]. These wave define disease invasion of a susceptible population, and their properties especially the minimum wave speed are important determinants of outbreak severity and containment effectiveness [20,22,23,28,29,31,35].

We achieve this by integrating the upper-lower solution method and monotone iteration schemes, which have been demonstrated to be effective in the analysis of nonlocal delayed systems [11,13,30].We also establish conditions under which the minimum wave speed can be calculated explicitly.

This work constitutes a contribution to the new literature on delayed and spatially structured epidemic models through providing a formal mathematical framework to examine the combined effect of delay, nonlocality, and multi-class infection structure on disease propagation.

To this end, we consider the following nonlocal dispersal delayed SI_1I_2R epidemic model.

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} &= d_1(J*S(x,t)-S(x,t)) + \Delta - L_1(S(x,t),I_1(x,t)) - L_2(S(x,t),I_2(x,t)) - \mu S(x,t), \\ \frac{\partial I_1(x,t)}{\partial t} &= d_2(J*I_1(x,t)-I_1(x,t)) + L_1(S(x,t-\vartheta),I_1(x,t-\vartheta)) - (\delta+\mu)I((x,t), \\ \frac{\partial I_2(x,t)}{\partial t} &= d_3(J*I_2(x,t)-I_2(x,t)) + L_2(S(x,t-\vartheta),I_2(x,t-\vartheta)) - (\mu+\rho)I_2(x,t), \\ \frac{\partial R(x,t)}{\partial t} &= d_4(J*R(x,t)-R(x,t)) + \delta I_1(x,t) + \rho I_2(x,t) - \mu R(x,t), \end{cases} \tag{1.1}$$

with $t > 0$ and $x \in \mathbb{R}$. We make the assumptions on the parameters

- **(A)** d_1, d_2, d_3 are positive, and $\Delta, \mu, \vartheta, \delta, \rho > 0$.
- **(J)** $J \in C^1(\mathbb{R})$, $J(0) > 0$, $J(x) = J(-x) \geq 0$ $\forall x \in \mathbb{R}, \int_{\mathbb{R}} J(x)dx = 1$; $\lim_{\xi\to+\infty} \frac{1}{\xi}\int_{\mathbb{R}} J(y)e^{-\xi y}dy = +\infty$.
- **(H)** : $L_i(S,0) = L_i(0,I_i) = 0$, $\frac{\partial L_i(S,I_i)}{\partial I_i} > 0$, $\frac{\partial^2 L_i(S,I_i)}{\partial I_i^2} < 0$ and $\frac{\partial L_i(S,I_i)}{\partial S} > 0$ for all $S, I_i > 0$.

2. EES, R_0, and MWS

We must identify the constant equilibria of (1.1) in order to examine the TWS of the system (1.1). Let $S^0 = \frac{\Delta}{\mu}$. It is easy to see that $(S_0, 0, 0)$, also known as the IFES of (1.1). To demonstrate the existence and uniqueness of EES, we examine the temporal model of (1.1), as they are equivalent, that is

$$\begin{cases} S_i' &= \Delta - \mu S(t) - L_1(S(t),I_1(t)) - L_2(S(t),I_2(t)), \\ I_1' &= L_1(S(t-\vartheta),I_1(t-\vartheta)) - (\mu+\delta)I_1(t), \\ I_2' &= L_2(S(t-\vartheta),I_2(t-\vartheta)) - (\mu+\rho)I_2(t), \\ S(\sigma) &= S_0(\sigma), \quad I_i(\sigma) = I_{i0}(\sigma), i = 1,2, \end{cases} \tag{2.1}$$

with $(S_0, I_{i0}) \in \mathbb{C}([-\tau,0],\mathbb{R}^+) \times \mathbb{C}([-\tau,0],\mathbb{R}^+)$. The standard results of theory of differential equations imply that (2.1) has a unique positive solution. For showing that the solution is globally defined, we sum up the three equations of (2.1), for each $i = 1,2$, to obtain

$$(S(t) + I_i(t+\vartheta))' \leq \Delta - \mu(S(t) + I_i(t+\vartheta) + R(t)),$$

then

$$\limsup_{t\to+\infty}(S(t) + I_i(t+\vartheta)) \leq \frac{\Delta}{\mu}.$$

Therefore, the solution is globally defined. Then,

$$\limsup_{t\to+\infty} S(t) \leq \frac{\Delta}{\mu}.$$

Next, we derive the results that follows

Parameter	Biological Meaning
$S(x,t)$	Density of susceptible individuals at location x and time t.
$I_1(x,t)$	Density of infected individuals with epidemic 1.
$I_2(x,t)$	Density of infected individuals with epidemic 2 .
$R(x,t)$	Density of recovered individuals.
d_1	Diffusion rate for susceptible individuals.
d_2	Diffusion rate for infected class I_1.
d_3	Diffusion rate for infected class I_2.
d_4	Diffusion rate for recovered individuals.
$J(x)$	Dispersal kernel: describes the probability distribution of movement from one location to another.
$J*u(x,t)$	Nonlocal convolution: $J*u(x,t)=\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy$.
Λ	Recruitment or birth rate of susceptible individuals.
μ	Natural death rate (applies to all classes).
δ	Recovery rate of infected individuals in class I_1.
ρ	Recovery rate of infected individuals in class I_2.
τ	Time delay accounting for disease incubation or delayed response.
$L_1(S,I_1)$	Incidence function for infections due to I_1 (e.g., $\frac{\beta_1 SI_1}{1+\alpha I_1}$)
$L_2(S,I_2)$	Incidence function for infections due to I_2 (e.g., $\frac{\beta_2 SI_2^2}{1+\alpha I_2^2}$).

Table 1: Biological meaning of the parameters in the nonlocal delayed epidemic model.

Theorem 2.1 *Let $(S_0, I_{i0}) \in \mathbb{C}([-\tau,0],\mathbb{R}^+)\times\mathbb{C}([-\tau,0],\mathbb{R}^+)$, and assume that $(S(t), I_i(t))$ is the solution of (2.1), thus, $S(t)>0,\ I_i(t)>0,\ R(t)>0$ for all finite $t\geq 0$. Moreover, the following set*

$$\Omega=\{(S(t),I_i(t)),S(t)\geq 0,I_i(t)\geq 0,R(t)\geq 0,S(t)+I_i(t+\vartheta)+R(t)\leq \frac{\Delta}{\mu}\},$$

is a positively invariant set.

Clearly Ω is a positively invariant set for (2.1). For simplicity reasons, we denote $\frac{\partial L_i(S,I_i)}{\partial S}=\partial_1 L_i(S,I_i)$, and $\frac{\partial L_i(S,I_i)}{\partial I_i}=\partial_2 L_i(S,I_i)$. The BRN, R_0, associated to (2.1) is the spectral radius of $\mathcal{M}$, that is,

$$R_0=\rho(\mathcal{M}),$$

where

$$\mathcal{M}=\begin{bmatrix}\frac{\partial_2 L_1(S_0,0)}{(\mu+\delta)} & 0\\ 0 & \frac{\partial_2 L_2(S_0,0)}{(\mu+\rho)}\end{bmatrix}.$$

Now, we only focus on the reduced system of (2.1),

$$\begin{cases}\frac{dS(t)}{dt} &= \Delta-\mu S(t)-L_1(S(t),I_1(t))-L_2(S(t),I_2(t)),\\ \frac{dI_1(t)}{dt} &= L_1(S(t-\vartheta),I_1(t-\vartheta))-(\mu+\delta)I_1(t),\\ \frac{dI_2(t)}{dt} &= L_2(S(t-\vartheta),I_2(t-\vartheta))-(\mu+\rho)I_2(t).\end{cases} \tag{2.2}$$

The following theorem guarantees the existence and uniqueness of the EES for (2.2), which also functions as an EES for the reaction-diffusion system (1.1). In the following, we shall always assume $R_0>1$. The system (2.1) permits two equilibriums: E_0 and E^*. Our primary goal is to confirm the existence of traveling wave solutions of (1.1) between the two equilibria E_0 and E^*. The traveling wave solution of (1.1) has the following special form:

$$(S(\chi),I_i(\chi)),\quad \chi=x+\alpha t\in\mathbb{R} \tag{2.3}$$

Plugging (2.3) into (1.1), we get the wave form equations as

$$\begin{cases} \alpha S'(\chi) &= d_1(J*S(\chi)-S(\chi))+\Delta-\mu S(\chi)-L_1(S(\chi),I_1(\chi))-L_2(S(\chi),I_2(\chi)),\\ \alpha I_1'(\chi) &= d_2(J*I_1(\chi)-I_1(\chi))+L_1(S(\chi-\alpha\vartheta),I_1(\chi-\alpha\vartheta))-(\mu+\delta)I_1(\chi),\\ \alpha I_2'(\chi) &= d_3(J*I_2(\chi)-I_2(\chi))+L_2(S(\chi-\alpha\vartheta),I_2(\chi-\alpha\vartheta))-(\mu+\rho)I_2(\chi), \end{cases} \tag{2.4}$$

with the boundary conditions

$$(S,I_1,I_2)(-\infty)=(\frac{\Delta}{\mu},0,0),\qquad (S,I_1,I_2)(+\infty)=(S^*,I_1^*,I_2^*). \tag{2.5}$$

We aim to show that (2.4) has a positive solution that satisfies (2.5). The linearized equations of the second equation of (2.4) with $i=1,2$ at E_0 provide the following decoupled system.

$$\begin{cases} \alpha I_1'(\chi) &= d_2(J*I_1(\chi)-I_1(\chi))+\partial_2L_1(S^0,0)I_1(\chi-\alpha\vartheta)-(\mu+\delta)I_1(\chi),\\ \alpha I_2'(\chi) &= d_3(J*I_2(\chi)-I_2(\chi))+\partial_2L_2(S^0,0)I_2(\chi-\alpha\vartheta)-(\mu+\rho)I_2(\chi). \end{cases}$$

Letting $I_1(\chi)=\kappa_1\mathrm{e}^{\xi\chi}$, and $I_2(\chi)=\kappa_2\mathrm{e}^{\xi\chi}$, we get

$$\begin{cases} \alpha\xi\kappa_1 &= d_2\displaystyle\int_{-\infty}^{+\infty}\kappa_1J(y)\mathrm{e}^{-\xi y}dy+\partial_2L_1(S^0,0)\kappa_1\mathrm{e}^{-\alpha\xi\vartheta}-(\mu+\delta+d_2)\kappa_1,\\ \alpha\xi\kappa_2 &= d_3\displaystyle\int_{-\infty}^{+\infty}J(y)\kappa_2\mathrm{e}^{-\xi y}dy+\partial_2L_2(S^0,0)\kappa_2\mathrm{e}^{-\alpha\xi\vartheta}-(\mu+\rho+d_3)\kappa_2. \end{cases} \tag{2.6}$$

Let

$$\mathcal{A}=\begin{bmatrix} d_2 & 0\\ 0 & d_3\end{bmatrix},\quad \mathcal{B}=\begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix},\quad \mathcal{D}=\begin{bmatrix}\mu+\delta+d_2 & 0\\ 0 & \mu+\rho+d_3\end{bmatrix},$$

and

$$\mathcal{E}=\begin{bmatrix}\partial_2L_1(S^0,0)\mathrm{e}^{-\alpha\xi\vartheta} & 0\\ 0 & \partial_2L_2(S^0,0)\mathrm{e}^{-\alpha\xi\vartheta}\end{bmatrix}.$$

Where $\mathcal{J}(\xi)=\int_{-\infty}^{+\infty}J(y)\mathrm{e}^{-\xi y}dy$. Denote $p(\xi,\alpha)=\mathcal{J}(\xi)\mathcal{A}-\xi\mathcal{B}-\mathcal{D}+\mathcal{E}$. Thus, (2.6) reduces into

$$p(\xi,\alpha)\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}=0.$$

Let $A'=\mathcal{D}^{-1}\mathcal{A}, B'=\mathcal{D}^{-1}\mathcal{B}$ and $F'=\mathcal{D}^{-1}\mathcal{E}$, thus $p(\xi,\alpha)$ becomes

$$\Big(-A'\mathcal{J}(\xi)+B'\xi+I\Big)^{-1}F'\kappa=\kappa, \tag{2.7}$$

where $\kappa=\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}$, $m_1(\xi,\alpha)=-d_2\mathcal{J}(\xi)+\alpha\xi+(\mu+\delta+d_2), m_2(\xi,\alpha)=-d_3\mathcal{J}(\xi)+\alpha\xi+(\mu+\rho+d_3)$ and

$$\Big(-A'\mathcal{J}(\xi)+B'\xi+I\Big)^{-1}F'=\begin{bmatrix}\frac{\partial_2L_1(S^0,0)\mathrm{e}^{-\alpha\xi\vartheta}}{m_1(\xi,\alpha)} & 0\\ 0 & \frac{\partial_2L_1(S^0,0)\mathrm{e}^{-\alpha\xi\vartheta}}{m_2(\xi,\alpha)}\end{bmatrix}.$$

Let $N(\xi,\alpha)=\Big(-A'\mathcal{J}(\xi)+B'\xi+I\Big)^{-1}F'$, thus (2.16) becomes

$$N(\xi,\alpha)\kappa=\kappa.$$

Let $L(\xi,\alpha)$ represent the primary eigenvalue of $N(\xi,\alpha)$. Now, we solve $m_1(\xi,\alpha)=0$ in ξ. Clearly, $m_1(0,\alpha)=(\mu+\eta+d_2)>0$; similarly, $m_2>0$ and $m_i(+\infty,\alpha)=-\infty$ by assumption **(J)**,

$$\frac{\partial m_i(\xi,\alpha)}{\partial\xi}\Big|_{\xi=0}=\alpha>0,\quad \frac{\partial^2m_1(\xi,\alpha)}{\partial\xi^2}=-d_2\int_{\mathbb{R}}y^2J(y)e^{-\xi y}dy<0\quad \frac{\partial m_2(\xi,\alpha)}{\partial\xi^2}=-d_3\int_{\mathbb{R}}y^2J(y)e^{-\xi y}dy<0.$$

Therefore, there is always $\xi_\alpha^i > 0$ satisfying $m_i(\xi_\alpha^i, \alpha) = 0$, for all $\alpha > 0$, $i = 1, 2$. We let

$$\xi_\alpha = \min\left\{\xi_\alpha^1, \xi_\alpha^2\right\}.$$

For $\alpha \geq 0$, $\xi \in [0, \xi_\alpha)$, we get

$$\begin{aligned} L(\xi,\alpha) \quad = \quad & \tfrac{e^{-\xi\alpha\vartheta}}{2}\Bigg[\left(\tfrac{\partial_2 L_1(S^0,0)}{m_1(\xi,\alpha)} + \tfrac{\partial_2 L_2(S^0,0)}{m_2(\xi,\alpha)}\right) + \Bigg\{\left(\tfrac{\partial_2 L_1(S^0,0)}{m_1(\xi,\alpha)} - \tfrac{\partial_2 L_2(S^0,0)}{m_2(\xi,\alpha)}\right)^2 \\ & + \tfrac{4\partial_2 L_1(S^0,0)\partial_2 L_2(S^0,0)e^{-\alpha\xi\vartheta}}{m_1(\xi,\alpha)m_2(\xi,\alpha)}\Bigg\}^{\frac{1}{2}}\Bigg]. \end{aligned} \tag{2.8}$$

This following proposition is true for any $L(\xi, \alpha)$.

Proposition 2.1 *The three claims that follow are true:*

(i) ξ_α *is increasing in* $\alpha \in [0, +\infty)$, *and* $\lim_{\alpha\to+\infty} \xi_\alpha = +\infty$;

(ii) $L(0, \alpha) = R_0$, $\alpha \geq 0$, $L(\xi, \alpha)$ *is decreasing in* $\xi \in [0, \xi_0)$, *and* $\lim_{\xi\to\xi_\alpha} L(\xi, \alpha) = 0$, $\forall\alpha \geq 0$;

(iii) $\forall\xi \in (0, \xi_\alpha)$, $\frac{\partial}{\partial\alpha} L(\xi, \alpha) < 0$.

Proof: For (i), is easy to check, therefore, we omit it here. Fixing $\xi \in (0, \xi_\alpha)$, and differentiate $L(\xi, \alpha)$ in α to get

$$\tfrac{\partial L(\xi,\alpha)}{\partial\alpha} \quad = \quad \Big\{\partial_2 L_1(S^0, 0)\tfrac{\partial}{\partial\alpha}\left[\tfrac{e^{-\xi\alpha\vartheta}}{m_1(\xi,\alpha)}\right](V_1(\xi, \alpha) + v(\xi, \alpha)) + \partial_2 L_2(S^0, 0)\tfrac{\partial}{\partial\alpha}\left[\tfrac{e^{-\xi\alpha\vartheta}}{m_2(\xi,\alpha)}\right](V_1(\xi, \alpha) - v(\xi, \alpha)),$$

where

$$v(\xi, \alpha) = \partial_2 L_1(S^0, 0)\left[\frac{e^{-\xi\alpha\vartheta}}{m_1(\xi, \alpha)}\right] - \partial_2 L_2(S^0, 0)\left[\frac{e^{-\xi\alpha\vartheta}}{m_2(\xi, \alpha)}\right],$$

and

$$V_1(\xi, \alpha) = \left\{\left(\partial_2 L_1(S^0, 0)\left[\frac{e^{-\xi\alpha\vartheta}}{m_1(\xi, \alpha)}\right] - \partial_2 L_2(S^0, 0)\left[\frac{e^{-\xi\alpha\vartheta}}{m_2(\xi, \alpha)}\right]\right)^2,\right.$$

As both $m_i(\xi, \alpha) > 0$ and $\mathcal{J}(\xi) > 0$, $\forall\xi \in (0, \xi(\alpha))$, we obtain $V_1(\xi, \alpha) + v(\xi, \alpha) > 0$, $V_1(\xi, \alpha) - v(\xi, \alpha) > 0$ and

$$\frac{\partial}{\partial\alpha}\left[\frac{e^{-\xi\alpha\vartheta}}{m_i(\xi, \alpha)}\right] = -\frac{\xi e^{-\xi\alpha\vartheta}(\vartheta m_i(\xi, \alpha) + 1)}{(m_i(\xi, \alpha))^2} < 0 \quad i = 1, 2.$$

Hence,

$$\frac{\partial L(\xi, \alpha)}{\partial\alpha} < 0.$$

Thus, (iii) is checked.

Clearly, $L(0, \alpha) = R_0$, $\forall\alpha \geq 0$. Now, we differentiate $L(\xi, 0)$ in $\xi \in (0, \xi(0))$ to get

$$\tfrac{\partial L(\xi,\alpha)}{\partial\xi} \quad = \quad \Big\{\partial_2 L_1(S^0, 0)\tfrac{\partial}{\partial\xi}\left[\tfrac{1}{m_1(\xi,0)}\right](V_1(\xi, 0) + v(\xi, 0)) + \partial_2 L_2(S^0, 0)\tfrac{\partial}{\partial\xi}\left[\tfrac{1}{m_2(\xi,\alpha)}\right](V_1(\xi, 0) - v(\xi, 0)),$$

As $m_i(\xi, \alpha) > 0$, $\mathcal{J}(\xi) > 0$, $V_1(\xi, 0) + v(\xi, 0) > 0$, $V_1(\xi, 0) - v(\xi, 0) > 0$ and

$$\frac{\partial}{\partial\xi}\left[\frac{1}{m_1(\xi, 0)}\right] = \frac{d_2\mathcal{J}'(\xi)}{m_1(\xi, \alpha)^2} > 0,$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{m_2(\xi, 0)}\right] = \frac{d_3\mathcal{J}'(\xi)}{m_2(\xi, \alpha)^2} > 0 \quad i = 1, 2.$$

Thus

$$\frac{\partial L(\xi, 0)}{\partial\xi} > 0.$$

Since that

$$\lim_{\xi\to\xi_\alpha-0}\max\left\{\frac{1}{m_1(\xi,\alpha)},\frac{1}{m_2(\xi,\alpha)}\right\}=+\infty,$$

for all $\alpha\geq 0$, we deduce that $\lim_{\xi\to\xi_\alpha-0}L(\xi,\alpha)=+\infty$. This concludes (ii). □

In the light of Proposition 2.1, we let

$$\tilde{\xi}(\alpha)=\min_{\xi\in[0,\xi_\alpha)}L(\xi,\alpha)\quad for\quad \alpha\geq 0.$$

Thus $\tilde{\xi}(0)=L(\xi,0)=R_0$, $\lim_{\alpha\to+\infty}\tilde{\xi}(\alpha)=0$ and $\tilde{\xi}(\alpha)$ is continuous and decreasing in $c\in[0,\infty)$. When $R_0>1$, there is a constant $c^*>0$ verifying $\tilde{\xi}(\alpha^*)=1,\tilde{\xi}(\alpha)>1$, $\forall\alpha\in[0,\alpha^*)$ and $\tilde{\xi}(\alpha)<1$, $\forall\alpha\in(\alpha^*,\infty)$. Let

$$\xi^*=\inf\left\{\xi\in[0,\xi_{\alpha^*}):L(\xi,\alpha)=1\right\}.$$

Hence, $L(\xi^*,\alpha^*)=1$, and $L(\xi^*,\alpha)<1$, $\forall c>c^*$. Denote

$$\xi_1(\alpha)=\sup\{\xi\in(0,\xi^*):L(\xi,\alpha)=1,L(\xi',\alpha)\geq 1\quad\forall\xi'\in(0,\xi)\}.$$

As $L(\xi^*,\alpha)<1$, $\forall\alpha>\alpha^*$, we have the following results

Proposition 2.2 *If $R_0>1$, then there is $\alpha^*>0$, $\xi^*\in(0,\xi_{\alpha^*})$ satisfying*

(i) $L(\xi,\alpha)>1$, $\forall 0\leq\alpha<\alpha^*$, $\forall\xi\in(0,\xi_\alpha)$, *where* $\xi_\alpha\in[0,+\infty)$;

(ii) $L(\xi^*,\alpha^*)=1$, $L(\xi,\alpha^*)>1$ *when* $\xi\in(0,\xi^*)$, *and* $L(\xi,\alpha^*)\geq 0$ *when* $\xi\in(0,\xi_{\alpha^*})$;

(iii) $\forall\alpha>\alpha^*$, *there is* $\xi_1(\alpha)\in(0,\xi^*)$ *satisfies* $L(\xi_1(\alpha),\alpha)=1$, $L(\xi,\alpha)\geq 1$ *for* $\xi\in(0,\vartheta_1(\alpha))$, *and* $L(\xi_1(\alpha)+\varepsilon_n(\alpha),\alpha)<1$ *for some decreasing sequences* $\{\varepsilon_n(\alpha)\}$ *verifying* $\lim_{n\to\infty}\varepsilon_n=0$ *and* $\varepsilon_n+\xi_1(\alpha)<\xi^*$, $\forall n\in\mathbb{N}$. *Particularly,* $\xi_1(\alpha)$ *decreases in* $\alpha\in(\alpha^*,\infty)$.

As $N(\xi,\alpha)$ an irreducible nonnegative matrix for $\xi\in[0,\xi_\alpha)$, we obtain by using the Perron-Frobenius theorem.

Proposition 2.3 *Suppose that $R_0>1$. When $\alpha>\alpha^*$, there exists positive unit vectors $\kappa(\alpha)=(\kappa_1(\alpha),\kappa_2(\alpha))^T$ and $\xi^n(\alpha)=(\xi_1^n(\alpha),\xi_2^n(\alpha))^T(n\in\mathbb{N})$ verifying*

$$N(\xi_1(\alpha),\alpha)\kappa(\alpha)=\kappa(\alpha),$$

$$N(\xi_1(\alpha)+\varepsilon_n(\alpha),\alpha)\xi^n(\alpha)=L(\xi_1(\alpha)+\varepsilon_n(\alpha),\alpha)\xi^n(\alpha),n\in\mathbb{N}.$$

Now, let fix $\alpha>\alpha^*$. Suppose that $\xi_1(\alpha),\kappa(\alpha)=(\kappa_1(\alpha),\kappa_2(\alpha))^T,\varepsilon_n(\alpha)$, and $\xi^n(\alpha)=(\xi_1^n(\alpha),\xi_2^n(\alpha))^T(n\in\mathbb{N})$ are as provided in Propositions 2.2 and 2.3. Without loss of generality, we substitute $\xi_1(\alpha),\xi(\alpha)=(\xi_1(\alpha),\xi_2(\alpha))^T,\varepsilon_n(\alpha)$, and $\xi^n(\alpha)=(\xi_1^n(\alpha),\xi_2^n(\alpha))^T(n\in\mathbb{N})$ for $\xi_1,\kappa=(\kappa_1,\kappa_2)^T$, ε_n and $\xi^n(\alpha)=(\xi_1^n(\alpha),\xi_2^n(\alpha))^T(n\in\mathbb{N})$. Given that $L(\xi_1+\varepsilon_n,\alpha)<1$, Proposition 2.3 implies that

$$\begin{cases}-m_1(\xi_1,\alpha)\kappa_1+\Bigg(\partial_2L_1(S^0,0)\kappa_1\Bigg)e^{-c\xi_1\vartheta} & = & 0,\\ -m_1(\xi_1,\alpha)\kappa_2+\Bigg(\partial_2L_1(S^0,0)\kappa_2\Bigg)e^{-c\xi_1\vartheta} & = & 0,\end{cases}$$

and

$$\begin{cases}-m_1(\xi_1+\varepsilon_n,c)\xi_1^n+\partial_2L_1(S^0,0)\kappa_1e^{-\alpha(\xi_1+\varepsilon_n)\vartheta} & < & 0,\\ -m_2(\xi_1+\varepsilon_n,\alpha)\xi_2^n+\partial_2L_2(S^0,0)\kappa_2e^{-\alpha(\xi_1+\varepsilon_n)\vartheta} & < & 0,\end{cases}$$

$\forall n\in\mathbb{N}$.

Lemma 2.1 *Let* $K(\chi) = (k_1(\chi), k_2(\chi))^T$ *with* $k_i(\chi) = \kappa_i \mathrm{e}^{\xi_1 \chi}$*, satisfies*

$$\begin{cases} \alpha p_1'(\chi) &= d_2 \displaystyle\int_{-\infty}^{+\infty} J(y)\mathrm{e}^{-\xi y} dy + \partial_2 L_1(S^0, 0) p_1(\chi - \alpha\vartheta) - (\mu + \delta + d_2) p_1(\chi), \\ \alpha p_2'(\chi) &= d_3 \displaystyle\int_{-\infty}^{+\infty} J(y)\mathrm{e}^{-\xi y} dy + \partial_2 L_2(S^0, 0) p_1(\chi - \alpha\vartheta) - (\mu + \rho + d_3) p_2(\chi). \end{cases}$$

for any $\chi \in \mathbb{R}$.

2.1. Sub-Super solution

We let $\alpha > \alpha^*$, to construct the sub and sup solution of (2.4) using an iteration procedure, we consider the following definition

Definition 2.1 (S^+, I_1^+, I_2^+) *and* (S^-, I_1^-, I_2^-) *are the super and sub solutions of* (2.4)*, respectively, that verify*

$$-\alpha(S^+)'(\chi) + d_2(J * S^+(\chi) - S^+(\chi)) + \Delta - \mu S^+(\chi) - L_1(S^+(\chi), I_1^-(\chi)) - L_2(S^+(\chi), I_2^-(\chi)) \leq 0, \tag{2.9}$$

$$-\alpha(S^-)'(\chi) + d_2(J * S^-(\chi) - S^-(\chi)) + \Delta - \mu S^-(\chi) - L_1(S^-(\chi), I_1^+(\chi)) - L_2(S^-(\chi), I_2^+(\chi)) \geq 0, \tag{2.10}$$

$$-\alpha(I_1^+)'(\chi) + d_2(J * I_1^+(\chi) - I_1^+(\chi)) + L_1(S^-(\chi - \alpha\vartheta), I_1^+(\chi - \alpha\vartheta)) - (\mu + \delta) I_1^+(\chi) \leq 0, \tag{2.11}$$

$$-\alpha(I_2^+)'(\chi) + d_3(J * I_2^+(\chi) - I_2^+(\chi)) + L_2(S^-(\chi - \alpha\vartheta), I_2^+(\chi - \alpha\vartheta)) - (\mu + \rho) I_2^+(\chi) \leq 0, \tag{2.12}$$

$$-\alpha(I_1^-)'(\chi) + d_2(J * I_1^-(\chi) - I_1^-(\chi)) + L_1(S^-(\chi - \alpha\vartheta), I_1^-(\chi - \alpha\vartheta)) - (\mu + \delta) I_1^-(\chi) \geq 0, \tag{2.13}$$

$$-\alpha(I_2^-)'(\chi) + d_3(J * I_2^-(\chi) - I_2^-(\chi)) + L_2(S^-(\chi - \alpha\vartheta), I_2^-(\chi - \alpha\vartheta)) - (\mu + \rho) I_2^-(\chi) \geq 0,, \tag{2.14}$$

except for finite points of $\chi \in \mathbb{R}$.

In what follows, we always suppose that $R_0 > 1$.

Proposition 2.4 *The following algebraic system*

$$\begin{cases} L_1(S^0, \mathcal{B}_1) - (\mu + \delta)\mathcal{B}_1 &= 0, \\ L_2(S^0, \mathcal{B}_2) - (\mu + \rho)\mathcal{B}_2 &= 0, \end{cases} \tag{2.15}$$

Proof: By the first equation of (2.15) we have the following observation. The equation $H(q) := L_1(s^0, q_1) = (\mu + \delta) q_1$ has a unique positive root $\mathcal{B}_1$ because: we have $H(0) = 0$, and $\frac{\partial L_1(S_0, 0)}{\partial I_1} - (\mu + \delta) = (\mu + \delta)(\frac{\frac{\partial L_1(S_0,0)}{\partial I_1}}{\mu + \delta} - 1) > 0$ when $R_0 = \rho(\mathcal{M}) > 1$ then, by the concavity of H guarantees uniqueness (see [9, Fig. 2]), bu the same way we can proof the second equation of (2.15). □

The following result illustrates the validation of the sub and super solution by using the prior definition.

Lemma 2.2 *Let* $R_0 > 1$*,* $\alpha > \alpha^*$*. Define*

$$\begin{aligned} S^+(\chi) &= S^0, & I_i^+ &= \min\{\kappa_i \mathrm{e}^{\xi_1 \chi}, \mathcal{B}_i\}, \\ S^- &= \max\left\{S^0 - M\mathrm{e}^{\gamma\chi}, 0\right\}, & I_i^-(\chi) &= \max\{\kappa_i e^{\xi_1 \chi}(1 - J_i e^{\eta\chi}), 0\}. \end{aligned}$$

For positive constants γ, J_i *and* M $(i = 1, 2)$ *to be found later,* (2.9)-(2.14) *are met.*

Proof: The following elements are utilised to establish evidence.

(i): Clearly $S^+(\chi) = S^0$ satisfies

$$-\alpha(S^+)'(\chi) + d_2(J * S^+(\chi) - S^+(\chi)) + \Delta - \mu S^+(\chi) - L_1(S^+(\chi), I_1^-(\chi)) - L_2(S^+(\chi), I_2^-(\chi)) \le 0, \tag{2.16}$$

then, the proof of (2.9) is clear.

(ii) If $\chi < \chi_0$, where $\chi_0 = \frac{ln\mathcal{B}_1}{\xi_1}$, we have $I_1^+(\chi) = \mathcal{B}_1$, and then $I_1^+(\chi - \alpha\vartheta) \le \mathcal{B}_1$. So, we have

$$d_2(J * I_1(\chi) - I_1(\chi)) + L_1(S^+(\chi - \alpha\vartheta), I_1^+(\chi - \alpha\vartheta)) - (\mu + \delta)I_1^+(\chi) - \alpha(I_1^+)'(\chi)$$
$$\le L_1(S^0, \mathcal{B}_1) - (\mu + \eta)\mathcal{B}_1 = 0.$$

For $\chi > \chi_0$, we obtain $I_1^+(\chi) = \kappa_1 e^{\xi_1\chi}$, which we demonstrate fulfils (2.13). It is easy to check that.

$$d_2(J * I_1^+(\chi) - I_1^+(\chi)) + L_1((s^+)(\chi - \alpha\vartheta), I_1^+(\chi - \alpha\vartheta)) - (\mu + \eta)(I_1^+)(\chi) - \alpha(I_1^+)'(\chi), \tag{2.17}$$

$$\begin{aligned}
&\le d_2(J * I_1^+(\chi) - I_1^+(\chi)) + \tfrac{\partial L_1(S^0,0)}{\partial I_1}(I_1^+)(\chi - \alpha\vartheta) - (\mu + \eta)(I_1^+)(\chi) - \alpha(I_1^+)'(\chi),\\
&\le -\alpha(I_1^+)'(\chi) + d_2\int_{-\infty}^{+\infty} J(y)e^{-\xi_1 y}dy + \frac{\partial L_1(S^0,0)}{\partial I_1}I_1^+(\chi - \alpha\vartheta) - (\mu + \eta + d_2)I_1^+(\chi),\\
&= d_2\int_{-\infty}^{+\infty} J(y)e^{-\xi_1 y}dy + \frac{\partial L_1(S^0,0)}{\partial I_1}\kappa_1 e^{\xi_1(\chi - \alpha\vartheta)} - (\mu + \eta + d_2)\kappa_1 e^{\xi_1\chi} - \alpha\xi_1 e^{\xi_1\chi},\\
&= e^{\xi_1\chi}p(\xi_1, \alpha),\\
&= 0,
\end{aligned}$$

with ξ_1 is defined in Proposition 2.2. By similar method we prove (2.12)

(iii) Take $0 < \gamma < \min\left\{\xi_1, \frac{\alpha}{d_1}\right\}$. Let $\chi \ne \frac{1}{\gamma}ln\frac{1}{M} := \chi^*$, and claiming that S^- verifies

$$-\alpha(S^-)'(\chi) + d_1(J * S^-(\chi) - S^-(\chi)) + L - \mu(S^-)(\chi) - L_1(S^-, I_1^+) - L_2(S^-, I_2^+) \ge 0.$$

To demonstrate the validity of this claim, we begin with the situation $\chi > \chi^*$, resulting in $S^-(\chi) = 0$ in (χ^*, ∞), indicating that it is satisfied. If $\chi < \chi^*$, we have $S^-(\chi) = S^0 - Me^{\gamma\chi}$. Clearly, $L_1(S(\chi), I_1(\chi)) \le L_1(S^0, 0)I_1(\chi), L_2(S(\chi), I_1(\chi)) \le L_2(S^0, 0)I_2(\chi)$. Then, we have

$$\begin{aligned}
&-\alpha(S^-)'(\chi) + d_1(J * S^-(\chi) - S^-(\chi)) + L - \mu(S^-)(\chi) - L_1(S^-, I_1^+) - L_2(S^-, I_2^+),\\
\ge\ &\alpha M\gamma e^{\gamma\chi} - d_1 Me^{\gamma\chi}\int_{-\infty}^{+\infty} J(x)e^{-\gamma x}dy + \Delta - \mu(S^0 - Me^{\gamma\chi})\\
&-\partial_2 L_1(S^0, 0)(\kappa_1 e^{\xi\chi}) - \partial_2 L_2(S^0, 0)(\kappa_2 e^{\xi\chi}),\\
\ge\ &e^{\gamma\chi}\left[\alpha M\gamma - d_1 M\int_{-\infty}^{+\infty} J(x)e^{-\gamma x}dy + d_1 M - \partial_2 L_1(S^0, 0)(\kappa_1 e^{(\xi-\gamma)\chi}) - \partial_2 L_2(S^0, 0)(\kappa_2 e^{(\xi-\gamma)\chi})\right]\\
\ge\ &e^{\gamma\chi}\left[\alpha M\gamma - d_1 M\int_{-\infty}^{+\infty} J(x)e^{-\gamma x}dy + d_1 M - \kappa_1\left(\frac{S^0}{M}\right)^{\frac{\xi-\gamma}{\gamma}} - \kappa_2\left(\frac{S^0}{M}\right)^{\frac{\xi-\gamma}{\gamma}}\right].
\end{aligned}$$

Here we use

$$e^{\xi\chi} < \left(\frac{S^0}{M}\right)^{\frac{\xi-\gamma}{\gamma}} \quad \text{for} \quad \chi < \chi^*.$$

Maintaining $\gamma M = 1$ and allowing $M \to \infty$ for any $M_> S^0$ big enough and γ small enough, we obtain

$$-\alpha(S^-)'(\chi) + d_1(J * S^-(\chi) - S^-(\chi)) + L - \mu(S^-)(\chi) - L_1(S^-, I_1^+) - L_2(S^-, I_2^+) \ge 0.$$

(iv) Selecting $0 < \eta < \min\{\xi_2 - \xi_1, \xi_1\}$ and $L > 0$ suitably big. Next, we assert that $i^-(\chi)$ satisfies

$$-\alpha(I_1^-)'(\chi) + d_2(J * I_1(\chi) - I_1(\chi)) +_1 (S^-(\chi c\vartheta), I_1^-(\chi - \alpha\vartheta)) - (\mu + \eta)I_1^-(\chi) \ge 0, \tag{2.18}$$

with $\chi \neq \chi_2 := \frac{-lnL_1}{\eta}$.

We show this claim in two separate cases: $\chi > \chi_2$ and $\chi < \chi_2$. If $\chi > \chi_2$, then $I_1^-(\chi) = 0$, which means that (2.18) is fulfilled. If $\chi < \chi_2$, then $i^-(\chi) = e^{\xi_1\chi}(1 - L_1e^{\eta\chi})$. In this case, we show that (2.18) holds for sufficiently large L_1, which will be defined later. The inequality in (2.18) may be expressed as follows.

$$\begin{aligned} \frac{\partial L_1(S^0,0)}{\partial I_1} I_1^-(\chi-\alpha\vartheta) - L_1(S^-(\chi-\alpha\vartheta), I_1^-(\chi-\alpha\vartheta)) \quad &\leq \quad -\alpha(I_1^-)'(\chi) + d_2(J * I_1^-(\chi) - I_1^-(\chi)) \\ &\quad + \frac{\partial L_1(S^0,0)}{\partial I_1} I_1^-(\chi-\alpha\vartheta) - (\mu+\eta)I_1^-(\chi), \\ &\leq \quad -Lp(\xi_1+\eta,\alpha)\kappa_1 e^{(\xi_1+\eta)\chi}. \end{aligned} \tag{2.19}$$

For each $\xi \in (0, \frac{\partial L_1(S,I_1)}{\partial I_1}|_{(S,I_1)=(S^0,0)})$, $L_1(S, I_1)/I_1$ is a decreasing function on $(0, \infty)$. Given that I_1^- is a bounded function for $\chi < \chi_2$, there exists $\delta_0 > 0$ such that $0 < I_1^- < \delta_0$ for each $\chi < \chi_2$. The boundedness of I_1^- for $\chi < \chi_2$, as well as the fact that $\frac{\partial L_1(S,I_1)}{\partial I_1}|_{(S,I_1)=(\frac{\Delta}{\mu},0)} > 0$, indicates the existence of $\xi > 0$ small enough to satisfy the following inequality. For each $0 < I_1^- < \delta_0$,

$$L_1(S^-, I_1^-)/I_1^- > \frac{\partial L_1(S^0,0)}{\partial I_1} - \xi > 0$$

. Using the fact that $0 < I_1^- < \delta_0$, we obtain

$$\begin{aligned} &\partial_2 L_1(S^0,0)I_1^-(\chi-\alpha\vartheta) - L_1(S^-(\chi-\alpha\vartheta), I_1^-(\chi-\alpha\vartheta)) \\ = \quad &\left(\frac{\partial L_1(\frac{\Delta}{\mu},0)}{\partial I_1} - \frac{L_1(S^-(\chi-\alpha\vartheta), I_1^-(\chi-\alpha\vartheta))}{I_1^-(\chi-\alpha\vartheta)}\right) i^-(\chi-\alpha\vartheta), \\ \leq \quad &\left(\frac{\frac{\partial L_1(\frac{\Delta}{\mu},0)}{\partial I_1} - \frac{L_1(S^-(\chi-\alpha\vartheta), I_1^-(\chi-\alpha\vartheta))}{I_1^-(\chi-\alpha\vartheta)} + I_1^-(\chi-\alpha\vartheta)}{2}\right)^2 \\ \leq \quad &\left[\frac{\partial L_1(S^0,0)}{\partial I_1} - \left(\frac{\partial L_1(S^0,0)}{\partial I_1} - \xi\right) + I_1^-(\chi-\alpha\vartheta)\right]^2. \end{aligned} \tag{2.20}$$

Then, we have

$$\frac{\partial L_1(S^0,0)}{\partial I_1} I_1^-(\chi-\alpha\vartheta) - L_1(s^-(\chi-\alpha\vartheta), I_1^-(\chi-\alpha\vartheta)) \leq I_1^{-2}(\chi-\alpha\vartheta).$$

Therefore, to prove the inequality (2.19), it is sufficient to show that

$$(I_1^{-2})(\chi-\alpha\vartheta) \leq -Lp(\lambda_1+\eta, c)\kappa_1 e^{(\xi_1+\eta)\chi} \tag{2.21}$$

Noting that $I_1^- \leq I_1^+$, then we have $(I_1^-(\chi-\alpha\vartheta))^2 \leq e^{2\xi_1\chi}$. To ensure (2.21), we show that

$$e^{2\xi_1\chi} \leq -Lp(\xi_1+\eta,\alpha)\kappa_1 e^{(\xi_1+\eta)\chi}. \tag{2.22}$$

The inequality (2.22) is true for sufficiently large M as both sides are restricted for all $\chi < \chi_2$ and trend to 0 as $\chi \to -\infty$. The proof is complete. Similarly, we prove $I_2^-(\chi)$.

□

2.2. Truncated problem

We put $\Theta > \max\{|\chi^*|, |\chi_0|, r\}$, we take the following bounded set

$$\begin{aligned} \Gamma_\Theta(\chi) \quad = \quad &\Big\{ (\phi(\chi)\varphi(\chi)) \in C\left([-\Theta,\Theta], \mathbb{R}^2\right) \Big| \phi(-\Theta) = S(-\Theta), \\ &\varphi_i(-\Theta) = I_i(-\Theta), \quad S^-(\chi) \leq \phi(\chi) \leq S^0, \quad I_i^-(\chi) \leq \varphi_i(\chi) \leq I_i^+(\chi), \\ &\chi \in [-\Theta,\Theta] \Big\}. \end{aligned}$$

For any $(\phi(\chi), \varphi_i(\chi)) \in \Gamma_\Theta(\chi)$, we define

$$\hat{\phi}(\chi) = \begin{cases} \phi(\Theta), & \chi > \Theta, \\ \phi(\chi), & |\chi| \leq \Theta, \\ S^-(-\Theta), & \chi < -\Theta, \end{cases} \qquad \hat{\varphi}(\chi) = \begin{cases} \varphi_i(\Theta), & \chi > \Theta, \\ \varphi_i(\chi), & |\chi| \leq \Theta, \\ I_i^-(-\Theta), & \chi < -\Theta, \end{cases}$$

Clearly, $\Gamma_\Theta(\chi)$ is a closed and convex set. The expression $\left(\hat{\phi}(\chi), \hat{\varphi}_i(\chi)\right)$ is

$$S^-(\chi) \leq \hat{\phi}(\chi) \leq S^0, \quad I_i^-(\chi) \leq \hat{\varphi}_i(\chi) \leq I_i^+(\chi), \quad \chi \in \mathbb{R}.$$

Letting the truncated problem

$$\begin{cases} \alpha S'(\chi) &= d_1((J * \hat{\phi})(\chi) - s(\chi)) + \Delta - \mu S(\chi) - L_1(S(\chi), \varphi_1(\chi)) - L_2(S(\chi), \varphi_2(\chi)), \\ \alpha I_1'(\chi) &= d_2((J * \hat{\varphi}_1)(\chi) - I_1(\chi)) + L_1(\hat{\phi}(\chi - \alpha\vartheta), \hat{\varphi}_1(\chi - \alpha\vartheta)) - (\mu + \eta) I_1(\chi), \\ \alpha I_2'(\chi) &= d_3((J * \hat{\varphi}_2)(\chi) - I_2(\chi)) + L_2(\hat{\phi}(\chi - \alpha\vartheta), \hat{\varphi}_2(\chi - \alpha\vartheta)) - (\mu + \rho) I_2(\chi), \end{cases} \tag{2.23}$$

with

$$S(-\Theta) = S^-(-\Theta), \quad I_i(-\Theta) = I_i^-(-\Theta), \quad i = 1, 2. \tag{2.24}$$

The general results of differential equations ensure that the starting value problems (2.23 and (2.24) permit a single nonnegative solution $(S_\Theta(\chi), I_{i,\Theta}(\chi))$ defined for $\chi \in [-\Theta, \Theta]$. Therefore, we let the solution map $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_{i2})$ on $\Gamma_\Theta(\chi)$ by

$$\mathcal{X}_1(\phi, \varphi_i) = S_\Theta, \qquad \mathcal{X}_{i2}(\phi, \varphi_i) = I_{i,\Theta}.$$

Lemma 2.3 *For all $\Theta > \max\{|\chi^*|, |\chi_0|, r\}$, map $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_{i2}) : \Gamma_\Theta(\chi) \to \Gamma_\Theta(\chi)$.*

Lemma 2.3 may be inferred from Lemma 2.2 and the comparison principle. For instance, we can refer to [12, Proposition 2.1]

Lemma 2.4 *The map $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_{i2}) : \Gamma_\Theta(\chi) \to \Gamma_\Theta(\chi)$ is completely continuous.*

Proof: For any $(\phi, \varphi_i) \in \Gamma_\Theta(\chi)$, we readily establish from (2.23) that $(S_\Theta(\chi), I_{i,\Theta}(\chi)) \in C^1([-\Theta, \Theta], \mathbb{R}^2)$. The Arzelà-Ascoli theorem implies that map $\mathcal{X}$ is compact.

Next, we look at the continuity of $\mathcal{X}$. Define $S_{\Theta,k}(\chi) = \mathcal{X}_1(\phi_k, \varphi_{i,k})(\chi)$. , $I_{i,\Theta,k}(\chi) = \mathcal{X}_2(\phi_k, \varphi_{i,k})(\chi)$ where
$(\phi_k(\chi), \varphi_{i,k}(\chi)) \in \Gamma_\Theta(\chi)$ $(i, k = 1, 2)$ for $\chi \in [-\Theta, \Theta]$. We first examine the continuity of $\mathcal{X}_1$. The first equation of (2.23) produces

$$\begin{aligned} & \alpha(S'_{\Theta,1}(\chi) - S'_{\Theta,2}(\chi)) + (d_1 + \mu)(s_{\Theta,1}(\chi) - S_{\Theta,2}(\chi)) \\ = \; & d_1 \int_{\mathbb{R}} J(y)(\hat{\phi}_1(\chi - y) - \hat{\phi}_2(\chi - y))dy + L_1(S_{\Theta,2}(\chi), \varphi_{1,2}(\chi)) - L_1(S_{\Theta,1}(\chi), \varphi_{1,1}(\chi)) \\ & + L_2(S_{\Theta,2}(\chi), \varphi_{2,2}(\chi)) - L_2(S_{\Theta,1}(\chi), \varphi_{2,1}(\chi)). \end{aligned} \tag{2.25}$$

Since

$$\int_{\mathbb{R}} J(y)\hat{\phi}(\chi - y)dy = \int_{-\infty}^{-\Theta} J(\chi - y)s(y)dy + \int_{-\Theta}^{\Theta} J(\chi - y)\phi(y)dy + \int_{\Theta}^{+\infty} J(\chi - y)\phi(\Theta)dy,$$

we have

$$\left| \int_{\mathbb{R}} J(y)(\hat{\phi}_1(\chi - y) - \hat{\phi}_2(\chi - y))dy \right| \leq 2 \max_{y \in [-\Theta, \Theta]} |\phi_1(y) - \phi_2(y)|. \tag{2.26}$$

For $(\phi_1, \varphi_1), (\phi_2, \varphi_2) \in \Gamma_\Theta(\chi)$, since $I_i^+(\chi) \leq \mathcal{B}_i$ for $\chi \in [-\Theta, \Theta]$, then

$$\begin{aligned}&\Big|L_1(\phi_1(\chi),\varphi_{1,1}(\chi))-L_1(\phi_2(\chi),\varphi_{1,2}(\chi))+L_2(\phi_1(\chi),\varphi_{2,1}(\chi))-L_2(\phi_2(\chi),\varphi_{2,2}(\chi))\Big|\\&\le M_4\Big[\big|\phi_1(\chi)-\phi_2(\chi)\big|+\big|\varphi_{1,1}(\chi)-\varphi_{1,2}(\chi)+\varphi_{2,1}(\chi)-\varphi_{2,2}(\chi)\big|\Big],\end{aligned} \tag{2.27}$$

where $M_4=\sup\left\{\frac{\partial L_1(S^0,0)}{\partial I_1},L_1(\sigma,\mathcal{B}_1),frac\partial L_2(S^0,0)\partial I_2,L_2(\sigma,\mathcal{B}_2):0\le\sigma\le S^0\right\}$.

Let $u(\chi)=\alpha|S_{\Theta,1}(\chi)-S_{\Theta,2}(\chi)|$. Then, from (2.25)–(2.27), we obtain

$$\begin{aligned}u'(\chi)&=\alpha\,\mathrm{sign}(S_{\Theta,1}(\chi)-S_{\Theta,2}(\chi))(S'_{\Theta,1}(\chi)-S'_{\Theta,2}(\chi)),\\&\le 2d_1\max_{y\in[-\Theta,\Theta]}|\phi_1(y)-\phi_2(y)|-(d_1+\mu-M_4)|S_{\Theta,1}(\chi)-S_{\Theta,2}(\chi)|\\&\quad+M_4|\varphi_{1,2}(\chi)-\varphi_{1,1}(\chi)+\varphi_{2,2}(\chi)-\varphi_{2,1}(\chi)|,\\&=\left(\frac{d_1+\mu+}{\alpha}+\frac{M_4}{\alpha}\right)u(\chi)+2d_1\max_{y\in[-\Theta,\Theta]}|\phi_1(y)-\phi_2(y)|\\&\quad+M_4|\varphi_{1,2}(\chi)-\varphi_{1,1}(\chi)+\varphi_{2,2}(\chi)-\varphi_{2,1}(\chi)|.\end{aligned}$$

Thus, $\forall\chi\in[-\Theta,\Theta]$, we obtain

$$\begin{aligned}u(\chi)\;&\le u(-\Theta)e^{-(\frac{d_1+\mu}{\alpha}+\frac{M_4}{\alpha})(\chi+\Theta)}+\int_{-\Theta}^{\chi}\Bigg[\left(2d_1\max_{y\in[-\Theta,\Theta]}|\phi_1(y)-\phi_2(y)|\right)\\&\qquad+M_4\max_{y\in[-\Theta,\Theta]}|\varphi_{1,1}(y)-\varphi_{1,2}(y)+\varphi_{2,1}(y)-\varphi_{2,2}(y)|\Bigg]e^{-(\frac{d_1+\mu}{\alpha}+\frac{M_4}{\alpha})(\chi-\tau)}d\tau.\end{aligned}$$

From (2.28), we obtain $\|u(\chi)\|_{\Gamma_\Theta(\chi)}\to 0$ as $\|(\phi_2,\varphi_{1,2},\varphi_{2,2})-(\phi_1,\varphi_{1,1},\varphi_{2,1})\|_{\Gamma_\Theta(\chi)}\to 0$. Therefore, $\mathcal{X}_1$ is continuous on $\Gamma_\Theta(\chi)$. Similar logic suggests that $\mathcal{X}_{i,2}$ is continuous. □

Where $\Gamma_\Theta(\chi)$ is closed and convex, and by Lemmas 2.3 and 2.4, Schauder's fixed point theorem, the following result is valid.

Theorem 2.2 *$\mathcal{X}$ admits at least one fixed point $(S^*_\Theta(\chi),(I_i)^*_\Theta(\chi))\in\Gamma_\Theta(\chi)$.*

We present some previous estimates for the fixed point $(S^*_\Theta(\chi),(I_i)^*_\Theta(\chi))$. for $\mathcal{X}$ in $C^{1,1}([-\Theta,\Theta],\mathbb{R}^2)$, where.

$$C^{1,1}([-\Theta,\Theta])=\{u\in C^1([-\Theta,\Theta],\mathbb{R}^2):u\text{ and }u'\text{ are Lipschitz continuous}\},$$

endowed with the norm

$$\|u\|_{C^{1,1}([-\Theta,\Theta])}=\max_{x\in[-\Theta,\Theta]}|u(x)|+\max_{x\in[-\Theta,\Theta]}|u'(x)|+\sup_{x,y\in[-\Theta,\Theta],x\ne y}\frac{|u'(x)-u'(y)|}{|x-y|}. \tag{2.28}$$

Next, we have the outcomes shown below.

Lemma 2.5 *Put $(S^*_\Theta(\chi),(I_i)^*_\Theta(\chi))$ be the fixed point of map $\mathcal{F}$, therefore there is a constant $C>0$ independent of Θ satisfying $\|S^*_\Theta(\chi)\|_{C^{1,1}([-\Theta,\Theta])}\le C$ and $\|I^*_{i,\Theta}(\chi)\|_{C^{1,1}([-\Theta,\Theta])}\le C$, $\forall\Theta>\max\{|\chi^*|,|\chi_0|,r\}$.*

Proof: Obviously, we have

$$\begin{cases}\alpha S^{*'}_\Theta(\chi)&=d_1J*S^*_\Theta(\chi)-d_1S^*_\Theta(\chi)+\Delta-\mu S^*_\Theta(\chi)-L_1(S^*_\Theta(\chi),(I_1)^*_\Theta(\chi))-L_2(S^*_\Theta(\chi),(I_2)^*_\Theta(\chi)),\\\alpha(I_1)^{*'}_\Theta(\chi)&=d_2J*(I_1)^*_\Theta(\chi)-(d_2+\mu+\eta)(I_1)^*_\Theta(\chi)+L_1(S^*_\Theta(\chi-\alpha\vartheta),(I_1)^*_\Theta(\chi-\alpha\vartheta)),\\\alpha(I_2)^{*'}_\Theta(\chi)&=d_2J*(I_2)^*_\Theta(\chi)-(d_3+\mu+\rho)(I_2)^*_\Theta(\chi)+L_2(S^*_\Theta(\chi-\alpha\vartheta),(I_2)^*_\Theta(\chi-\alpha\vartheta)),\end{cases} \tag{2.29}$$

for $\chi \in [-\Theta, \Theta]$, where

$$\hat{S}_\Theta(\chi) = \begin{cases} S^*_\Theta(\Theta), \chi > \Theta, \\ S^*_\Theta(\chi), |\chi| \leq \Theta, \\ S^-(\Theta), \chi < -\Theta, \end{cases} \quad (\hat{I}_i)_\Theta(\chi) = \begin{cases} (I_i)^*_\Theta(\Theta), \chi > \Theta, \\ (I_i)^*_\Theta(\chi), |\chi| \leq \Theta, \\ (I_i)^-(\Theta), \chi < -\Theta. \end{cases}$$

Since $S^*_\Theta(\chi) \leq s^0$ and $(I_i)^*_\Theta(\chi) \leq \mathcal{B}_i$ for $\chi \in [-\Theta, \Theta]$, and (2.29), we can obtain

$$|S^{*'}_\Theta(\chi)| \leq \frac{1}{\alpha}(2d_1 S^0 + \Delta + \mu S^0 + (\frac{\partial L_1(S^0,0)}{\partial I_1})\mathcal{B}_1 + (\frac{\partial L_2(S^0,0)}{\partial I_2})\mathcal{B}_2) := L_1, \tag{2.30}$$

$$\begin{array}{lll} |(I_1)^{*'}_X(\chi)| & \leq & \frac{1}{\alpha}(2d_2\mathcal{B}_1 + (\mu+\eta)\mathcal{B}_1 + (\frac{\partial L_1(S^0,0)}{\partial I_1})\mathcal{B}_1) := L_2, \\ |(I_2)^{*'}_X(\chi)| & \leq & \frac{1}{\alpha}(2d_3\mathcal{B}_2 + (\mu+\rho)\mathcal{B}_2 + (\frac{\partial L_2(S^0,0)}{\partial I_2})\mathcal{B}_2) := L_3, \end{array} \tag{2.31}$$

Thus,

$$|S^*_\Theta(\chi) - S^*_\Theta(\eta)| \leq L_1|\chi-\eta|, \quad |(I_1)^*_\Theta(\chi) - (I_1)^*_\Theta(\eta)| \leq L_2|\chi-\eta|, \quad |(I_2)^*_\Theta(\chi) - (I_2)^*_\Theta(\eta)| \leq L_3|\chi-\eta|. \tag{2.32}$$

By (2.29), (2.30) and (2.31), we further have

$$\begin{array}{lll} \alpha|S^{*'}_\Theta(\chi) - S^{*'}_\Theta(\eta)| & \leq & d_1 \int_{-\infty}^{+\infty} J(y)(\hat{S}_\Theta(\chi-y) - \hat{S}_\Theta(\eta-y))\,dy + (d_1+\mu)|S^*_\Theta(\chi) - S^*_\Theta(\eta)| \\ & & +\Big|L_1(S^*_\Theta(\chi),(I_1)^*_\Theta(\chi)) - L_1(S^*_\Theta(\eta),(I_1)^*_\Theta(\eta)) \\ & & +L_2(S^*_\Theta(\chi),(I_2)^*_\Theta(\chi)) - L_2(S^*_\Theta(\eta),(I_2)^*_\Theta(\eta))\Big|, \end{array} \tag{2.33}$$

and

$$\begin{array}{lll} \alpha|(I_1)^{*'}_\Theta(\chi) - (I_1)^{*'}_\Theta(\eta)| & \leq & d_2 \int_{-\infty}^{+\infty} J(y)((\hat{I}_1)_\Theta(\chi-y) - (\hat{I}_1)_\Theta(\eta-y))\,dy + (d_2+\mu+\eta)|(I_1)^*_\Theta(\chi) \\ & & -(I_1)^*_\Theta(\eta)| + \Big|L_1(S^*_\Theta(\chi-\alpha\vartheta),(I_1)^*_\Theta(\chi-\alpha\vartheta)) \\ & & -L_1(S^*_\Theta(\eta-\alpha\vartheta),(I_1)^*_\Theta(\eta-\alpha\vartheta))\Big|. \end{array} \tag{2.34}$$

$$\begin{array}{lll} \alpha|(I_2)^{*'}_\Theta(\chi) - (I_2)^{*'}_\Theta(\eta)| & \leq & d_3 \int_{-\infty}^{+\infty} J(y)((\hat{I}_2)_\Theta(\chi-y) - (\hat{I}_2)_\Theta(\eta-y))\,dy + (d_3+\mu+\rho)|(I_2)^*_\Theta(\chi) \\ & & -(I_2)^*_\Theta(\eta)| + \Big|L_2(S^*_\Theta(\chi-\alpha\vartheta),(I_2)^*_\Theta(\chi-\alpha\vartheta)) \\ & & -L_2(S^*_\Theta(\eta-\alpha\vartheta),(I_2)^*_\Theta(\eta-\alpha\vartheta))\Big|. \end{array} \tag{2.35}$$

Let $[-r, r]$ denote the compact support of $J(x)$. Since $J(x)$ is a C^1- function, there is a constant $L_J, r > 0$ that confirms $J(x) \leq L_J$. So $|J_n(x_1) - J_n(x_2)| \leq L_J|x_1 - x_2|$,1 $\forall x_1, x_2 \in [-r, r]$. Thus, we infer that

$$\begin{array}{lll} \int_{-\infty}^{+\infty} J(y)\hat{S}_\Theta(\chi-y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{S}_\Theta(\eta-y)\,dy & = & \int_{\eta-r}^{\chi-r} J(y)S_\Theta(y)\,dy + \int_{\chi+r}^{\eta+r} J(y)s_\Theta(y)\,dy \\ & & + \int_{\eta-r}^{\chi+r} (J(y-\eta) - J(y-\chi))s_\Theta(y)\,dy \\ & \leq & 4L_J r S^0|\chi-\eta|. \end{array}$$

Likewise, we obtain

$$\int_{-\infty}^{+\infty} J(y)(\hat{I}_i)_\Theta(\chi-y)\,dy - \int_{-\infty}^{+\infty} J(y)(\hat{I}_i)_\Theta(\eta-y)\,dy \leq 4L_J r\mathcal{B}_i|\chi-\eta|.$$

Then, it follows from (2.27) and (2.32) that

$$\begin{aligned}&\Big|L_1(S^*_\Theta(\chi-\alpha\vartheta),(I_1)^*_\Theta(\chi-\alpha\vartheta))-L_1(S^*_\Theta(\eta-\alpha\vartheta),(I_1)^*_\Theta(\eta-\alpha\vartheta))\\&+L_2(S^*_\Theta(\chi-\alpha\vartheta),(I_2)^*_\Theta(\chi-\alpha\vartheta))-L_2(S^*_\Theta(\eta-\alpha\vartheta),(I_2)^*_\Theta(\eta-\alpha\vartheta))\Big|\\ \leq\ & M_4(L_1+L_2+L_3)|\chi-\eta|.\end{aligned} \tag{2.36}$$

Combining (2.33)-(2.36), we know

$$|S^{*'}_\Theta(\chi)-S^{*'}_\Theta(\eta)|\leq C_S|\chi-\eta| \quad \text{and} \quad |(I_i)^{*'}_\Theta(\chi)-(I_i)^{*'}_\Theta(\eta)|\leq (C_I)_i|\chi-\eta|,$$

where

$$C_S=\frac{1}{\alpha}(4d_1L_JrS^0+(d_1+(\mu)L_1+M_4(L_1+L_2+L_3)),$$

$$(C_I)_1=\frac{1}{\alpha}(4d_2L_Jr\mathcal{B}_1+(d_2+\mu+\eta)L_2+M_4(L_1+L_2)).$$

$$(C_I)_2=\frac{1}{\alpha}(4d_3L_Jr\mathcal{B}_2+(d_2+\mu+\rho)L_2+M_4(L_1+L_3)).$$

Consequently, we have that

$$\|S^*_\Theta(\chi)\|_{C^{1,\chi}([-\Theta,\Theta])}\leq C$$

and

$$\|(I_i)^*_\Theta(\chi)\|_{C^{1,\chi}([-\Theta,\Theta])}\leq C_{I_i},$$

where $C=\max\{S^0+L_1+C_S,\mathcal{B}_1+L_2+C_{I_1},\mathcal{B}_2+L_3+C_{I_2}\}$. □

2.3. Existence of a noncritical traveling wave solution

Theorem 2.3 *Let $R_0>1$ and $\alpha>\alpha^*$, then (2.4) has a solution $(S^*(\chi),I_i^*(\chi))$ defined for $\chi\in\mathbb{R}$ satisfying $S^-(\chi)\leq S^*(\chi)\leq S^0$, $I_i^-(\chi)\leq I_i^*(\chi)\leq I_i^+(\chi)$ for $\chi\in\mathbb{R}$.*

Proof: $\{X_n\}_{n=1}^\infty$, with $X_n>\max\{|\chi^*|,|\chi_0|,r\}$ and $\lim_{n\to\infty}X_n=+\infty$. Schauder's fixed point theorem states that a fixed point $(S^*_{X_n}(\chi),I^*_{i,X_n}(\chi))\in\Gamma_{X_n}(\chi)$ of the map $\mathcal{X}$ exists for every X_n. Lemma 2.5 implies that $\|S^*_{X_n}(\chi)\|_{\mathbb{C}[1-\alpha,X_n,X_n]}\leq C_S$ and $\|I^*_{i,X_n}(\chi)\|_{\mathbb{C}[1-\alpha,X_n,X_n]}\leq C_{I_i}$, $n=1,2,\ldots$. For any integer k, $\{(S^*_{X_n}(\chi),I^*_{i,X_n}(\chi))\}$ and
$\{(S^{*'}_{X_n}(\chi),I^{*'}_{i,X_n}(\chi))\}$, with $n\geq k$, are uniformly bounded and equicontinuous on $[-X_k,X_k]$. The Arzelà-Ascoli theorem and the diagonal extraction strategy guarantee that a subsequence $\{(S^*_{X_m}(\chi),I^*_{i,X_m}(\chi))\}$ converges uniformly in each $[-X_k,X_k]$, $(k=1,2,\ldots)$, as $m\to\infty$.

Assuming $\lim_{m\to\infty}(S^*_{X_m}(\chi),I^*_{i,X_m}(\chi))=(S^*(\chi),I_i^*(\chi))$, we have
$\lim_{m\to\infty}(S^{*'}_{X_m}(\chi),I^{*'}_{i,X_m}(\chi)) = (S^{*'}(\chi),I_i^{*'}(\chi))$. Let r be the supported radius of $J(\chi)$. Since $(S^*_{X_m}(\chi),I^*_{i,X_m}(\chi))\leq(S^{+*}(\chi),I_i^{+*}(\chi))$ for $\chi\in\mathbb{R}$ and $i,m=1,2,\ldots$, using the Lebesgue dominated convergence theorem, it follows that

$$\lim_{m\to\infty}\int_{\mathbb{R}}J(\chi)S^*_{X_m}(\chi-y)d\chi=\lim_{m\to\infty}\int_{-r}^{r}J(\chi)S^*_{X_m}(\chi-y)d\chi=J*S^*(\chi).$$

By the same way, we can obtain $\lim_{m\to\infty}I^*_{i,X_m}(\chi)=J*I_i^*(\chi)$. Then, $(S^*(\chi),I_i^*(\chi))$ satisfies (2.4) and $S^-(\chi)\leq S^*(\chi)\leq S^0$ and $I_i^-(\chi)\leq I_i^*(\chi)\leq I_i^+(\chi)$ for $\chi\in\mathbb{R}$.

Now, we prove $S^0>S^*(\chi)>0$ and $I_i^*(\chi)>0$. Since $S(-\infty)=S^0>0$, assume that there is $\chi_{00}\in\mathbb{R}$ verifying $S(\chi_{00})=0$ and $S(\chi)>0$, $\forall\chi\in(-\infty,\chi_{00})$, then $S'(\chi_{00})\leq 0$. The first equation of (2.4) gives

$$d_1\int_{-\infty}^{+\infty}J(y)S(\chi_{00}-y)dy+\Delta\leq 0.$$

This is a contradiction. Thus, $S^*(\chi) > 0$, $\forall\chi \in \mathbb{R}$. Likewise, we obtain $I_i^*(\chi) > 0$, $\forall\chi \in \mathbb{R}$. Now, we prove $S^*(\chi) < S^0$. Assuming that there is $\chi_{00} \in \mathbb{R}$ satisfying $S^*(\chi_{00}) = S^0$, then, $s^{*'}(\chi_{00}) \geq 0$. Together with the first equation of (2.4) yields

$$d_1 \int_{-\infty}^{+\infty} J(y)(s(\chi_{00} - y) - s^0)dy + \Delta - \mu S^0 - L_1(S^*(\chi_{00}), I_1^*(\chi_{00})) - L_2(S^*(\chi_{00}), I_2^*(\chi_{00})) \geq 0,$$

that is,

$$d_1 \int_{-\infty}^{+\infty} J(y)(S(\chi_{00} - y) - S^0)dy - L_1(S^*(\chi_{00}), I_1^*(\chi_{00})) - L_2(S^*(\chi_{00}), I_2^*(\chi_{00})) \geq 0,$$

this is a contradiction with $s^*(\chi_{00}-y)-S^0 \leq 0$ and $-L_1(S^*(\chi_{00}), I_1^*(\chi_{00})), L_2(S^*(\chi_{00}), I_2^*(\chi_{00})) > 0$. Thus, $S^*(\chi) < S^0$, $\forall\chi \in \mathbb{R}$. □

Theorem 2.4 *Let $R_0 > 1$ and $\alpha > \alpha^*$, then (2.4) has a solution $(S^*(\chi), I_i^*(\chi))$ defined for $\chi \in \mathbb{R}$ satisfying $\lim_{\chi\to\infty}(S^*(\chi), I_i^*(\chi)) = (S^0, 0)$, $0 < S^*(\chi) \leq S^0$, and $I_i^*(\chi) > 0$ for $\chi \in \mathbb{R}$.*

Proof: By Theorem 2.3, there is a solution sequence $\Phi_n(\chi) = (S_n^*(\chi), I_{i,n}^*(\chi))$, $n \in \mathbb{N}^*$ and $\chi \in \mathbb{R}$, verifying

$$\begin{cases} \alpha S_n^{*'}(\chi) &= d_1 J * S_n^*(\chi) - d_1 S_n^*(\chi) + \Delta - \mu S_n^*(\chi) - L_1(S_n^*(\chi), I_{1,n}^*(\chi)) - L_2(S_n^*(\chi), I_{2,n}^*(\chi)), \\ \alpha I_{1,n}^{*'}(\chi) &= d_2 J * I_{1,n}^*(\chi) - (d_2 + \mu + \eta) I_{1,n}^*(\chi) + L_1(S_n^*(\chi - \alpha\vartheta), I_{1,n}^*(\chi\alpha\vartheta)), \\ \alpha I_{2,n}^{*'}(\chi) &= d_3 J * I_{2,n}^*(\chi) - (d_3 + \mu + \rho) I_{2,n}^*(\chi) + L_2(S_n^*(\chi - \alpha\vartheta), I_{2,n}^*(\chi\alpha\vartheta)), \end{cases} \tag{2.37}$$

and

$$S(\chi) < S_n^*(\chi) \leq S^0, \quad I_i(\chi) \leq I_{i,n}^*(\chi) \leq I_i(\chi), \quad S_n^*(\chi) > 0, \quad I_{i,n}^*(\chi) > 0, \quad \chi \in \mathbb{R}.$$

In $[-k-1, k-1]$, we select subsequences $\{\Phi_{k-1,m}(\chi)\}$ of $\{\Phi_{k-2,m}(\chi)\}$ satisfying $\{\Phi_{k-1,m}(\chi)\}$ and $\{\Phi'_{k-1,m}(\chi)\}$ converge uniformly on $[-k-1, k-1]$ when $m \to \infty$.

We also have $\mathcal{B}_{i,k-1,m} \leq \kappa_i e^{1+\chi}$ for all $\chi \in [-k-1, k-1]$. As $\mathcal{B}_{i,k-1,m}$ is uniformly restricted on $[-k, k]$, we may argue that $I_{i,k,m}^*(\chi) \leq \kappa_i e^{1+\chi}$ for all $\chi \in [-k, k]$. Thus, for $m > m_k$, $\{\Phi_{k,m}(\chi)\}$ is uniformly restricted on $[-k, k]$.

The proof of Lemma 2.4 shows that both $\{\Phi_{k-1,m}(\chi)\}$ and $\{\Phi'_{k-1,m}(\chi)\}$ are equicontinuous and uniformly bounded on $[-k, k]$. $\{\Phi_{k,m}(\chi)\}$ is a subsequence of $\{\Phi_{k-1,m}(\chi)\}$ that converges uniformly on $[-k, k]$ for both $\{\Phi_{k,m}(\chi)\}$. as $m \to \infty$. Moreover, $I_{i,k,m}^*(\chi) \leq e^{\xi_1\chi}$, $\forall\chi \in [-k, k]$.

The diagonal extraction approach implies that there are subsequences $\{\Phi_{m,m}(\chi)\}$ and $\{\Phi'_{m,m}(\chi)\}$ that converge uniformly on each $[-k, k]$ (k = 1, 2, 3,...). Consider $\{\Phi_{m,m}(\chi)\} \to (S^*(\chi), I_i^*(\chi))$ as $m \to +\infty$. Thus, $\{\Phi'_{m,m}(\chi)\} \to (S^{*'}(\chi), I_i^{*'}(\chi))$ and $m \to +\infty$. Because for each $m \in \mathbb{N}^*$, we have

$$\begin{cases} \alpha S_{m,m}^{*'}(\chi) &= d_1 J * S_{m,m}^*(\chi) - d_1 S_{m,m}^*(\chi) + \Delta - \mu S_{m,m}^*(\chi) - L_1(S_{m,m}^*(\chi), I_{1,m,m}^*(\chi)) \\ & \quad -L_2(S_{m,m}^*(\chi), I_{2,m,m}^*(\chi), \\ \alpha I_{1,m,m}^{*'}(\chi) &= d_2 J * I_{1,m,m}^*(\chi) - (d_2 + \mu + \eta) I_{1,m,m}^*(\chi) + -L_1(S_{m,m}^*(\chi - \alpha\vartheta), I_{1,m,m}^*(\chi - \alpha\vartheta) \\ \alpha I_{2,m,m}^{*'}(\chi) &= d_3 J * I_{2,m,m}^*(\chi) - (d_3 + \mu + \rho) I_{2,m,m}^*(\chi) + -L_2(S_{m,m}^*(\chi - \alpha\vartheta), I_{2,m,m}^*(\chi - \alpha\vartheta). \end{cases} \tag{2.38}$$

When $m \to +\infty$, utilising the continuity of $L_i(S(\chi), I_i(\chi)), i = 1, 2,$ functions and the dominated convergence theorem yields

$$\begin{cases} \alpha S^{*'}(\chi) &= d_1 J * S^*(\chi) - d_1 S^*(\chi) + \Delta - \mu S^*(\chi) - L_1(S^*(\chi), I_1^*(\chi)) - L_2(S^*(\chi), I_2^*(\chi)), \\ \alpha I_1^{*'}(\chi) &= d_2 J * I_1^*(\chi) - (d_2 + \mu + \eta) I_1^*(\chi) + L_1(S^*(\chi - \alpha\vartheta), I_1^*(\chi - \alpha\vartheta)), \\ \alpha I_2^{*'}(\chi) &= d_3 J * I_2^*(\chi) - (d_3 + \mu + \rho) I_2^*(\chi) + L_2(S^*(\chi - \alpha\vartheta), I_2^*(\chi - \alpha\vartheta)). \end{cases} \tag{2.39}$$

For every $\chi \in \mathbb{R}$. That is, $(S^*(\chi), I_i^*(\chi))$ is the solution of (2.4) for $\chi \in \mathbb{R}$. (2.37) gives $S(\chi) < S^*(\chi) \leq s^0$ and $I_i(\chi) \leq I_i^*(\chi)$ for $\chi \in \mathbb{R}$. For any integers $k > 0$ and $m \geq k$, $I_{i,m,m}^*(\chi) \leq \kappa_i e^{\xi_1\chi}$, $\forall\chi \in [-k, k]$, we further get $I_i^*(\chi) \le \kappa_i e^{\xi_1\chi}$, $\chi \in \mathbb{R}$. The upper-lower solutions show that $(S^*(\chi), I_i^*(\chi))$ satisfies $\lim_{\chi\to-\infty}(S^*(\chi), I_i^*(\chi)) = (S^0, 0)$. Similar to Theorem 2.3, we obtain $0 < S^*(\chi) < S^0$ for every $\chi \in \mathbb{R}$. Assume $\chi' \in \mathbb{R}$, $I_i^*(\chi') = 0$, $I_i^*(\chi) > 0$, and $\forall \chi \in (-\infty, \chi')$. Clearly, for $\chi' > \chi_0$, $I_i^{*'}(\chi') \le 0$. Using the second equation of (2.39), we obtain

$$\alpha I_1^{*'}(\chi') = d_2 J * I_1^*(\chi') - (d_2 + \mu + \eta) I_1^*(\chi') + L_1(S^*(\chi' - \alpha\vartheta), I_1^*(\chi' - \alpha\vartheta)) > 0.$$

This is a contradiction. Then, $I_1^*(\chi) > 0$, $\forall \chi \in \mathbb{R}$. the same with $I_2^*(\chi)$, so we get $I_2^*(\chi) > 0$ □

Let's define $(S^*(\chi), I_i^*(\chi))$ as in Theorem 2.4. To calculate the asymptotic boundary condition $(S^*(\chi), I_i^*(\chi)) \to (s^*, i^*)$ as $\chi \to +\infty$. To establish the existence of non-critical TWS, we need to demonstrate that $(S^*(\chi), I_i^*(\chi)) \to (S^*, I_i^*)$ as $\chi \to \infty$ using the Lyapunov-LaSalle Theorem. The findings are emphasised as follows:

Lemma 2.6 *$(S^*(\chi), I_i^*(\chi)) \to (S^*, I_i^*)$ uniformly as $\chi \to +\infty$.*

Proof: Motivated by [33], we let the following assumption for all $S \ne S^*$

$$\left(\frac{S}{S^*} - 1\right)\left[\frac{L_i(S, I_i^*)}{L_i(S^*, I_i^*)} - 1\right] > 0, \tag{2.40}$$

and for all $S, I_i > 0, 1 \le i \le 2$,

$$(L_i(S^*, I_i^*)L_i(S, I_i) - L_i(S^*, I_i^*)L_i(S, I_i^*))\left(\frac{L_i(S^*, I_i^*)L_i(S, I_i)}{I_i} - \frac{L_i(S^*, I_i^*)L_i(S, I_i^*)}{I_i^*}\right) \le 0, \tag{2.41}$$

Define

$$V_i(\chi) = V_i^1(\chi) + V_i^2(\chi),$$

where

$$\begin{aligned} V_i^1(\chi) &= \int_{S^*}^{S(\chi)} \left(1 - \frac{L_i(S, I_i)}{L_i(\xi, I_i)}\right) d\xi + I_i^* h\left(\frac{I_i(\chi)}{I_i^*}\right) + d_1 S^* F_1(\chi) + d_2 F_2(\chi) + d_2 F_3(\chi), \\ V_i^2(\chi) &= \sum_{i=1}^{2} L_i(S^*, I_i^*) \int_0^{\vartheta} h\left(\frac{L_i(S(\chi - \alpha\sigma), I_i(\chi - \alpha\sigma))}{L_i(S^*, I_i^*)}\right) d\sigma. \end{aligned}$$

with

$$\begin{aligned} F_1(\chi) = &\int_0^{+\infty} b^+(y)\left[h\left(\frac{S(\chi - y)}{S^*}\right) - h\left(\frac{L_i(S^*, I_i^*)S(\chi - y)}{L_i(S(\chi), I_i^*)S^*}\right)\right] dy - \int_{-\infty}^{0} b^-(y)\left[h\left(\frac{S(\chi - y)}{S^*}\right)\right. \\ &\left. - h\left(\frac{L_i(S^*, I_i^*)S(\chi - y)}{L_i(S(\chi), I_i^*)S^*}\right)\right] dy \end{aligned}$$

and

$$F_2(\chi) = \int_0^{+\infty} b^+ h\left(\frac{I_1(\chi - y)}{I_1^*}\right) dy - \int_{-\infty}^{0} b^- h\left(\frac{I_1(\chi - y)}{I_1^*}\right) dy$$

$$F_3(\chi) = \int_0^{+\infty} b^+ h\left(\frac{I_2(\chi - y)}{I_2^*}\right) dy - \int_{-\infty}^{0} b^- h\left(\frac{I_2(\chi - y)}{I_2^*}\right) dy$$

Thanks to [21, Theorem 1], $S(\chi) > 0, I(\chi) > 0$ and by the proof of Theorem **??**, we can have that $F_1(\chi), F_2(\chi), F_3(\chi)$ are bounded from below. Thus $V_i(S, I_1, I_2)(\chi)$ is well defined and bounded from below. Noting that $b^{\pm}$ satisfies $b^{\pm}(0) = \frac{1}{2}$, $\frac{db^+(y)}{dy} = J(y)$ and $\frac{db^-(y)}{dy} = -J(y)$, we have

$$\begin{aligned}
\frac{dF_1(\chi)}{d\chi} &= \frac{d}{d\chi}\int_0^{+\infty} b^+(y)\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy \\
&\quad -\frac{d}{d\chi}\int_{-\infty}^0 b^-(y)\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S_i^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy, \\
&= \int_0^{+\infty} b^+(y)\frac{d}{d\chi}\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy \\
&\quad -\int_{-\infty}^0 b^-(y)\frac{d}{\chi}\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy, \\
&= -\int_0^{+\infty} b^+(y)\frac{d}{dy}\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy \\
&\quad +\int_{-\infty}^0 b^-(y)\frac{d}{dy}\Big[h\Big(\frac{S(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy, \\
&= h\Big(\frac{S(\chi)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big) - \int_0^{+\infty} J(y)\Big[h\Big(\frac{S0(\chi-y)}{S^*}\Big) - h\Big(\frac{L_i(S^*,I_i^*)S(\chi-y)}{L_i(S(\chi),I_i^*)S^*}\Big)\Big]dy.
\end{aligned}$$

Similarly,

$$\frac{dF_2}{d\chi} = h\Big(\frac{I_1}{I_1*}\Big) - \int_{-\infty}^{+\infty} J(y)h\Big(\frac{I_1(\chi-y)}{I_1^*}\Big)dy.$$

$$\frac{dF_3}{d\chi} = h\Big(\frac{I_2}{I_2*}\Big) - \int_{-\infty}^{+\infty} J(y)h\Big(\frac{I_2(\chi-y)}{I_2^*}\Big)dy.$$

By some calculations, it can be shown that

$$\begin{aligned}
\frac{dV_i^2(z)}{dz} &= -\sum_{i=1}^{2} L_i(S^*,I_i^*)\Big[\Big(\frac{L_i(S(\chi-\alpha\vartheta),I_i(\chi-\alpha\vartheta))}{L_i(S^*,I_i^*)}\Big) - 1 - ln\Big(\frac{L_i(S(\chi-\alpha\vartheta),I_i(\chi-\alpha\vartheta))}{L_i(S^*,I_i^*)}\Big) \\
&\quad -\Big(\frac{L_i(S(\chi),I_i(\chi))}{L_i(S^*,I_i^*)}\Big) + 1 + ln\Big(\frac{L_i(S(\chi),I_i(\chi))}{L_i(S^*,I_i^*)}\Big)\Big].
\end{aligned}$$

Therefore, by a simple calculation and with the same way with proof of second part of Theorem 2.2 [28], we get

$$\begin{aligned}
\frac{dV_i(t)}{dt} &= \mu S^*\Big(1 - \frac{L_i(S^*,I_i^*)}{L_i(S(\chi),I_i^*)}\Big)\Big(1-\frac{S}{S^*}\Big) + \textstyle\sum_{i=1}^2 L_i(S^*,I_i^*)\Big[-h\Big(\frac{L_i(S(\chi-\alpha\vartheta),I_i(\chi-\alpha\vartheta))I_i^*}{L_i(S^*,I_i^*)I_i}\Big)\Big] \\
&\quad + \textstyle\sum_{i=1}^2 L_i(S^*,I_i^*)\Big[2 - \frac{L_i(S^*,I_i^*)}{P_i(S(\chi),I_i^*)} - \frac{L_i(S(\chi),I_i(\chi))I_i^*}{L_i(S^*,I_i^*)I_i(\chi)} - \frac{I_i(\chi)}{I_i^*} + \frac{L_i(S^*,I_i^*)L_i(S(\chi),I_i(\chi))}{L_i(S(\chi),I_i^*)L_i(S^*,I_i^*)} \\
&\quad + h\Big(\frac{L_i(S(\chi),I_i(\chi))I_i^*}{L_i(S^*,I_i^*)I_i(\chi)}\Big)\Big] - \int_{-\infty}^{+\infty} J(y)h\Big(\frac{I_i(\chi-y)}{I_i(\chi)}\Big)dy.
\end{aligned}$$

Therefore, we can show that $V_i, F_i(S^*, I_i^*)$ verify the assumptions of [46, Theorem 3.1, Corollary 3.3]. Consequently, $V = \sum_{i=1}^2 c_i V_i$ as defined in the Theorem 3.1 of [46] is a Lyapunov function for the system(2.4), namely, $V_i'(\chi) \le 0$ for all (S, I_1, I_2) . Hence, $V'(\chi) \le 0$, and $V(\chi) = 0$ if and only if $S = S^*$, $I_1 = I_1^*$ and $I_2 = I_2^*$.

Finally, we deduce that $(S(\chi), I_1(\chi), I_2(\chi)) \to (S^*, I_1^*, I_2^*)$, We let the set D corresponding to (2.4) as follows:

$$D = \Big\{(S(\chi), I_1(\chi), I_2(\chi)) | 0 < S < S^0, \quad 0 < I_i < I_i^+\Big\}.$$

Then, D is an invariant set (2.4), $\forall \chi \ge 0$. Since that L has a non-positive orbital derivative along $\Psi(\chi)$, and bounded from below. By the Lyapunov-LaSalle Theorem we obtain $\Psi(\chi) \to (S^*, I_1^*, I_2^*)$ as $z \to \infty$, and as a result,

$$(S(\chi), I_1(\chi), I_2(\chi)) \to (S^*, I_1^*, I_2^*)$$

when $\chi \to +\infty$. This concludes the proof. □

It worth noting that the solution of (2.4) satisfies $S^- \leq S \leq S^+$, and $I_i^- \leq I_i \leq I_i^+$, and $(S(\chi), I_1(\chi), I_2(\chi)) \to (S^0, 0, 0)$ as $z \to -\infty$. By Lemma 2.6, we have $(S(\chi), I_1(\chi), I_2(\chi)) \to (S^*, I_1^*, I_2^*)$ as $z \to +\infty$. Consequently, (2.4) has positive solution which verifies (2.5), that is a TWS of (1.1).

Remark 2.1 *The assumptions* (2.40)-(2.41) *are based on the results [33] (condition (7) and (8)). The proof of global stability is prompted by the findings of [27] for the delay terms, [33] for the multigroup system, and [34] for the nonlinear incidence function.*

3. Non-existence of TWS

3.1. Case I: $R_0 < 1$

In this subsection, we assume that $R_0 < 1$, then we by the contradiction we prove the non-existence of TWS for (1.1).

Theorem 3.1 *If $R_0 < 1$, then there exists no nonnegative bounded solution $(S(\chi)), I_1(\chi), I_2(\chi))$, of (2.4) satisfying (2.5).*

Proof: Assume that there is $(S(\chi), I_1(\chi), I_2(\chi))$ that solves (2.4)-(2.5). Let $(I_1)_{sup} = \sup_{\chi\in\mathbb{R}} I_1(\chi)$ and $(I_2)_{sup} = \sup_{\chi\in\mathbb{R}} I_2(\chi)$. The comparison principle implies that

$$\begin{pmatrix} I_1(\chi) \\ I_2(\chi) \end{pmatrix} \leq \mathcal{G} \begin{pmatrix} (I_1)_{sup}(\chi) \\ (I_2)_{sup}(\chi) \end{pmatrix} \quad \forall \chi \in \mathbb{R},$$

by the definition of $\mathcal{G}$ and R_0 in the section 2. Clearly, $\mathcal{G}$ is nonnegative and irreducible. The Perron-Frobenius theorem ensures the existence of a vector $P = (p_1, p_2)^T \in \mathbb{R}^2$, $p_1 > 0$, $p_2 > 0$, satisfies $\mathcal{G}P = R_0 P$. Noting that there is a constant $\epsilon > 0$ large enough, that satisfies

$$\begin{pmatrix} (I_1)_{sup} \\ (I_2)_{sup} \end{pmatrix} \leq R_0 P,$$

$$\begin{pmatrix} (I_1)_{sup} \\ (I_2)_{sup} \end{pmatrix} \leq \mathcal{G}^n \begin{pmatrix} (I_1)_{sup} \\ (I_2)_{sup} \end{pmatrix} \leq \epsilon \mathcal{G}^n P = \epsilon R_0^n P \to 0,$$

for $n \to \infty$, that contradicts $I_1(\chi) > 0$ and $I_2(\chi) > 0$, $\forall \chi \in \mathbb{R}$. □

3.2. Case II: $R_0 > 1$ and $0 < \alpha < \alpha^*$

In the next theorem, we showcase when (2.4) does not admit any TWS.

Theorem 3.2 *If $R_0 > 1$ and $0 < \alpha < \alpha^*$, hence,* (2.4) *has no TWS of the form $(S(\chi), I_1(\chi)I_2(\chi))$ that satisfies* (2.5).

Proof: Suppose that there is a TWS $(S^*(\chi), I_1^*(\chi), I_2^*(\chi))$ of (2.4) satisfying (2.5) for some $0 < \alpha < \alpha^*$. From (2.5) and $R_0 > 1$, for $\epsilon > 0$, there is some $M_\epsilon > 0$ large large as necessary to satisfy $S^0 - \epsilon \leq S^*(\chi) < S^0$ for all $\chi \leq -M_\epsilon$. Combining the second equation of (2.4), then

$$\begin{aligned} \alpha I_1^*(\chi) &= d_2(J * I_1^*(\chi) - I_1^*(\chi)) + L_1(S(\chi - \alpha\vartheta), I_1(\chi - \alpha\vartheta)) - (\mu + \eta) I_1(\chi) \\ &\geq d_2(J * I_1^*(\chi) - I_1^*(\chi)) + \Delta_1(S^0 - \epsilon, I_1(\chi - \alpha\vartheta)) - (\mu + \eta) I_1(\chi), \end{aligned} \tag{3.1}$$

for $\chi < -M_\epsilon$. Notice that by the continuity and (2.5), there is some constants $\delta_0 > 0$ and $M_i^0 > 0$ to satisfy $S^*(\chi) \geq \delta_0$ and $I_i^*(\chi) \leq M_i^0$ for all $\chi \in \mathbb{R}$. We have $L_i(S^*(\chi), I_i^*(\chi))$ quadratic continuously differentiable, non-decreasing, we obtain that

$$\frac{L_i((S^0 - \epsilon, I_i^*(\chi - \alpha\vartheta))}{L_i(S^*(\chi - \alpha\vartheta), I_i^*(\chi - \alpha\vartheta))} \leq \frac{L_i(S^0 - \epsilon, I_i^*(\chi - \alpha\vartheta))}{(\delta_0, I_i^*(\chi - \alpha\vartheta))} = \frac{L_i(S^0 - \epsilon, I_i^*(\chi - \alpha\vartheta)) I_i^*(\chi - \alpha\vartheta)}{L_i(\delta_0, I_i^*(\chi - \alpha\vartheta)) I_i^*(\chi - \alpha\vartheta)}$$

$$\leq \frac{M_i^0}{L_i(\delta_0, M_i^0)} \partial_2 L_i(S^0, 0) < \infty, \quad \chi > -M_\epsilon.$$

Noting that $I_i^*(\chi) > 0$, $\chi \in \mathbb{R}$ and $I_i^*(+\infty) = I_i^* > 0$, there is a positive constant $I_i^-(\chi) > 0$ satisfying $I_i^*(\chi) \geq I_i^-$, $\forall \chi \geq -M_\epsilon$. Therefore, we select some constants $h > 1$ to satisfy

$$\frac{L_i(S^0 - \epsilon, I_i^*(\chi - \alpha\vartheta))}{(1 + I_i^*(\chi - \alpha\vartheta))^h} \leq L_i(S^*(\chi - \alpha\vartheta), I_i^*(\chi - \alpha\vartheta)) \quad \text{for} \quad \chi > -M_\epsilon.$$

Then, for $\chi > -M_\epsilon$, the following the inequality holds:

$$\alpha I_1^*(\chi) \geq d_2(J * I_1^*(\chi) - I^*(\chi)) + \frac{L_1(S^0 - \epsilon, I_1^*(\chi - \alpha\vartheta))}{(1 + I_1^*(\chi - \alpha\vartheta))^h} - (\mu + \eta) I_1^*(\chi). \tag{3.2}$$

By (3.1) and (3.2), we get

$$\alpha I_1^*(\chi) \geq d_2(J * I_1^*(\chi) - I_1^*(\chi)) + \frac{L_1(S^0 - \epsilon, I_j^*(\chi - \alpha\vartheta))}{(1 + I_1^*(\chi - \alpha\vartheta))^h} - (\mu + \eta) I_1^*(\chi), \quad z \in \mathbb{R} \tag{3.3}$$

By the same way with $I_2(\chi)$. Let $b_i(u_i) = \inf_{u_i \leq v_i \leq M^0} \frac{L_i(S^0 - \epsilon, v_i)}{(1+v_i)^h}$, and $u_i(x,t) = I_i^*(x + \alpha\vartheta)$. (3.3) yields

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} \geq d_2(J * u_1^*(x,t) - u_1(x,t)) + b_1(u_1(x, t - \vartheta)) - (\mu + \eta) u_1(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} \geq d_3(J * u_2^*(x,t) - u_2(x,t)) + b_2(u_2(x, t - \vartheta)) - (\mu + \rho) u_2(x,t), \\ u_i(x,t) = I_i^*(x + \alpha\vartheta), \quad x \in \mathbb{R}, \vartheta > 0. \end{cases}$$

The comparison principle [40] gives

$$u_i(x,t) \geq v_i(x,t), \quad x \in \mathbb{R}, t \geq 0, \tag{3.4}$$

with $v_i(x,t)$ solves

$$\begin{cases} \frac{\partial v_1(x,t)}{\partial t} &= d_2(J * v_1^*(x,t) - v_1(x,t)) + L_1(v_1(x, t - \vartheta)) - (\mu + \eta) v_1(x,t), \\ \frac{\partial v_2(x,t)}{\partial t} &= d_2(J * v_2^*(x,t) - v_2(x,t)) + b_2(v_2(x, t - \vartheta)) - (\mu + \rho) v_2(x,t), \\ & v_i(x,t) = I_i^*(x + \alpha\vartheta), \quad x \in \mathbb{R}, \vartheta > 0. \end{cases} \tag{3.5}$$

Next, we claim that for any $\hat{\alpha} \in (0, \alpha^*)$, we have

$$\lim_{t \to \infty} \inf_{|x| \leq \hat{\alpha} t} v_i(x,t) > 0. \tag{3.6}$$

By [44], we have that $J * \cdot$ generates a C_0-semigroup [38,43]. Clearly, (3.5) has two equilibriums $(v_1, v_2) = (0,0)$ and a ES $(v_1, v_2)(= v_1^*, v_2^*)$ verifying the system $b_1(v_1^*) - (\mu + \eta) v_1^* = 0$ $b_2(v_2^*) - (\mu + \rho) v_2^* = 0$,. Let $C := C(\mathbb{R} \times [-\vartheta, 0])$ and $C_{v_i^*} := \{v_i \in C : 0 \leq v_i \leq v_i^*\}$. By the semigroup theory [38,41], we have that (3.5) generates a monotone semi-flow $Q_i^t : C_{v_i^*} \to C_{v_i^*}$ induced by

$$Q_i^t(\psi_i)(x) = v_i(x, t + s), \quad x \in \mathbb{R}, t \geq 0, s \in [-\vartheta, 0], \psi_i \in C_{v_i^*},$$

with $v_i(x,t)$ is the unique solution of (3.5) satisfying $v_i(x,s) = \psi_i$.

Define $\tilde{C} = C([-\vartheta, 0])$ and $\tilde{C}_{v_i^*} = \{v_i \in \tilde{C} : 0 \leq v_i \leq v_i^*\}$. Let $\tilde{Q}_i^t : \tilde{C}_{v_i^*} \to \tilde{C}_{v_i^*}$ be the solution semi-flow associated to

$$\begin{cases} \frac{dv_1(t)}{dt} &= b_1(v_1(t - \vartheta)) - (\mu + \eta) v_1(t), \\ \frac{dv_2(t)}{dt} &= b_2(v_2(t - \vartheta)) - (\mu + \rho) v_2(t), \quad t > 0. \end{cases}$$

with $v_i^0 = \psi_i^0 \in \tilde{C}_{v_i^*}$, and $v_i^t = v_i(t + s)$, $s \in [-\vartheta, 0]$. [39, Corollary 5.3.5] yields that $\tilde{Q}_t$ is eventually strongly monotone on $\tilde{C}_{v_i^*}$. Moreover, the Dancer-Hess connecting orbit lemma [45] implies that $\tilde{Q}_t$ also is a strongly monotone full orbit connecting 0 to v_i^*. Therefore, the assumption (A5) in [40] is valid. Indeed, for each $t > 0$, we have that $\tilde{Q}_t$ satisfies (A1)-(A5) in [40]. Also, $\tilde{Q}_t$ satisfies equation (3.5). thus, $\tilde{Q}_t$ also is the restriction of Q_t to $\tilde{C}_{v_i^*}$. This ensures that [40, Theorem 2.17] is applicable. Consequently, we get that (3.6) is valid.

Selecting $\alpha_0 \in (\alpha, \alpha^*)$ and let $x = -\alpha_0 t$. (3.4)-(3.6) yields

$$\liminf_{t\to\infty} u_i(x,t) \geq \liminf_{t\to\infty, |x|\leq\alpha_0 t} v_i(x,t) > 0. \tag{3.7}$$

As $\chi = x + \alpha t = (\alpha - \alpha_0)t \to -\infty$ as $t \to \infty$, we get

$$\lim_{t\to\infty} u_i(x,t) = \lim_{t\to\infty} I_i^*(x+\alpha t) = \lim_{t\to\infty} I_i^*((\alpha-\alpha_0)t) = \lim_{\chi\to-\infty} I_i^*(\chi) = 0.$$

That is a contradiction with (3.7). □

4. Numerical Simulation

First, we present some numerical representations to verify the TWS of (2.4) that connect the two equilibria. To do this, we consider the following initial conditions

$$(S)_0(x) = \begin{cases} 0.65 & \text{if } x \in [-50, 0], \\ 0.15 & \text{if } x \in [0, 50[, \end{cases}$$

$$(I_1)_0(x) = \begin{cases} 0 & \text{if } x \in [-50, 0], \\ 2.9 & \text{if } x \in [0, 50[. \end{cases}$$

$$(I_2)_0(x) = \begin{cases} 0 & \text{if } x \in [-50, 0], \\ 2.2 & \text{if } x \in [0, 50[. \end{cases}$$

We also adopt the kernel function as

$$J(x) = \begin{cases} Ce^{\frac{1}{4x^2-1}}, & -0.5 < x < 0.5 \\ 0, & \text{otherwise}, \end{cases}$$

with C is chosen to satisfy $\int_{-0.5}^{0.5} J(x)dx = 1$.

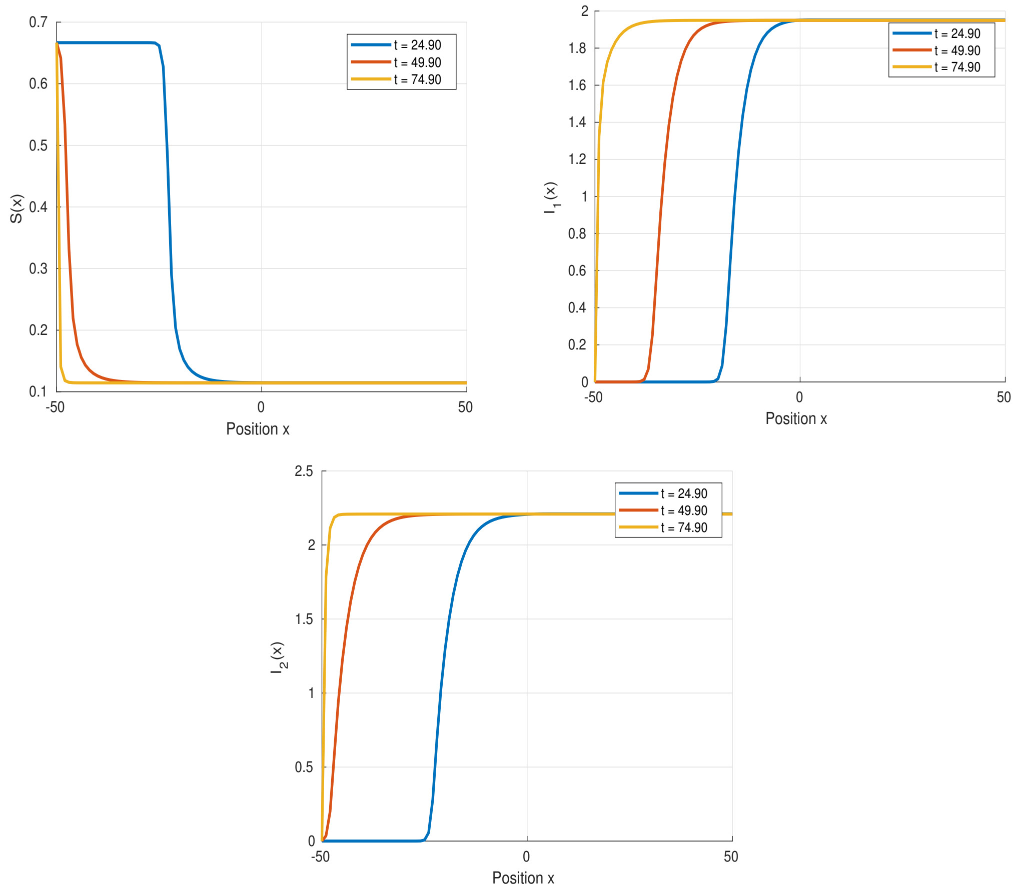

Figure 1: Cross section curves of solutions of the model (2.4) for different time values and $L_i(S, I_i) = \frac{\beta_i S I_i}{1+\alpha I_i}$, which ensures the existence of a TWS with $\Delta = 0.08$,$\beta_1 = 3$,$\beta_2 = 2$, $\mu = 0.1$.

Figure 1 shows the cross-sectional solution profiles of the model (2.4) at different time values for the incidence function $L_1(S, I_1) = \frac{\beta_1 S I_1}{1+\alpha_1 I_1}$. The plots reveal the formation and progression of traveling wave solutions (TWS) connecting two steady states. The susceptible population S decreases from left to right, while the infected populations I_1 and I_2 increase, forming smooth wavefronts. As time progresses, the curves stabilize, confirming the theoretical existence of TWS. The chosen parameters $\Delta = 0.08, \beta_1 = 3, \beta_2 = 2, \mu = 0.1$ allow for coherent wave propagation, aligning with analytical expectations. This validates the model's ability to reproduce realistic epidemic wave behavior under nonlinear incidence.

Now we put

$$(S)_0(x) = \begin{cases} 0.65 & \text{if } x \in [-50, 0], \\ 0.3 & \text{if } x \in [0, 50[, \end{cases}$$

$$(I_1)_0(x) = \begin{cases} 0 & \text{if } x \in [-50, 0], \\ 0.35 & \text{if } x \in [0, 50[. \end{cases}$$

$$(I_2)_0(x) = \begin{cases} 0 & \text{if } x \in [-50, 0], \\ 0.33 & \text{if } x \in [0, 50[. \end{cases}$$

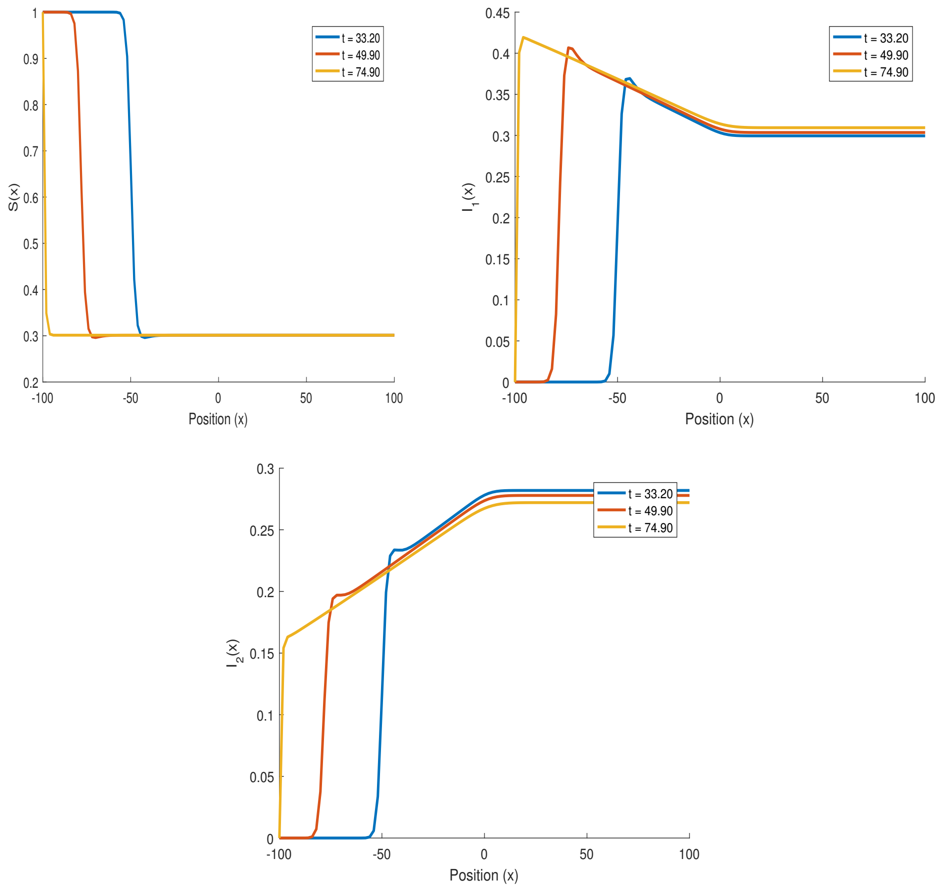

Figure 2: Cross section curves of solutions of the model (2.4) for different time values and $L_1(S, I_1) = \frac{\beta_1 S I_1}{1+\alpha I_1}$ and $L_2(S, I_2) = \frac{\beta_2 S I_2}{1+\alpha I_2+\xi S}$, which ensures the existence of a TWS with $\Delta = 0.08$,$\beta_1 = 3$,$\beta_2 = 2$, $\mu = 0.1$.

Figure 2 presents the numerical evolution of the model (2.4) with modified initial conditions and a more complex incidence structure, where $L_1(S, I_1) = \frac{\beta_1 S I_1}{1+\alpha_1 I_1}$ and $L_2(S, I_2) = \frac{\beta_2 S I_2}{1+\alpha_2 I_2+\kappa S}$. Nevertheless, the presence of traveling wave patterns remains evident, reinforcing the robustness of TWS existence even under stronger nonlinear interactions. The parameters $\Delta = 0.08, \beta_1 = 3, \beta_2 = 2, \mu = 0.1$ continue to support this dynamic. These simulations underline the influence of nonlinear incidence on wave behavior and further support the analytical framework developed.

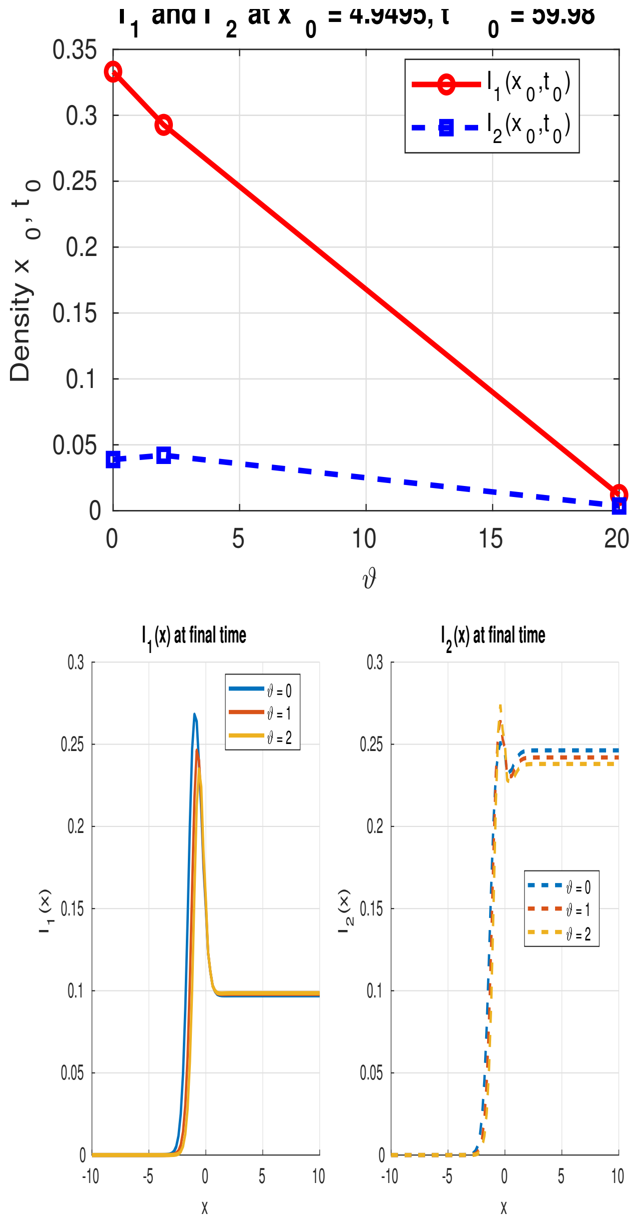

Figure 3: The effect of the delay ϑ in the infected population of the model (2.4) I_1 and I_2.

When the time delay (ϑ) in the disease transmission process increases, the infected population (both I_1 and I_2) typically decreases in magnitude. This occurs because delays slow down the interaction between susceptible (S) and infected individuals, reducing the instantaneous force of infection (see the Fig 3. In biological terms, a longer delay represents a slower progression from exposure to infectiousness, which can arise from factors such as prolonged incubation periods, delayed symptom onset, or slower pathogen replication.

The precise impact depends on the balance between delay-induced suppression and other factors like diffusion rates and recovery times.

References

1. J.D. Murray, *Mathematical Biology: I. An Introduction*, Springer, 2002.
2. Z.C. Wang and X.Q. Zhao, Traveling waves of reaction–diffusion systems for competing species with nonlocal diffusion, *J. Math. Biol.*, 49(5):620–645, 2004.
3. W.T. Li, Z.C. Wang, and X.Q. Zhao, Spreading speeds and traveling waves for non-cooperative reaction–diffusion systems, *J. Differential Equations*, 248(1):1–27, 2010.
4. M.X. Wang and W.T. Li, Nonlocal dispersal and bistable dynamics: convergence to traveling waves, *SIAM J. Math. Anal.*, 44(6):4055–4089, 2012.
5. W. Wang and X.Q. Zhao, Spreading speed and traveling waves for a delayed reaction-diffusion equation in ecology, *J. Math. Anal. Appl.*, 342(2):895–908, 2008.
6. W.S.C. Gurney, R.M. Nisbet, and J.H. Lawton, Oscillations in epidemics: a field study and mathematical model, *Nature*, 287:45–47, 1980.
7. M.Y. Li and H.F. Weinberger, Existence of traveling wavefronts in delayed reaction–diffusion systems with nonlocal terms, *Nonlinearity*, 25(2):451, 2012.
8. R. Li, S. Pei, B. Chen, et al., Substantial undocumented infection facilitates the rapid dissemination of novel coronavirus (SARS-CoV-2), *Science*, 368(6490):489–493, 2020.
9. A Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission. Bulletin of Mathematical biology 68 (2006) 615-626.
10. X. He, et al., Temporal dynamics in viral shedding and transmissibility of COVID-19, *Nature Medicine*, 26:672–675, 2020.
11. J. Fang and X.Q. Zhao, Monotone traveling waves for delayed reaction-diffusion systems with nonlocal effects, *J. Dynam. Differential Equations*, 23(2):199–219, 2011.
12. J Zhou, J Xu, J Wei, H Xu, Existence and non-existence of traveling wave solutions for a nonlocal dispersal SIR epidemic model with nonlinear incidence rate. Nonlinear Analysis: Real World Applications 41 (2018) 204-231.
13. J.S. Guo and C.H. Wu, Traveling waves in reaction–diffusion equations with nonlocal delay, *J. Differential Equations*, 244(8):1867–1891, 2008.
14. Z.C. Wang and J.H. Wu, Traveling waves of a diffusive Kermack–McKendrick epidemic model with non-local delayed transmission, *Proc. R. Soc. A*, 466:237–261, 2010.
15. G. Zhang and Y. Wang, Minimal wave speed of a nonlocal diffusive epidemic model with temporal delay, *Rocky Mountain J. Math.*, 44(1):329–348, 2014.
16. L. Zhao, L. Zhang, and H. Huo, Traveling wave solutions of a diffusive SEIR epidemic model with nonlinear incidence rate, *Taiwanese J. Math.*, 23(4):951–980, 2019.
17. X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Adv. Differential Equations*, 2:125–160, 1997.
18. J. Coville, J. Dvila, and S. Martnez, Nonlocal anisotropic dispersal with monostable nonlinearity, *J. Differential Equations*, 244:3080–3118, 2008.
19. A. A. Mohsen and R. K. Naji. Dynamical Analysis Within-Host and Between-Host for HIV AIDS with the Application of Optimal Control Strategy. (2020). Iraqi Journal of Science, 61(5), 1173-1189. https://doi.org/10.24996/ijs.2020.61.5.25.
20. B. Guenad, R. Darazirar, S. Djilali, I. Alraddadi. Traveling waves in a delayed reaction–diffusion SIR epidemic model with a generalized incidence function, Nonlinear Dynamics, (2024).
21. G Huang, Y Takeuchi, W Ma, D Wei, Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate. Bulletin of mathematical biology 72 (2010) 1192-1207.
22. A. Ducrot, P. Magal, and S. Ruan, Travelling wave solutions in multigroup age-structured epidemic models, *Arch. Rational Mech. Anal.*, 195:311–331, 2010.
23. S. Djilali, G. Sarmad, A. Tridane. Dynamics and asymptotic profiles of a local-nonlocal dispersal SIR epidemic model with spatial heterogeneity. Infectious Disease Modelling, 10(2), 387-409 (2025).
24. J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
25. A. A. Mohsen , H. F. Al-Husseiny and K. Hattaf, "The awareness effect of the dynamical behavior of SIS epidemic model with Crowley-Martin incidence rate and holling type III treatment function," International Journal of Nonlinear Analysis and Applications, 12 2 (2021): 1083-1097, doi: 10.22075/ijnaa.2021.5177.
26. M. Naim, A. Zeb, A. A. Mohsen, Y. Sabbar, et al., . Local and global stability of a fractional viral infection model with two routes of propagation, cure rate and non-lytic humoral immunity. Mathematical Modelling and Numerical Simulation With Applications, 4(5), (2024): 94-115. https://doi.org/10.53391/mmnsa.1517325
27. C. C. McCluskey, Using Lyapunov functions to construct Lyapunov functionals for delay differential equations. SIAM Journal on Applied Dynamical Systems 14.1 (2015) 1-24.

28. S.Djilali, R. Darazirar, I. Alraddadi. Traveling wave solution for a delayed reaction-diffusion two-group SIR epidemic model with a generalized nonlinear incidence function. J. Appl. Math. Comput, s12190-025-02474-4. (2025).

29. Y. Hosono and B. Ilyas, Traveling waves for a simple diffusive epidemic model, *Math. Models Methods Appl. Sci.*, 5:935–966, 1995.

30. H. Toumi, K. Boudjema Djeffal, A. Mezouaghi, R. Darazirar, S. Djilali. Traveling Waves in a Reaction-Diffusion Addiction Epidemic Model with Distributed Delays. Acta Applicandae Mathematicae, 198(1), 1-28 (2025).

31. J. Huang and X. Zou, Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays, *J. Math. Anal. Appl.*, 271:455–466, 2002.

32. C.Y. Kao, Y. Lou, and W. Shen, Random dispersal vs. nonlocal dispersal, *Discrete Contin. Dyn. Syst.*, 26:551–596, 2010.

33. R. Sun, Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence. Computers and Mathematics with Applications 60 (2010) 2286–2291.

34. A. Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission. Bulletin of Mathematical Biology 68 (2006) 615-626.

35. W.O. Kermack and A.G. McKendrick, A contribution to the mathematical theory of epidemics, *Proc. R. Soc. Lond. Ser. A*, 115:700–721, 1927.

36. K. Li and X. Li, Traveling waves in a nonlocal delayed epidemic model with diffusion, *Math. Methods Appl. Sci.*, 44(13):10823–10836, 2021.

37. K. Wu and K. Zhou, Traveling waves in a nonlocal dispersal SIR model with standard incidence rate and nonlocal delayed transmission, *Mathematics*, 7(7):641, 2019.

38. A Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag 44 (1983).

39. H Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems.

40. L Smith, X Zhao, Global asymptotic stability of traveling waves in delayed reaction-diffusion equations." SIAM Journal on Mathematical Analysis 31.3 (2000) 514-534.

41. W Wu, Z Teng, Traveling wave solutions in a nonlocal dispersal SIR epidemic model with general nonlinear incidence. Acta Applicandae Mathematicae 175.1 (2021) 4.

42. W Wu, Z Teng, Traveling waves in nonlocal dispersal SIR epidemic model with nonlinear incidence and distributed latent delay. Advances in Difference Equations 2020.1 (2020) 614.

43. J Wu, Theory and applications of partial functional differential equations. Vol. 119. Springer Science & Business Media, 2012.

44. X Liang, X Zhao. Asymptotic spread and travelling wave speeds in monotone semiflows with applications. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 60.1 (2007) 1-40.

45. X Zhao, Dynamical systems in population biology. Vol. 16. New York: Springer, 2003.

46. M.Y. Li, Z. Shuai, Global-stability problem for coupled systems of differential equations on networks. J. Differential Equations 248 , 1–20 (2010).

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