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Distance-Based Connectivity Indices for Chain Polygonal Cactus Networks and their Applications in Wireless Communication Systems

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ABSTRACT: This paper develops two novel distance-based connectivity indices for Chain Polygonal Cactus (CPC) networks, a class of graphs formed by linking cycles via articulation points. We define the d-connectivity index, which characterizes short-range communication potential through degree–distance analysis, and the e-connectivity index, which reflects long-range connectivity based on eccentricity–distance relationships. Exact analytical formulas for these indices are derived for various structural configurations of CPCs by systematically partitioning the vertex sets according to their combinatorial and distance properties. The theoretical results not only advance the study of distance-based invariants in graph theory but also motivate applications in wireless communication systems, where CPC-like topologies can enhance network efficiency and resilience. Our findings provide a mathematical framework for assessing structural robustness and accessibility in networked systems.

Keywords: Distance-based indices, cactus graphs, eccentricity, connectivity measures, graph invariants, degree–distance analysis, network robustness, wireless communication modeling.

Contents

1. Introduction

Graph-theoretic modeling plays a pivotal role in designing efficient communication networks. In particular, cactus structures, due to their sparse yet cyclic configuration, offer a compelling framework for minimizing redundancy while maintaining reliability. A *cactus network* is a connected graph in which no edge lies in more than one cycle. Among various classes of cactus networks, the *Chain Polygonal Cactus* (CPC) networks, where cycles are arranged linearly with shared articulation points, provides a tractable and robust topology for modeling modular wireless networks.

In this study, we develop a new perspective on connectivity within CPC networks by introducing two distance-based indices: the *d-connectivity* index and the *e-connectivity* index. These indices, derived from degree, d-number, eccentricity and e-number, enable mathematical estimation of node-to-node and node-to-network communication potential. The *d-number* facilitates modeling of localized, short-range communication, while the *e-number* captures long-range interaction necessary for network-wide data flow.

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By quantifying these metrics across various CPC configurations, we establish a theoretical framework for optimizing access point placement to ensure maximal coverage and minimum isolation in wireless networks. Although the application discussion remains theoretical, the results offer meaningful implications for the planning and deployment of network infrastructures, especially in topologies with inherent sparsity and modularity.

Chain polygonal cactus structures have been extensively studied in mathematical literature. Došlić and others examined matchings and independent sets in hexagonal and polyphenylene cacti [3,4,5,6,7,8]; Jiachang et al. evaluated extremal CPCs for bond incident degree indices and general Sombor indices [9,10]; Klamsakul et al. analyzed maximal independent sets in polygonal cacti [11]; Song et al. contributed by finding Zagreb eccentricity indices of cacti [12]; Zeng et al. contributed results for Wiener and Kirchhoff indices [13]; recent work by Salman et al. explored metric identification in such networks [14]. These contributions underscore the analytical richness and real-world modeling capability of cactus structures.

The remainder of this paper is organized as follows: In Section 2, we develop the graph theoretical formulation for the subsequent part of the paper. Sections 3 and 4 explore the concepts of d- and e-connectivity of CPC. and their corresponding indices. Section 5 outlines the potential applications of these indices in theoretical wireless communication models. Finally, Section 6 concludes the paper with a summary of our findings and future directions.

2. Preliminaries and Definitions

Let $\mathcal{N} = (V, E)$ be a simple, connected network, where $V = V(\mathcal{N})$ denotes the set of nodes (vertices) and $E = E(\mathcal{N})$ the set of links (edges). Two nodes $u, v \in V$ are called neighbors if there exists an edge $uv \in E$. The set of neighbors of u is called the (open) neighborhood $N(u)$, while the set $N[u] = N(u) \cup \{u\}$ is called the closed neighborhood of u. The *degree* of a node u, denoted by $d(u)$, is the number of edges incident to u. The *distance* $d(u, v)$ between two nodes u and v is the length of the shortest path connecting them. The *eccentricity* of a node u, denoted by $\mathrm{ecc}(u)$, is defined as the maximum distance from u to any other node in $\mathcal{N}$. The *radius* and *diameter* of $\mathcal{N}$ are defined as $\mathrm{rad}(\mathcal{N}) = \min\{\mathrm{ecc}(u) : u \in V\}$ and $\mathrm{diam}(\mathcal{N}) = \max\{\mathrm{ecc}(u) : u \in V\}$, respectively [2]. We define two special distance types in $\mathcal{N}$:

- The d-distance between nodes u and v occurs when $d(u, v) \in \{d(u), d(v)\}$.
- The e-distance between nodes u and v occurs when $d(u, v) \in \{\mathrm{ecc}(u), \mathrm{ecc}(v)\}$.

A node $u \in V$ is an *articulation point* (cut-node) if its removal increases the number of connected components in $\mathcal{N}$. A *cactus* is a connected graph in which every block (maximal 2-connected subgraph) is either a cycle or a single edge, and no edge belongs to more than one block. If each block is a cycle, the graph is called a *polygonal cactus* network. A *k-polygon* refers to a cycle of length k, denoted by C_k. A *Chain Polygonal Cactus* (CPC) network is a polygonal cactus where each cycle is a C_k and every pair of consecutive cycles shares exactly one articulation point. If a CPC network contains n such polygons, it is denoted by $T_{n,k}$. The graph $T_{n,k}$ contains $n(k-1)+1$ nodes. Figure 1 depicts the structure of $T_{6,8}$ CPC to elaborate its four types.

From the definition, $T_{n,k}$ has exactly $n-2$ non-pendent polygons and two pendent polygons. $T_{n,3}$ is the unique CPC, whereas $T_{n,k}$ is not unique for each $k \geq 4$. In a CPC, an articulation point will be symbolized as c_i, where $1 \leq i \leq n-1$. For any $l \in \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$, whenever $d(c_i, c_{i+1}) = l$ (the length between any two consecutive articulation points) for each $1 \leq i \leq n-1$, the corresponding $T_{n,k}$ will be called an l-type CPC.

These definitions serve as the foundation for constructing the d- and e-connectivity indices in CPC networks.

3. d-Connectivity in CPC Networks

Let u be any node in a connected network $\mathcal{N}$, and let $d = d(u)$ be the degree of u. A node $v \in V(\mathcal{N})$ is called a *d-neighbor* of u if $d(u, v) = d$. The set of all d-neighbors of u is denoted by $N_d(u)$ and referred to as the *d-neighborhood* of u. The *d-number* of u is defined as:

$$n_d(u) = |N_d(u)|.$$

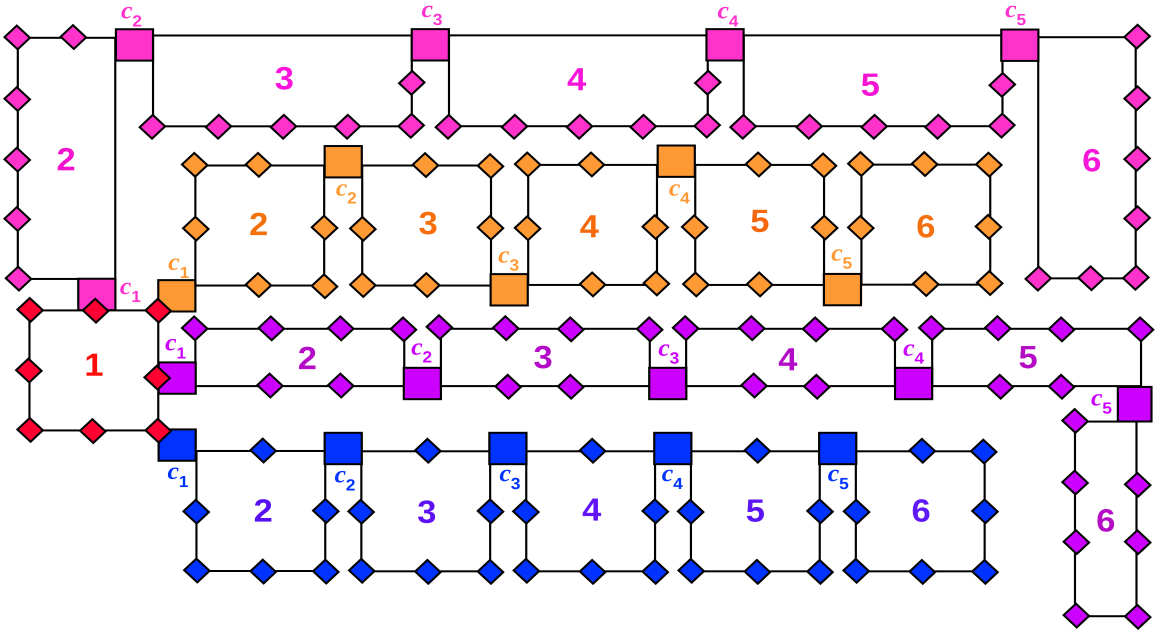

Figure 1: Illustrating, in one drawing, the structure and types of a CPC network with 6 polygons each has 8 nodes. A node acting as an articulation point is shown large. Starting with the red polygon and following the given numeration, we have 1-type CPC in pink, 2-type CPC in blue, 3-type CPC in purple, and 4-type CPC in mustered.

Using this, we define the *d-connectivity index* of $\mathcal{N}$ as:

$$C_d(\mathcal{N}) = \sum_{u \in V(\mathcal{N})} n_d(u) \cdot d(u).$$

This index reflects the total weighted influence of each node's d-neighborhood in terms of both quantity (number of d-neighbors) and quality (node degree), offering a measure of short-range communication potential in networked systems.

3.1. Partitioning of Nodes

Let $T_{n,k}$ be a CPC network with $n \geq 3$ k-polygons. Denote by $C = \{c_i \mid 1 \leq i \leq n-1\}$ the set of articulation points, and let X be the set of all other nodes. In any CPC, we observe:

- $d(v) = 4$ if $v \in C$,
- $d(v) = 2$ if $v \in X$.

Thus, the d-connectivity index becomes:

$$C_d(T_{n,k}) = 4 \sum_{v \in C} n_d(v) + 2 \sum_{v \in X} n_d(v).$$

To compute this index, we classify nodes into sets $V_{a,b} = \{v \in V \mid d(v) = a, n_d(v) = b\}$ for $a \in \{2, 4\}$. Then,

$$C_d(T_{n,k}) = 4 \sum_{v \in V_{4,b}} b + 2 \sum_{v \in V_{2,b}} b.$$

The detailed evaluation of $C_d(T_{n,k})$ for various values of k and CPC types will follow in the next subsection.

3.2. Evaluation for Specific CPC Configurations

We now evaluate the d-connectivity index for specific values of k and CPC types. We begin with the case $k = 3$.

Theorem 3.1 *For $n \geq 3$, the d-connectivity index of $T_{n,3}$ is given by:*

$$C_d(T_{n,3}) = 24n - 64.$$

Proof: Let $V_{4,2}$ denote the set of articulation points $c_1, c_2, c_3, c_{n-1}, c_{n-2}, c_{n-3}$, each having $n_d = 2$. Let $V_{4,4}$ include the remaining $n-7$ articulation points with $n_d = 4$. Non-articulation points in end polygons contribute to $V_{2,2}$ with 4 such nodes. All other non-articulation points belong to $V_{2,4}$ with cardinality $n-2$. Therefore:

$$\begin{aligned} C_d(T_{n,3}) &= 2(2|V_{2,2}| + 4|V_{2,4}|) + 4(2|V_{4,2}| + 4|V_{4,4}|) \\ &= 2(2 \cdot 4 + 4(n-2)) + 4(2 \cdot 6 + 4(n-7)) \\ &= 2(8 + 4n - 8) + 4(12 + 4n - 28) \\ &= 8n + 4(4n - 16) \\ &= 8n + 16n - 64 \\ &= 24n - 64. \end{aligned}$$

□

Theorem 3.2 *For $n \geq 3$, let $T_{n,4}$ be an l-type CPC. Then the d-connectivity index is given by:*

$$C_d(T_{n,4}) = \begin{cases} 36n - 84, & \text{if } l = 1, \\ 28n - 28, & \text{if } l = 2. \end{cases}$$

Proof: **Case 1:** $l = 1$. The articulation points $c_1, c_2, c_{n-1}, c_{n-2}$ contribute to $V_{4,3}$ (each with $n_d = 3$); c_3, c_{n-3} contribute to $V_{4,4}$; and all remaining articulation points to $V_{4,6}$. That gives:

- $|V_{4,3}| = 4$, $|V_{4,4}| = 2$, $|V_{4,6}| = n - 7$;
- Non-articulation points: $|V_{2,1}| = 2$, $|V_{2,3}| = 2n$.

Substituting into the formula:

$$\begin{aligned} C_d(T_{n,4}) &= 2(1 \cdot 2 + 3 \cdot 2n) + 4(3 \cdot 4 + 4 \cdot 2 + 6(n-7)) \\ &= 2(2 + 6n) + 4(12 + 8 + 6n - 42) \\ &= 2(6n + 2) + 4(6n - 22) \\ &= 12n + 4 + 24n - 88 \\ &= 36n - 84. \end{aligned}$$

Case 2: $l = 2$. Articulation points c_1, c_{n-1} go to $V_{4,1}$, the rest to $V_{4,2}$. For non-articulation nodes: $|V_{2,1}| = 2$, $|V_{2,3}| = 4$, and $|V_{2,5}| = 2n - 4$.

$$\begin{aligned} C_d(T_{n,4}) &= 2(1 \cdot 2 + 3 \cdot 4 + 5(2n-4)) + 4(1 \cdot 2 + 2(n-3)) \\ &= 2(2 + 12 + 10n - 20) + 4(2 + 2n - 6) \\ &= 2(10n - 6) + 4(2n - 4) \\ &= 20n - 12 + 8n - 16 \\ &= 28n - 28. \end{aligned}$$

□

Theorem 3.3 *For $n \geq 3$, let $T_{n,5}$ be an l-type CPC. Then the d-connectivity index is given by:*

$$C_d(T_{n,5}) = \begin{cases} 52n - 104, & \textit{if } l = 1, \\ 44n - 40, & \textit{if } l = 2. \end{cases}$$

Proof: Case 1: $l = 1$. Articulation points $c_1, c_2, c_{n-1}, c_{n-2}$ belong to $V_{4,4}$, c_3, c_{n-3} to $V_{4,6}$, and the remaining to $V_{4,8}$. Non-articulation points are split into:

- $|V_{2,2}| = n + 2$,
- $|V_{2,4}| = 2n$,
- $|V_{4,4}| = 4$, $|V_{4,6}| = 2$, $|V_{4,8}| = n - 7$.

Substituting:

$$\begin{aligned} C_d(T_{n,5}) &= 2(2 \cdot (n+2) + 4 \cdot 2n) + 4(4 \cdot 4 + 6 \cdot 2 + 8(n-7)) \\ &= 2(2n + 4 + 8n) + 4(16 + 12 + 8n - 56) \\ &= 2(10n + 4) + 4(8n - 28) \\ &= 20n + 8 + 32n - 112 \\ &= 52n - 104. \end{aligned}$$

Case 2: $l = 2$. Articulation points c_1, c_{n-1} belong to $V_{4,2}$, and the rest to $V_{4,4}$. Non-articulation points include:

- $|V_{2,2}| = 4$, $|V_{2,4}| = 2n$, $|V_{2,6}| = n - 2$,
- $|V_{4,2}| = 2$, $|V_{4,4}| = n - 3$.

Thus:

$$\begin{aligned} C_d(T_{n,5}) &= 2(2 \cdot 4 + 4 \cdot 2n + 6(n-2)) + 4(2 \cdot 2 + 4(n-3)) \\ &= 2(8 + 8n + 6n - 12) + 4(4 + 4n - 12) \\ &= 2(14n - 4) + 4(4n - 8) \\ &= 28n - 8 + 16n - 32 \\ &= 44n - 40. \end{aligned}$$

□

Theorem 3.4 *For $n \geq 3$, let $T_{n,6}$ be an l-type CPC. Then the d-connectivity index is given by:*

$$C_d(T_{n,6}) = \begin{cases} 64n - 120, & \textit{if } l = 1, \\ 48n - 40, & \textit{if } l = 2, 3. \end{cases}$$

Proof: Case 1: $l = 1$. Articulation points are grouped as:

- $V_{4,5}$: c_1, c_{n-1},
- $V_{4,6}$: c_2, c_{n-2},
- $V_{4,8}$: c_3, c_{n-3},
- $V_{4,10}$: remaining $n - 7$ articulation points.

Non-articulation nodes:

- $|V_{2,2}| = 2n + 2$, $|V_{2,4}| = 2n$.

Substituting:

$$\begin{aligned} C_d(T_{n,6}) &= 2(2\cdot(2n+2)+4\cdot 2n)+4(5\cdot 2+6\cdot 2+8\cdot 2+10(n-7)) \\ &= 2(4n+4+8n)+4(10+12+16+10n-70) \\ &= 2(12n+4)+4(10n-32) \\ &= 24n+8+40n-128 \\ &= 64n-120. \end{aligned}$$

Case 2: $l=2$ or $l=3$. Articulation points:

- $V_{4,2}$: c_1, c_{n-1},
- $V_{4,4}$: remaining $n-3$ articulation points.

Non-articulation nodes:

- $|V_{2,2}|=6$, $|V_{2,4}|=4n-4$.

Substituting:

$$\begin{aligned} C_d(T_{n,6}) &= 2(2\cdot 6+4\cdot(4n-4))+4(2\cdot 2+4(n-3)) \\ &= 2(12+16n-16)+4(4+4n-12) \\ &= 2(16n-4)+4(4n-8) \\ &= 32n-8+16n-32 \\ &= 48n-40. \end{aligned}$$

□

Theorem 3.5 *For $n\geq 3$, let $T_{n,7}$ be an l-type CPC. Then the d-connectivity index is given by:*

$$C_d(T_{n,7}) = \begin{cases} 76n-136, & \text{if } l=1, \\ 56n-40, & \text{if } l=2,3. \end{cases}$$

Proof: Case 1: $l=1$. Articulation points are classified as:

- $V_{4,6}$: c_1, c_{n-1},
- $V_{4,8}$: c_2, c_{n-2},
- $V_{4,10}$: c_3, c_{n-3},
- $V_{4,12}$: remaining $n-7$ articulation points.

Non-articulation nodes:

- $|V_{2,2}|=3n+2$, $|V_{2,4}|=2n$.

Substituting:

$$\begin{aligned} C_d(T_{n,7}) &= 2(2\cdot(3n+2)+4\cdot 2n)+4(6\cdot 2+8\cdot 2+10\cdot 2+12(n-7)) \\ &= 2(6n+4+8n)+4(12+16+20+12n-84) \\ &= 2(14n+4)+4(12n-36) \\ &= 28n+8+48n-144 \\ &= 76n-136. \end{aligned}$$

Case 2: $l=2$ or $l=3$. Articulation points:

- $V_{4,2}$: c_1, c_{n-1},

- $V_{4,4}$: remaining $n-3$ articulation points.

Non-articulation nodes:

- $|V_{2,2}| = 6$, $|V_{2,4}| = 2n+2$, $|V_{2,6}| = 2n-4$.

Substituting:

$$\begin{aligned} C_d(T_{n,7}) &= 2(2\cdot 6 + 4(2n+2) + 6(2n-4)) + 4(2\cdot 2 + 4(n-3)) \\ &= 2(12 + 8n + 8 + 12n - 24) + 4(4 + 4n - 12) \\ &= 2(20n-4) + 4(4n-8) \\ &= 40n - 8 + 16n - 32 \\ &= 56n - 40. \end{aligned}$$

□

Theorem 3.6 *For $n \geq 3$, let $T_{n,8}$ be an l-type CPC. Then the d-connectivity index is given by:*

$$C_d(T_{n,8}) = \begin{cases} 88n - 144, & \text{if } l = 1, \\ 64n - 48, & \text{if } l = 2 \text{ or } 3, \\ 48n - 16, & \text{if } l = 4. \end{cases}$$

Proof: We now analyze different l-types by enumerating the node partitions:

Case 1: $l = 1$

- $V_{2,2} = V \setminus \bigcup_{i=1}^{n-1} N[c_i]$, $\quad |V_{2,2}| = 4n+2$,
- $V_{2,4} = \bigcup_{i=1}^{n-1} N(c_i) \setminus C$, $\quad |V_{2,4}| = 2n$,
- $V_{4,8} = \{c_1, c_{n-1}\}$, $\quad |V_{4,8}| = 2$,
- $V_{4,10} = \{c_2, c_{n-2}\}$, $\quad |V_{4,10}| = 2$,
- $V_{4,12} = \{c_3, c_{n-3}\}$, $\quad |V_{4,12}| = 2$,
- $V_{4,14} = C \setminus (V_{4,8} \cup V_{4,10} \cup V_{4,12})$, $\quad |V_{4,14}| = n-7$.

Thus,

$$C_d(T_{n,8}) = 2(2\cdot 4n + 2 + 4\cdot 2n) + 4(8\cdot 2 + 10\cdot 2 + 12\cdot 2 + 14(n-7)) = 88n - 144.$$

Case 2: $l = 2$ **or** $l = 3$

- $V_{2,2} = V \setminus \bigcup_{i=1}^{n-1} N[c_i]$, $\quad |V_{2,2}| = 2n+6$,
- $V_{2,4} = \bigcup_{i=1}^{n-1} N(c_i) \setminus C$, $\quad |V_{2,4}| = 4n-4$,
- $V_{4,4} = \{c_1, c_{n-1}\}$, $\quad |V_{4,4}| = 2$,
- $V_{4,6} = C \setminus V_{4,4}$, $\quad |V_{4,6}| = n-3$.

Then,

$$C_d(T_{n,8}) = 2(2\cdot(2n+6) + 4\cdot(4n-4)) + 4(4\cdot 2 + 6(n-3)) = 64n - 48.$$

Case 3: $l = 4$

- $V_{2,2} = V \setminus \bigcup_{i=1}^{n-1} N[c_i], \quad |V_{2,2}| = 2n + 6,$
- $V_{2,4} = \bigcup_{i=1}^{n-1} N(c_i), \quad |V_{2,4}| = 4n - 4,$
- $V_{4,2} = C, \quad |V_{4,2}| = n - 1.$

Then,

$$C_d(T_{n,8}) = 2(2 \cdot (2n + 6) + 4 \cdot (4n - 4)) + 4(2 \cdot (n - 1)) = 48n - 16.$$

□

Theorem 3.7 *For $n \geq 3$ and $k \geq 9$, let $T_{n,k}$ be an l-type CPC. Then:*

$$C_d(T_{n,k}) = \begin{cases} 4(nk + 16n - 38), & l = 1, \\ 4(nk + 10n - 14), & l = 2, 3, \\ 4(nk + 6n - 6), & l \geq 4. \end{cases}$$

Proof: We use the structure:

$$C_d(T_{n,k}) = 2 \sum_{v \in X} n_d(v) + 4 \sum_{v \in C} n_d(v).$$

Let us denote:

- $V_{2,2} = V \setminus \bigcup_{i=1}^{n-1} N[c_i]$, size depends on k and l,
- $V_{2,4} = \bigcup_{i=1}^{n-1} N(c_i) \setminus C$, typically $|V_{2,4}| = 2n$,
- $V_{4,b}$ = articulation points with degree 4 and varying $n_d(v)$ values, depending on l and k.

Case 1: $l = 1$

$$\begin{aligned} |V_{2,2}| &= nk - 4n + 2, \quad |V_{2,4}| = 2n, \\ |V_{4,10}| &= 2, \quad |V_{4,12}| = 2, \quad |V_{4,14}| = 2, \quad |V_{4,16}| = n - 7. \\ C_d(T_{n,k}) &= 2(2(nk - 4n + 2) + 4 \cdot 2n) + 4(10 \cdot 2 + 12 \cdot 2 + 14 \cdot 2 + 16(n - 7)) \\ &= 4nk + 64n - 152 = 4(nk + 16n - 38). \end{aligned}$$

Case 2: $l = 2, 3$

$$\begin{aligned} |V_{2,2}| &= kn - 6n + 6, \quad |V_{2,4}| = 4n - 4, \\ |V_{4,6}| &= 2, \quad |V_{4,8}| = n - 3. \\ C_d(T_{n,k}) &= 2(2(kn - 6n + 6) + 4(4n - 4)) + 4(6 \cdot 2 + 8(n - 3)) \\ &= 4kn + 40n - 56 = 4(nk + 10n - 14). \end{aligned}$$

Case 3: $l \geq 4$

$$\begin{aligned} |V_{2,2}| &= nk - 6n + 6, \quad |V_{2,4}| = 4n - 4, \quad |V_{4,4}| = n - 1. \\ C_d(T_{n,k}) &= 2(2(nk - 6n + 6) + 4 \cdot (4n - 4)) + 4(4(n - 1)) \\ &= 4nk + 24n - 24 = 4(nk + 6n - 6). \end{aligned}$$

□

4. e-Connectivity in CPC Networks

In wireless communication systems, long-range connectivity often hinges on the furthest reachable points within a network. To model such scenarios, we introduce a new eccentricity-based distance index, the *e-connectivity index* $\xi^{De}(G)$, which quantifies the long-range communication potential of nodes in CPC networks based on their eccentric neighbors. Unlike d-connectivity, which focuses on short-range connectivity via distance-degree, the e-connectivity captures long-range potential based on the eccentricity of each node. This section introduces the e-distance, defines the corresponding e-number and e-connectivity index (originally, known as the distance eccentric connectivity index in [1]), and provides a comprehensive mechanism for their computation in CPC networks.

Definition 4.1 Let $G = (V, E)$ be a connected network and let $\mathrm{ecc}(v)$ denote the eccentricity of a node $v \in V$, i.e., the greatest distance from v to any other node in G. A node $u \in V$ is an e-neighbor of v if $d(v, u) = \mathrm{ecc}(v)$. Let $n_e(v)$ be the number of such e-neighbors. The e-connectivity index of G is defined as:

$$\xi^{De}(G) = \sum_{v\in V} n_e(v) \cdot \mathrm{ecc}(v).$$

To evaluate ξ^{De} for a chain polygonal cactus (CPC) network $T_{n,k}$, we group nodes by their eccentricities and count of e-neighbors. This partitioning allows for systematic computation of the index based on network topology.

Proposition 4.1 *Let $T_{n,k}$ be a CPC network composed of $n \geq 3$ cycles of size $k \geq 3$, joined linearly such that each consecutive pair shares a single articulation point. Then the set $V = V(T_{n,k})$ can be partitioned into eccentricity classes $V_{e,b}$ where:*

- $e = ecc(v)$, $b = n_e(v)$,
- $V_{e,b} = \{v \in V \mid ecc(v) = e,\ n_e(v) = b\}$,
- *The e-connectivity index becomes:*

$$\xi^{De}(T_{n,k}) = \sum_{V_{e,b}\subseteq V} b \cdot e \cdot |V_{e,b}|. \tag{4.1}$$

4.1. Mechanism for Evaluating $\xi^{De}(T_{n,k})$

A node v of $T_{n,k}$ is said to be central if $ecc(v) = rad(T_{n,k})$ and is non-central otherwise [2]. To compute $\xi^{De}(T_{n,k})$, we:

1. Identify central and non-central nodes based on parity of n,
2. Determine $e = \mathrm{ecc}(v)$ and $b = n_e(v)$ for each node based on its location,
3. Group nodes into partitions $V_{e,b}$,
4. Apply the formula in equation (4.1).

We now present the exact values of $\xi^{De}(T_{n,k})$ for specific CPC networks with varying cycle lengths.

Theorem 4.1 *For $n \geq 3$, the e-connectivity index of the CPC network $T_{n,3}$ is given by:*

$$\xi^{De}(T_{n,3}) = \begin{cases} 3n^2 + 4n, & \text{if } n \text{ is even,} \\ 3n^2 + 4n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Table 1: Eccentricity-based node classification for $T_{n,3}$

e	***b***	**Frequency**	**Node type**
Case 1: Even n			
$\frac{n}{2}$	4	1	Central articulation point $c_{\frac{n}{2}}$
$\frac{n}{2}+i,\ 1 \leq i \leq \frac{n}{2}$	2	4 per level	Non-central nodes
Case 2: Odd n			
$\frac{n+1}{2}$	4	1	Central node $c_{\frac{n+1}{2}}$
$\frac{n+1}{2}$	2	2	Adjacent articulation points
$\frac{n+1}{2}+i,\ 1 \leq i \leq \frac{n-1}{2}$	2	4 per level	Non-central nodes

Proof: We classify nodes of $T_{n,3}$ based on whether n is even or odd.

Substituting the entries from Table 1 into equation (4.1) and applying the formula for the sum of a finite arithmetic sequence, we compute the index for both even and odd values of n.

Case 1: n **even.**

$$\xi^{De}(T_{n,3}) = 4 \cdot \frac{n}{2} + 8 \sum_{i=1}^{\frac{n}{2}} \left(\frac{n}{2} + i \right) = 2n + 8 \left[\frac{n}{2} \left(\frac{n}{2} + 1 \right) \right] = 3n^2 + 4n.$$

Case 2: n **odd.**

$$\begin{aligned} \xi^{De}(T_{n,3}) &= 4 \cdot \frac{n+1}{2} + 2 \cdot 2 \cdot \frac{n+1}{2} + 8 \sum_{i=1}^{\frac{n-1}{2}} \left(\frac{n+1}{2} + i \right) \\ &= 4n + 4 + 8 \left(\frac{n-1}{2} \cdot \left(\frac{n+3}{2} \right) \right) = 3n^2 + 4n + 1. \end{aligned}$$

□

Theorem 4.2 *Let $T_{n,4}$ be a 1-type CPC network for $n \geq 3$. Then*

$$\xi^{De}(T_{n,4}) = \frac{1}{4} \begin{cases} 9n^2 + 30n + 24, & \text{if } n \text{ is even,} \\ 9n^2 + 24n + 7, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Based on the eccentricities and e-neighbor counts of the central and non-central nodes, the set $V(T_{n,4})$ is partitioned as shown in Table 2.

Table 2: Node partitioning of the 1-type CPC $T_{n,4}$

e	***b***	**Frequency**	**Node type**
Even n			
$\frac{n}{2}+1$	2	1	Central
$\frac{n}{2}+2$	2	2	Non-central
$\frac{n}{2}+2$	1	2	Non-central
$\frac{n}{2}+1+i,\ 2 \leq i \leq \frac{n}{2}$	1	6	Non-central
$n+2$	1	2	Non-central
Odd n			
$\frac{n+3}{2}$	1	2	Central
$\frac{n+3}{2}+i,\ 1 \leq i \leq \frac{n+3}{2}-2$	1	6	Non-central
$n+2$	1	2	Non-central

Substituting the entries from Table 2 into equation (4.1) and applying the formula for the sum of a finite arithmetic sequence, we compute the index for both even and odd values of n.

Case 1: Even $n \geq 4$

$$\begin{aligned}\xi^{De}(T_{n,4}) &= 2\left(\frac{n}{2}+1\right)+2\cdot 2\left(\frac{n}{2}+2\right)+2\left(\frac{n}{2}+2\right)+2(n+2)+6\sum_{i=2}^{\frac{n}{2}}\left(\frac{n}{2}+1+i\right)\\ &= n+2+2n+8+n+4+2n+4+6\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+1\right)+6\sum_{i=2}^{\frac{n}{2}} i\\ &= \frac{1}{4}(9n^2+30n+24).\end{aligned}$$

Case 2: Odd $n \geq 3$

$$\begin{aligned}\xi^{De}(T_{n,4}) &= 2\left(\frac{n+3}{2}\right)+2(n+2)+6\sum_{i=1}^{\frac{n+3}{2}-2}\left(\frac{n+3}{2}+i\right)\\ &= n+3+2n+4+6\left(\frac{n-1}{2}\right)\left(\frac{n+3}{2}\right)+6\sum_{i=1}^{\frac{n+3}{2}-2} i\\ &= \frac{1}{4}(9n^2+24n+7).\end{aligned}$$

□

Theorem 4.3 *Let $T_{n,4}$ be a 2-type CPC network for $n \geq 3$. Then*

$$\xi^{De}(T_{n,4}) = \frac{1}{2}\begin{cases} 9n^2+6n, & \textit{if } n \textit{ is even,}\\ 9n^2+8n-1, & \textit{if } n \textit{ is odd.}\end{cases}$$

Proof: Based on the eccentricities and e-number values of the central and non-central nodes, the set $V(T_{n,4})$ can be partitioned as shown in Table 3.

Table 3: Node partitioning of the 2-type CPC $T_{n,4}$

$\boldsymbol{e}$	$\boldsymbol{b}$	**Frequency**	**Node type**
Even n			
n	2	1	Central
$n+2i-1,\ 1 \leq i \leq \frac{n}{2}$	1	4	Non-central
$n+2i,\ 1 \leq i \leq \frac{n}{2}$	1	2	Non-central
Odd n			
n	2	2	Central
$n+2i-1,\ 1 \leq i \leq \frac{n+1}{2}$	1	2	Non-central
$n+2i,\ 1 \leq i \leq \frac{n-1}{2}$	1	4	Non-central

Using Table 3 in formula (4.1) and applying the formula for summing finite arithmetic sequences, we compute the index values for both even and odd n:

Case 1: Even $n \geq 4$

$$\begin{aligned}
\xi^{De}(T_{n,4}) &= 2n + 4\sum_{i=1}^{\frac{n}{2}}(n+2i-1) + 2\sum_{i=1}^{\frac{n}{2}}(n+2i) \\
&= 2n + 4\cdot\frac{n}{2}\cdot n + 4\sum_{i=1}^{\frac{n}{2}}(2i-1) + 2\cdot\frac{n}{2}\cdot n + 2\sum_{i=1}^{\frac{n}{2}}2i \\
&= 2n + 2n^2 + 4\left(\frac{n}{2}\right)^2 + n^2 + 2\cdot\frac{n}{2}\left(\frac{n}{2}+1\right) \\
&= \frac{1}{2}(9n^2+6n).
\end{aligned}$$

Case 2: Odd $n \geq 3$

$$\begin{aligned}
\xi^{De}(T_{n,4}) &= 2\cdot 2n + 2\sum_{i=1}^{\frac{n+1}{2}}(n+2i-1) + 4\sum_{i=1}^{\frac{n-1}{2}}(n+2i) \\
&= 4n + 2\cdot\frac{n+1}{2}\cdot n + 2\sum_{i=1}^{\frac{n+1}{2}}(2i-1) + 4\cdot\frac{n-1}{2}\cdot n + 4\sum_{i=1}^{\frac{n-1}{2}}2i \\
&= 4n + (n+1)n + (n+1)^2/2 + 2(n-1)n + 4\cdot\frac{(n-1)(n+1)}{4} \\
&= \frac{1}{2}(9n^2+8n-1).
\end{aligned}$$

□

4.2. Mechanism for Subsequent Results

We begin by analyzing the center of $T_{n,k}$ for even $n \geq 4$ and any $k \geq 5$, as outlined in the following remark.

Remark 4.1 *For a fixed index j with $1 \leq j \leq n-2$, consider the geodesic path between articulation points c_j and c_{j+1} defined as:*

$$P_{j,j+1} : c_j – n(c_j) – \cdots – n(c_{j+1}) – c_{j+1},$$

of length l, where $1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor$, and $n(c_j)$ denotes a neighbor of c_j.

For any value of l, the constructed CPC network is symmetric with respect to the articulation point $c_{\frac{n}{2}}$. In this setting, the eccentricity of the central node is

$$ecc(c_{\frac{n}{2}}) = \left(\frac{n}{2}-1\right)l + \left\lfloor \frac{k}{2} \right\rfloor.$$

The eccentricities of each non-central node take the form

$$\left(\frac{n}{2}-1\right)l + \left\lfloor \frac{k}{2} \right\rfloor + i, \quad 1 \leq i \leq \left(\frac{n}{2}-1\right)l + \left\lfloor \frac{k}{2} \right\rfloor.$$

Hence, the radius and diameter of $T_{n,k}$ are given by:

$$rad(T_{n,k}) = \left(\frac{n}{2}-1\right)l + \left\lfloor \frac{k}{2} \right\rfloor, \quad diam(T_{n,k}) = 2\cdot rad(T_{n,k}).$$

Therefore, the center of $T_{n,k}$ consists solely of the articulation point $c_{\frac{n}{2}}$, and is thus a trivial center.

Now, consider an l-type CPC network $T_{n,k}$ with even $n \geq 4$, odd $k \geq 5$, and $1 \leq l \leq \frac{k-1}{2}$. Let the eccentricity of the central node $c_{\frac{n}{2}}$ be:

$$\mathrm{ecc}(c_{\frac{n}{2}}) = \left(\frac{n}{2} - 1\right) l + \left\lfloor \frac{k}{2} \right\rfloor = t.$$

Then,

$$\mathrm{rad}(T_{n,k}) = t, \quad \mathrm{diam}(T_{n,k}) = 2t,$$

and the eccentricities of the non-central nodes are of the form $t + i$ for $1 \leq i \leq t$.

The following observations will be used repeatedly in the results that follow:

- Each polygon in $T_{n,k}$ contributes twice to the total count of each non-central node's eccentricity.
- An eccentricity cannot occur $2n$-times in $T_{n,k}$ whenever $k \leq n - 1$.
- An eccentricity occurs exactly $2n$-times in an l-type CPC if and only if $1 \leq l \leq \left\lfloor \frac{k-1}{n-2} \right\rfloor$ and $l \neq \frac{k-1}{n-2}$.
- No eccentricity can occur more than $2n$-times in any CPC network.
- If an eccentricity occurs $2n$-times in $T_{n,k}$, then the number of such eccentricities is:

$$\frac{1}{2}\left(k - (l(n-2) + 1)\right).$$

- The eccentricities that occur $2n$-times are of the form:

$$t + i, \quad \text{for } \frac{l}{2}(n-2) + 1 \leq i \leq t - \frac{l}{2}(n-2).$$

- When an eccentricity occurs $2n$-times in $T_{n,k}$, other eccentricities occur $2m$-times for even values of m, where $2 \leq m \leq n - 2$.
- Out of the total t eccentricities for non-central nodes, there are $2l$ eccentricities of two types:

$$t + i \quad \text{and} \quad 2t - l + i, \quad \text{for } 1 \leq i \leq l,$$

  each occurring $2 \cdot 2$-times in $T_{n,k}$.
- The remaining $t - 2l$ eccentricities, $t + i$ for $l + 1 \leq i \leq t - l$, occur according to the following patterns:
  - If $1 \leq l \leq \left\lfloor \frac{k-1}{n-2} \right\rfloor$, then:

$$l(n-4) \text{ eccentricities occur } 2m\text{-times},$$

    for m even. These are:

$$t + i, \quad l + 1 \leq i \leq \frac{l}{2}(n-2), \quad \text{and} \quad t - \frac{l}{2}(n-2) + 1 \leq i \leq t - l.$$

  - If $\left\lfloor \frac{k+3}{4} \right\rfloor \leq l < \frac{k-1}{2}$, then the eccentricities occur in two ways:
    - $2(2)$-times for: $t + ql + i$, $1 \leq i \leq \frac{k-1}{2} - l$, $1 \leq q \leq \frac{n}{2} - 1$,
    - $2(4)$-times for: $t + ql + i$, $\frac{k-1}{2} - l + 1 \leq i \leq l$, $1 \leq q \leq \frac{n}{2} - 2$.
  - If $l = \frac{k-1}{4}$, then each eccentricity occurs $2(4)$-times.
  - If $l = \frac{k-1}{2}$, then each eccentricity occurs $2(2)$-times.

Table 4: Node partitioning of an l-type CPC $T_{n,k}$ for $1 \leq l \leq \left\lfloor \frac{k-1}{n-2} \right\rfloor$ and $l \neq \frac{k-1}{n-2}$

S.n	***e***	***b***	**Frequency**	**Node type**
(a)	t	4	1	Central
(b)	$t+i;\ 1 \leq i \leq l$	2	2(2)	Non-central
(c)	$2t-l+i;\ 1 \leq i \leq l$	2	2(2)	Non-central
(d)	$t+i;\ \frac{l(n-2)}{2}+1 \leq i \leq t-\frac{l(n-2)}{2}$	2	$2n$	Non-central
Remaining partitions with stepwise increasing multiplicities				
(e)	$t+ql+i;\ 1 \leq i \leq l$	2	$2(2q+2)$	Non-central for $1 \leq q \leq \frac{n}{2}-2$
Symmetric decreasing counterparts				
(f)	$2t-ql-i+1;\ 1 \leq i \leq l$	2	$2(2q+2)$	Non-central for $1 \leq q \leq \frac{n}{2}-2$

Theorem 4.4 *Let $T_{n,k}$ be an l-type CPC network for even $n \geq 4$ and odd $k \geq 5$, where $1 \leq l \leq \left\lfloor \frac{k-1}{n-2} \right\rfloor$ and $l \neq \frac{k-1}{n-2}$. Then, the e-connectivity index is given by:*

$$\xi^{De}(T_{n,k}) = 2nt(3t+1) + nl(n-2)\left(2l(n-1) - (3t+1)\right).$$

Proof: Based on the eccentricities and e-neighbor counts of the central and non-central nodes, the set $V(T_{n,k})$ is partitioned as shown in Table 4.

Substituting the eccentricities and counts from Table 4 into equation (4.1), we obtain:

$$\begin{aligned}\xi^{De}(T_{n,k}) = {} & 4t + 4(2)\sum_{i=1}^{l}(t+i) + 4(2)\sum_{i=1}^{l}(2t-l+i) \\ & + 4n \sum_{i=\frac{l(n-2)}{2}+1}^{t-\frac{l(n-2)}{2}} (t+i) + 8\sum_{q=1}^{\frac{n}{2}-2}\sum_{i=1}^{l}(q+1)(3t+1).\end{aligned} \tag{4.2}$$

We simplify each component of the expression:

(i) First and second sums:

$$4(2)\sum_{i=1}^{l}(t+i) = 8l(t) + 8\sum_{i=1}^{l} i = 4l(2t+l+1), \tag{4.3}$$

$$4(2)\sum_{i=1}^{l}(2t-l+i) = 8l(2t-l) + 8\sum_{i=1}^{l} i = 4l(4t-l+1). \tag{4.4}$$

(iii) Middle-band eccentricities:

$$4n \sum_{i=\frac{l(n-2)}{2}+1}^{t-\frac{l(n-2)}{2}} (t+i) = 2n\left(3t^2 - l(n-2)(3t+1) + l^2(n-1)(n-2) + t\right). \tag{4.5}$$

(iv) Outer symmetric layers:

$$8\sum_{q=1}^{\frac{n}{2}-2}\sum_{i=1}^{l}(q+1)(3t+1) = l(3t+1)(n+2)(n-4). \tag{4.6}$$

Substituting equations (4.3)-–(4.6) into equation (4.2) yields the desired closed-form expression:

$$\xi^{De}(T_{n,k}) = 2nt(3t+1) + nl(n-2)\left(2l(n-1) - (3t+1)\right).$$

□

Theorem 4.5 *Let $T_{n,k}$ be an l-type CPC network for even $n \geq 4$ and odd $k \geq 5$, where $l = \frac{k-1}{n-2}$. Then the e-connectivity index is given by:*

$$\xi^{De}(T_{n,k}) = 4t + nl(3t+1)(n-2).$$

Proof: In this case, the node partitioning of $T_{n,k}$ is the same as in Table 4, except that row (d) is excluded. Therefore, the index is computed by summing the expressions from equations (4.3), (4.4), and (4.6), and adding the central node contribution $4t$. This yields the desired result. □

Theorem 4.6 *Let $T_{n,k}$ be an l-type CPC network for even $n \geq 4$ and odd $k \geq 7$, where $\left\lfloor \frac{k+3}{4} \right\rfloor \leq l < \frac{k-1}{2}$. Then the e-connectivity index is given by:*

$$\xi^{De}(T_{n,k}) = 4(t+l(6t+2)) + (n-2)(k-2l-1)(4t+l(n-2)+k+1) + \frac{1}{2}(n-4)(4l-k+1)(4t+l(n+2)-k+3).$$

Proof: The set $V(T_{n,k})$ is partitioned according to the eccentricities and e-numbers as detailed in Table 5.

Table 5: Node partitioning for an l-type CPC $T_{n,k}$ with $\left\lfloor \frac{k+3}{4} \right\rfloor \leq l < \frac{k-1}{2}$

e	b	**Frequency**	**Node type**
t	4	1	Central
$t+i;\ 1 \leq i \leq l$	2	2(2)	Non-central
$2t-l+i;\ 1 \leq i \leq l$	2	2(2)	Non-central
$t+ql+i;\ 1 \leq i \leq \frac{k-1}{2} - l,\ 1 \leq q \leq \frac{n}{2} - 1$	2	2(4)	Non-central
$t+ql+i;\ \frac{k-1}{2} - l + 1 \leq i \leq l,\ 1 \leq q \leq \frac{n}{2} - 2$	2	2(2)	Non-central

Substituting the contributions from each group into equation (4.1), we get:

$$\begin{aligned}\xi^{De}(T_{n,k}) &= 4t + 4(2)\sum_{i=1}^{l}(t+i) + 4(2)\sum_{i=1}^{l}(2t-l+i) \\ &\quad + 16\sum_{q=1}^{\frac{n}{2}-1}\sum_{i=1}^{\frac{k-1}{2}-l}(t+ql+i) + 8\sum_{q=1}^{\frac{n}{2}-2}\sum_{i=\frac{k-1}{2}-l+1}^{l}(t+ql+i). \qquad (4.7)\end{aligned}$$

We simplify the key terms below:
(i) Central and symmetric bands:

$$4t + 4(2)\sum_{i=1}^{l}(t+i) + 4(2)\sum_{i=1}^{l}(2t-l+i) = 4(t+l(6t+2)).$$

(ii) High-frequency eccentricities:

$$16\sum_{q=1}^{\frac{n}{2}-1}\sum_{i=1}^{\frac{k-1}{2}-l}(t+ql+i) = (n-2)(k-2l-1)(4t+l(n-2)+k+1). \qquad (4.8)$$

(iii) Low-frequency eccentricities:

$$8\sum_{q=1}^{\frac{n}{2}-2}\sum_{i=\frac{k-1}{2}-l+1}^{l}(t+ql+i)=\frac{1}{2}(n-4)(4l-k+1)(4t+l(n+2)-k+3). \tag{4.9}$$

Combining the above expressions into equation (4.7) yields the required result. □

Theorem 4.7 *Let $T_{n,k}$ be an l-type CPC network for even $n \geq 4$ and odd $k \geq 5$, where $l=\frac{k-1}{4}$. Then the e-connectivity index is given by:*

$$\xi^{De}(T_{n,k})=4\left(3t(2t+1)-2l(3t+1)\right).$$

Proof: Based on the eccentricities and e-neighbor counts of the central and non-central nodes, the set $V(T_{n,k})$ is partitioned as shown in Table 6.

Table 6: Node partitioning for an l-type CPC $T_{n,k}$ with $l=\frac{k-1}{4}$

e	b	**Frequency**	**Node type**
t	4	1	Central
$t+i;\ 1\leq i\leq l$	2	4	Non-central
$2t-l+i;\ 1\leq i\leq l$	2	4	Non-central
$t+i;\ l+1\leq i\leq t-l$	2	8	Non-central

Using Table 6 in the formula (4.1) and applying the formula for arithmetic series, we obtain:

$$\xi^{De}(T_{n,k})=4t+4(2)\sum_{i=1}^{l}(t+i)+4(2)\sum_{i=1}^{l}(2t-l+i)+8(2)\sum_{i=l+1}^{t-l}(t+i). \tag{4.10}$$

Now, we compute the last term:

$$\begin{aligned}8(2)\sum_{i=l+1}^{t-l}(t+i)&=16(2t-l)t+16\sum_{i=l+1}^{t-l}i\\&=16t(2t-l)+16\left(\frac{t-2l}{2}\left(2(l+1)+(t-2l-1)\right)\right)\\&=8\left(3t^2-6lt-2l\right).\end{aligned} \tag{4.11}$$

Substituting expressions (4.3), (4.4), and (4.11) into equation (4.10) gives:

$$\xi^{De}(T_{n,k})=4\left(3t(2t+1)-2l(3t+1)\right).$$

□

Theorem 4.8 *Let $T_{n,k}$ be an l-type CPC network for even $n \geq 4$ and odd $k \geq 5$, where $l=\frac{k-1}{2}$. Then the e-connectivity index is given by:*

$$\xi^{De}(T_{n,k})=4t(3t+2).$$

Proof: The eccentricities and e-neighbor counts of all nodes are classified as shown in Table 7.

Using Table 7 in the formula (4.1), we compute:

$$\begin{aligned}\xi^{De}(T_{n,k})&=4t+4(2)\sum_{i=1}^{t}(t+i)\\&=4t+8t^2+8\left(1+2+\ldots+t\right)\\&=4t+8t^2+4t(t+1)=4t(3t+2).\end{aligned}$$

□

Table 7: Node partitioning for an l-type CPC $T_{n,k}$ with $l = \frac{k-1}{2}$

e	b	**Frequency**	**Node type**
t	4	1	Central
$t+i;\ 1 \leq i \leq t$	2	4	Non-central

5. Applications to Wireless Communication Networks

Effective communication in wireless networks critically depends on understanding the geographical coverage area and strategically placing access points. In wireless communication systems, maintaining robust and efficient connectivity across both short-range and long-range distances is a critical challenge. A well-structured network minimizes dead zones and ensures consistent signal strength across both short and long distances. Chain Polygonal Cactus (CPC) networks, due to their intrinsic topological regularity and modular symmetry, provide a compelling structure for modeling such systems. Due to the structural property that no two cycles in a chain polygonal cactus (CPC) share more than one node, CPC-based networks offer a naturally modular and symmetrical framework for designing fault-tolerant, efficient communication architectures. High-degree nodes in a CPC network, often articulation points, act as vital communication hubs or routing locations. These nodes enable robust short- and long-range interactions across various components of the network, enhancing resilience and load balancing.

Wireless networks typically operate over two scales:

- **Short-range communication**, such as Bluetooth and Wi-Fi, is ideal for local area networks and personal devices.
- **Long-range communication**, such as cellular or satellite, is essential for broader coverage and mobility.

In this context, two newly analyzed indices d-connectivity and e-connectivity offer theoretical foundations for optimizing node placement and communication flow within these networks:

- The d-number of a node measures the number of its short-range reachable neighbors.
- The e-number, based on eccentricity, captures the longest reliably reachable neighbors for long-range connectivity.
- The d-connectivity index, introduced in this paper, provides insight into short-range, high-frequency interactions. It identifies central regions within the network that are highly reachable and thus ideal for deploying access points or repeaters to maximize local coverage.
- The e-connectivity index captures the long-range communication potential of nodes by quantifying their extremal distances. This metric is particularly useful in determining the positions of backbone routers or strategic relay nodes in large-area sensor fields or low-power IoT environments.

Together, these indices support a hybrid communication model: dense, short-range clustering of signals around low-eccentricity nodes, and long-range, low-latency relays through high-eccentricity hubs. The analysis of e-number multiplicities further reveals how evenly traffic can be distributed among network paths, enhancing reliability and load-balancing in dynamic conditions.

Although our study is theoretical, the implications for the design and evaluation of physical and virtual communication infrastructures are substantial. These indices could inform the development of algorithms for energy-efficient routing, dynamic topology control, and resilience planning in real-world wireless networks.

The following illustrations visualize these concepts in CPC networks:

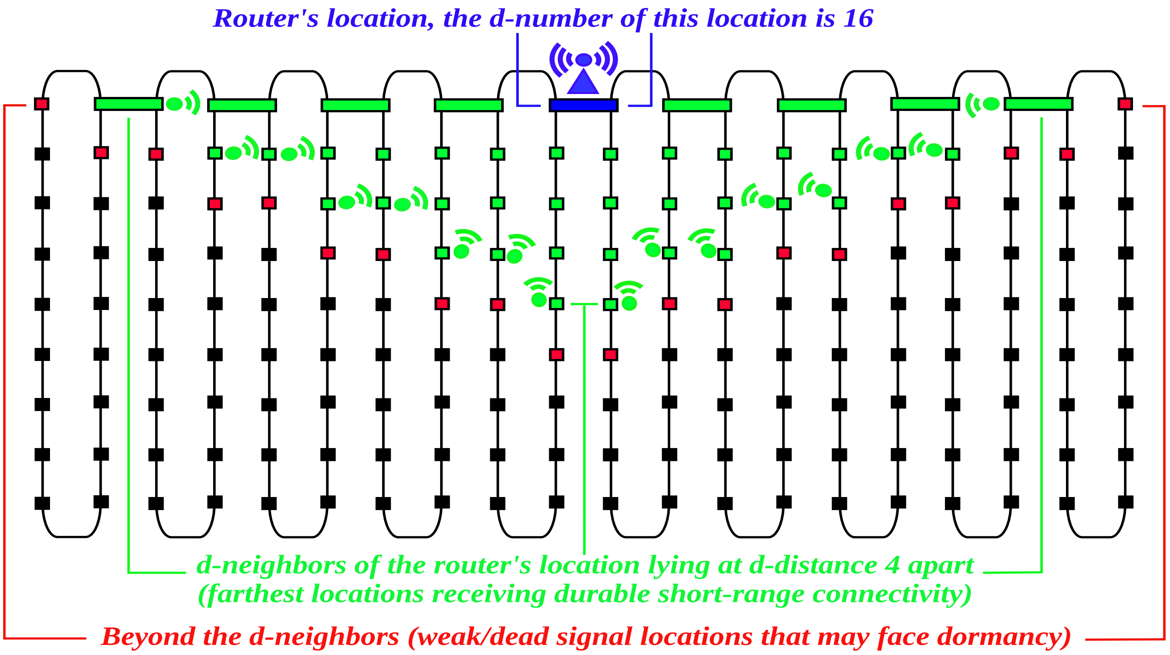

Figure 2: Short-range connectivity via d-distance in a 1-type CPC $T_{10,18}$. The router is placed at an articulation point. Farthest d-neighbors are at distance 4, receiving strong short-range connectivity. Nodes beyond this range may experience weak or dormant signal zones.

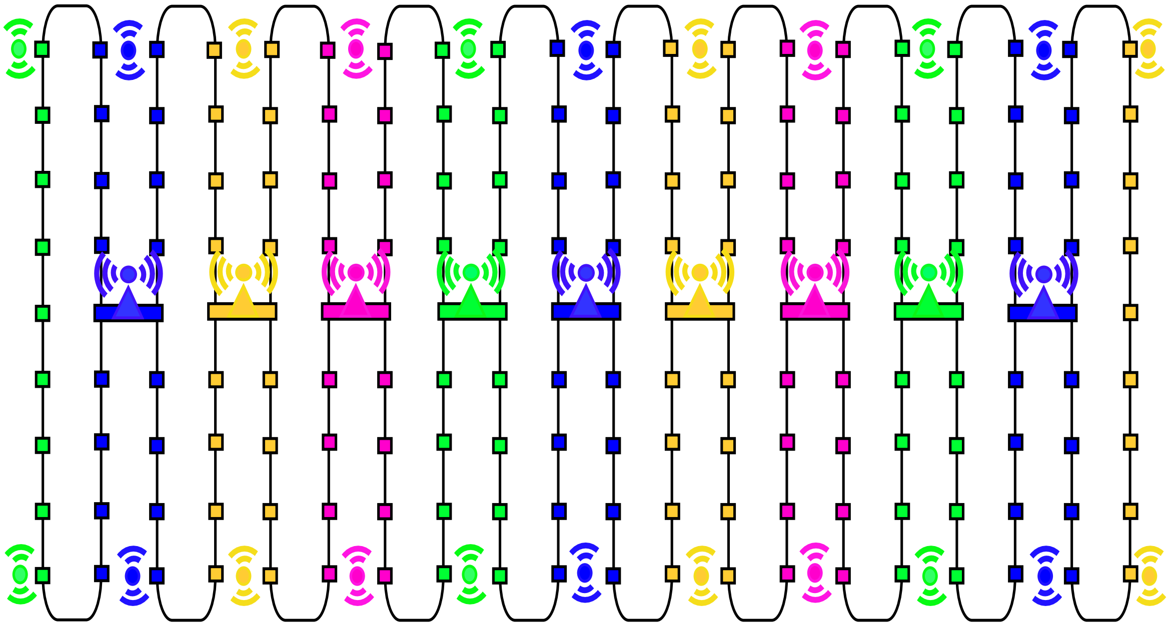

Figure 3: An optimized router deployment strategy using only articulation points in a CPC-based network. These locations yield maximal short-range connectivity through d-distance.

Similarly, long-range signal propagation is assessed using the e-number, capturing the extent of reliable connectivity from a centrally located antenna or tower.

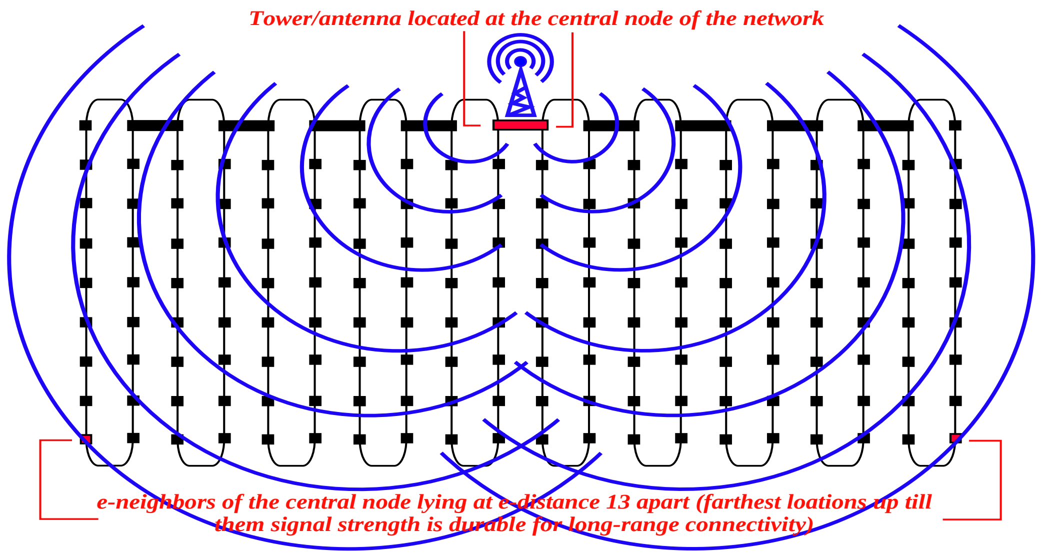

Figure 4: Long-range connectivity in a 1-type CPC $T_{10,18}$ with a tower placed at the central node. The e-neighbors extend up to e-distance 13, representing the boundary of strong signal range.

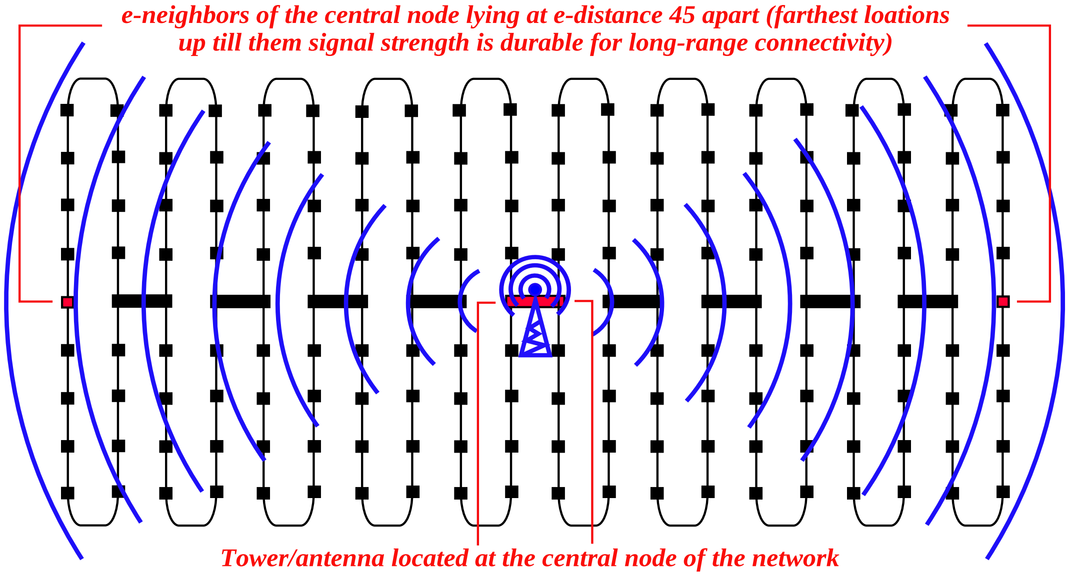

Figure 5: Long-range signal coverage in a 9-type CPC $T_{10,18}$, with the antenna placed at the central node. e-neighbors extend to e-distance 45, representing the limit of durable communication range.

These two metrics captured through the d-connectivity and e-connectivity indices enable:

- Strategic placement of routers or towers for optimal signal reach.
- Differentiated planning for dense local clusters versus sparse wide-area systems.
- Theoretical foundation for energy-efficient routing, robust network design, and dynamic topological control.

Thus, CPC-based models and the proposed indices provide a rigorous mathematical framework with direct relevance to the design, evaluation, and optimization of wireless communication networks.

6. Conclusion and Future Work

In this paper, we introduced and analyzed two graph-theoretic connectivity measures for Chain Polygonal Cactus (CPC) networks: the d-connectivity index, based on vertex degree d-number, and the e-connectivity index, based on vertex eccentricity and e-number. By analyzing the e-connectivity index, and contrasting it with the d-connectivity measure for CPC networks, we provided a rich characterization of node roles in both short-range and long-range CPC-based communication paradigms.

We derived closed-form expressions for the d-connectivity index for all $T_{n,k}$ with $n, k \geq 3$, and established exact formulae for the e-connectivity index in a range of cases, particularly when n is even and k is odd. These analytical results offer valuable insights into the structural behavior and applicability of CPCs in both theoretical and applied settings. We have systematically investigated the distance-based connectivity properties of Chain Polygonal Cactus (CPC) networks. Explicit formulas for $C_d(T_{n,k})$ and $\xi^{De}(T_{n,k})$ were established for various structural types of CPCs, classified by parameters n, k, and type l. Our approach combined precise eccentricity partitioning, arithmetic analysis of e-number occurrences, and combinatorial modeling of node multiplicities. The exact d- and e-connectivity indices were evaluated for a wide range of CPC types. Specifically:

- The d-connectivity index was successfully determined for all types of CPCs $T_{n,k}$ with $n, k \geq 3$.
- The e-connectivity index, while more complex due to increasing eccentricity values, was computed for various structured CPC classes, including:
  1. $n \geq 3, k = 3, l = 1$,
  2. $n \geq 3, k = 4, l = 1, 2$,
  3. $n \geq 4$ (even), $k \geq 5$ (odd), for $l \in \left\{1, 2, \ldots, \left\lfloor \frac{k-1}{n-2} \right\rfloor\right\} \cup \left\{\frac{k-1}{4}\right\} \cup \left\{\left[\frac{k+3}{4}\right], \left[\frac{k+3}{4}\right] + 1, \ldots, \frac{k-1}{2}\right\}$.

Our investigation revealed that the d-connectivity index effectively characterizes the strength and spread of short-range connectivity in CPCs, while the e-connectivity index captures the long-range communication potential within such networks. The central idea stems from analyzing how the structural features of CPCs, particularly articulation points and eccentricities, can influence network resilience and efficiency, especially in settings inspired by wireless communication systems. We demonstrated that the d-number of a node effectively captures short-range connectivity, which is particularly suitable for local Wi-Fi or Bluetooth-based communication models. Conversely, the e-number quantifies long-range connectivity performance, applicable to cellular or satellite-based communication layers. These concepts were then applied to theoretically model wireless networks with routers or towers placed at articulation points and central nodes in CPCs, providing efficient point-to-multipoint communication. The effectiveness of this strategy is illustrated in Figures 3, 4, and 5.

Moreover, the theoretical connectivity indices developed here provide a solid mathematical foundation to guide the simulation, optimization, and performance assessment of wireless communication networks in future research.

Future Work:

Due to the increasing structural complexity when handling eccentricity distributions, we leave open the determination of the e-connectivity index for the following unexplored configurations:

- $n \geq 4$ (even), $k \geq 5$ (odd), $\left\lfloor \frac{k-1}{n-2} \right\rfloor < l < \left\lfloor \frac{k+3}{4} \right\rfloor$,

- $n \geq 4$ (even), $k \geq 6$ (even), $1 \leq l \leq \frac{k}{2}$,
- $n \geq 3$ (odd), any $k \geq 5$, $1 \leq l \leq \lfloor \frac{k}{2} \rfloor$.

These cases offer fertile ground for further investigation and could serve as a framework for algorithmic implementation, optimization under constraints, or generalizations to weighted and dynamic CPC models.

The theoretical results presented here also provide a platform for practical application in wireless network planning, robust topology design, and fault-tolerant infrastructure development. These results not only advance the theoretical framework of distance-based indices but also open pathways and new directions for real-world engineering applications in future by:

- Extending these indices to broader classes of cactus-like graphs, including those with variable-sized cycles and non-linear branching patterns.
- Integrating weighted versions of d- and e-connectivity for scenarios involving traffic capacity, energy constraints, or probabilistic routing.
- Developing simulation frameworks to test and validate the application of these indices in dynamic wireless environments.
- Exploring inverse problems: deducing structural parameters of a CPC network from known connectivity indices.

The blend of topological rigor and applied motivation makes this work a valuable step toward bridging graph theory with modern network design challenges. We hope this work encourages further study on distance-based measures in cacti and contributes to the ongoing development of graph-theoretic models in communication and network science.

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References

1. Alkqesmah, A., Saleh, A., Rangarajan, R., Yurtas, A., Cangul, I.N., *Distance eccentric connectivity index of graphs.* Kyungpook Math. J. 61, 61-74, (2021).
2. Chartrand, G., Zhang, P., *Itroduction to graph theory.* Tata McGraw-Hill Companies Inc. New York (2006).
3. Došlić, T., Litz, M. S., *Matchings and independent sets in polyphenylene chain.* MATCH Commun. Math. Comput. Chem. 67, 314-329, (2011).
4. Došlić, T., Maloy, F., *Chain hexagonal cacti: Matching and independent sets.* Dis. Math. 310, 1676-1690, (2010).
5. Farrell, E. J., *Matchings in pentagonal cacti.* Int. J. Math. Sci. 11, 109-126, (2000).
6. Farrell, E. J., *Matchings in triangular cacti.* Int. J. Math. Sci. 11, 85-98, (2000).
7. Farrell, E. J., *Matchings in rectangular cacti.* Int. J. Math. Sci. 9, 163-183, (1998).
8. Farrell, E. J., *Matchings in hexagonal cacti.* Int. J. Math. Sci. 10, 321-338, (1987).
9. Jiachang, Y., Muhuo, L., Kinkar, C. D., *Extremal polygonal cacti for bond incident degree indices.* Discret. Appl. Math. 257, 289-298, (2019).
10. Jiachang, Y., Yuedan, Y., *A note on the zeroth-order genral Randić index of polygonal cati.* J. Discret. Appl. Math. 1, 1-7, (2018).
11. Klamsakul, N., Thengarnanchai, P., Suebtangjai, M., Kaewperm, P., Songsuwan, N., Kaemawichanurat, P., *Maximal independent sets in polygonal cacti.* arXive:2020:11294v2 [Math.Co] 20 May (2022).
12. Song, X., Li, J., He, W., *On Zagreb eccentricity indices of cacti.* Appl. Math. Comput. 383, 125361, (2020).
13. Zeng, M., Xiao, Q., Tang, Z., Deng, H., *Extremal polygonal cacti for Wiener index and Kirchhoff index.* Iranian J. Math. Chem. 11(3), 201-211, (2020).
14. Zhang, X., Salman, M., Rani, A., Tanveer, R., Ali, U., Shao, Z., *Metric identification of vertices in polygonal cacti.* CMES. 136, 883-899, (2023).

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