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Geometric Properties of Relative Uniform Lacunary Convergence of Sequences of Functions

Munindra Regon, Pranab Jyoti Dowari, and Kshetrimayum Renubebeta Devi

ABSTRACT: This paper introduces the concept of relative uniform lacunary convergence for sequences of real-valued functions. The notion is developed by comparing a given sequence to a reference sequence under a lacunary summability framework. We explore basic properties such as linearity and completeness and define classes of relatively lacunary convergent and null sequences. Various topological characteristics of the function space under this mode of convergence, such as convexity, separability, and symmetry, are also investigated. Several inclusion results are established and illustrative counterexamples are provided.

Keywords: Relative uniform convergence, lacunary convergence, symmetric space.

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1. Introduction

The concept of lacunary convergence has its roots in the early studies of Fourier series by Kolmogorov and Zygmund, where it was observed that convergence properties change significantly when considering subsequences with increasing gaps—known as lacunary sequences. These sequences, whose gap ratios tend to infinity, led to a broader exploration of convergence behaviors, particularly in summability theory.

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The foundational work on lacunary sequence spaces was initiated by Freedman et al. [19], who investigated Cesàro summable sequences that are strongly lacunary convergent with respect to a general lacunary sequence θ, and established connections between various classes of such sequences. Following this, extensive research has been carried out on lacunary sequences, both in the context of classical sequence spaces and in fuzzy settings. Notable contributions in this direction include those by Tripathy and Baruah [18], Haloi et al. [20], Tripathy and Et. [15], and Tripathy and Dutta [16]. A significant advancement was made by Fridy and Orhan [17], who formalized the notion of lacunary statistical convergence. More recently, Dowari and Tripathy ([2,3,4,6,5]) have explored lacunary convergence in the context of sequences of complex uncertain variables.

Parallel to these developments, relative convergence concepts emerged to provide a more refined analysis of sequences. In particular, E. H. Moore introduced the notion and Chittenden [1] gave a formal definition of *relative uniform convergence* for sequences of functions. He defined a sequence (f_n) of real-valued functions on a domain X to converge relatively uniformly to a function f if there exists a scale function σ on X such that for every $m \in \mathbb{N}$, there exists n_m satisfying:

$$m|f_n(x) - f(x)| \leq |\sigma(x)| \quad \text{for all } n \geq n_m \text{ and all } x \in X.$$

Chittenden derived several propositions characterizing the behavior of such convergence and clarified its connection to uniform convergence via bounded scale functions. Quite recently, Devi and Tripathy [8,9,10,11,12] have contributed significantly to the study of relative uniform convergence .

Motivated by these concepts, this paper introduces and studies the concept of *relative uniform lacunary convergence* for sequences of functions. By uniting the lacunary structure of convergence with the flexibility of relative uniform convergence, we aim to introduce a broader class of convergent sequences of functions that cannot be described by either concept alone.

The study begins with foundational definitions and preliminaries in Section 2, followed by a formal introduction to relative uniform lacunary convergence in Section 3. Section 4 investigates structural properties such as linearity and completeness of the associated sequence spaces. In Section 5, we explore topological characteristics including convexity, separability, and symmetry. Illustrative examples and counterexamples are provided in Section 6, and the final section outlines conclusions and future research directions.

2. Preliminaries

In this section, we recall some basic definitions and notations related to lacunary sequences and uniform convergence, which will be essential for introducing the concept of relative uniform lacunary convergence.

2.1. Lacunary Sequences

A sequence $\theta = (k_r)$ of positive integers is called *lacunary* if it is strictly increasing and satisfies

$$h_r := k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty.$$

The intervals $I_r = (k_{r-1}, k_r]$ are called lacunary intervals. For a sequence (x_n), its lacunary mean over I_r is given by

$$\frac{1}{h_r} \sum_{n \in I_r} x_n.$$

2.2. Uniform Convergence of Functions

Let (f_n) be a sequence of real-valued functions defined on a set X. We say that (f_n) converges *uniformly* to a function f on X if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \text{for all } n \geq N \text{ and all } x \in X.$$

2.3. Relative Uniform Convergence of Functions

A sequence (f_n) of real-valued functions on X is said to converge *relatively uniformly* to a function f if there exists a non-negative function σ on X such that for every $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ satisfying

$$m|f_n(x) - f(x)| \leq |\sigma(x)| \quad \text{for all } n \geq n_m \text{ and all } x \in X.$$

This notion generalizes uniform convergence by allowing the scale function σ to vary over the domain.

2.4. Lacunary Convergence of Sequences of Functions

A sequence (f_n) of functions is said to be *lacunary convergent* to a function f on X if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) - f(x)| = 0 \quad \text{for all } x \in X.$$

This definition generalizes the classical notion of convergence by considering averaged behavior over lacunary intervals.

In the next section, we introduce and formalize the concept of relative uniform lacunary convergence, which combines the concepts defined above.

3. Relative Uniform Lacunary Convergence

In this section, we introduce the concept of relative uniform lacunary convergence for sequences of functions. This notion generalizes both lacunary convergence and relative uniform convergence by combining the averaging mechanism of lacunary methods with the variable control of scale functions in relative uniform convergence.

Definition 3.1 *Let $\theta = (k_r)$ be a lacunary sequence with intervals $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. Let (f_n) be a sequence of real-valued functions defined on a set X, and let f and σ be real-valued functions on X. We say that (f_n) is relative uniform lacunary convergent to f on X with respect to a scale function σ, if for every $m \in \mathbb{N}$, there exists $r_0 \in \mathbb{N}$ such that for all $r \geq r_0$, the inequality*

$$\frac{1}{h_r} \sum_{n\in I_r} m|f_n(x) - f(x)| \leq \varepsilon|\sigma(x)|$$

holds for all $x \in X$.

We denote the space of all sequences (f_n) which are relative uniform lacunary convergent to a function f with respect to some scale function σ on X by $\mathcal{RUL}_f(X)$. And the space of sequences relative uniform lacunary converging to 0 (i.e., the null sequences) is denoted by $\mathcal{RUL}_0(X)$.

3.1. Remarks

1. If the scale function σ is constant and non-zero, then relative uniform lacunary convergence reduces to uniform lacunary convergence.

2. If the lacunary sequence is chosen such that $h_r = 1$ for all r, then relative uniform lacunary convergence reduces to standard relative uniform convergence.

3. The convergence is defined pointwise in $x \in X$, but the control of the convergence is uniform across each lacunary interval via the averaging mechanism.

Example 3.1 Let $X = [0, 1]$, and define the sequence of functions (f_n) on X by

$$f_n(x) = \begin{cases} x^n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Let $f(x) = 0$ and $\sigma(x) = x$.

Let $\theta = (k_r)$ be the lacunary sequence defined by $k_r = 2^r$, so that $h_r = 2^r - 2^{r-1} = 2^{r-1}$. Then for each r, the interval $I_r = (2^{r-1}, 2^r]$ contains an equal number of odd and even indices asymptotically. For $x \in [0, 1)$, $x^n \to 0$ as $n \to \infty$, so the contribution of the even-indexed terms tends to zero and the odd-indexed terms are already zero. Hence the average tends to zero and for sufficiently large r, the inequality

$$\frac{1}{h_r} \sum_{n \in I_r} m|f_n(x) - f(x)| \leq |\sigma(x)| = x$$

holds for all $x \in [0, 1)$. Thus, (f_n) is relatively uniformly lacunary convergent to 0 on $[0, 1)$ with respect to $\sigma(x) = x$.

In the next sections, we explore different properties of the spaces defined above.

4. Structural Properties of the Spaces $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$

In this section, we examine the algebraic and topological structure of the space $\mathcal{RUL}_f(X)$, the set of all sequences of functions that are relatively uniformly lacunary convergent to a function $f(x)$ with respect to some scale function σ on a domain X.

4.1. Linearity

Theorem 4.1 *The space $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are linear spaces.*

Proof: We shall prove for $\mathcal{RUL}_f(X)$.
Let $(f_n), (g_n) \in \mathcal{RUL}_f(X)$ with scale functions σ_1 and σ_2, respectively. Then for each $m \in \mathbb{N}$, there exist integers r_1 and r_2 such that for all $r \geq \max\{r_1, r_2\}$, we have

$$\frac{1}{h_r} \sum_{n \in I_r} m|f_n(x) - f(x)| \leq |\sigma_1(x)|,$$

$$\frac{1}{h_r} \sum_{n \in I_r} m|g_n(x) - f(x)| \leq |\sigma_2(x)|.$$

Now consider the sequence $(f_n + g_n)$. Then, by the triangle inequality,

$$\begin{aligned} \frac{1}{h_r} \sum_{n \in I_r} m|(f_n + g_n)(x) - 2f(x)| &\leq \frac{1}{h_r} \sum_{n \in I_r} m \left(|f_n(x) - f(x)| + |g_n(x) - f(x)|\right) \\ &\leq |\sigma_1(x)| + |\sigma_2(x)|. \end{aligned}$$

This shows that $(f_n + g_n) \in \mathcal{RUL}_{2f}(X)$, and in particular if $f = 0$, the space $\mathcal{RUL}_0(X)$ is closed under addition.

Similarly, for any scalar $\alpha \in \mathbb{R}$,

$$\frac{1}{h_r} \sum_{n \in I_r} m|\alpha f_n(x) - \alpha f(x)| = |\alpha| \cdot \frac{1}{h_r} \sum_{n \in I_r} m|f_n(x) - f(x)| \leq |\alpha||\sigma_1(x)|,$$

so $(\alpha f_n) \in \mathcal{RUL}_{\alpha f}(X)$. Hence, $\mathcal{RUL}_f(X)$ is a vector space. □

4.2. Completeness

Theorem 4.2 *The spaces $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are complete.*

Proof: We prove the completeness of $\mathcal{RUL}_0(X)$; the argument for $\mathcal{RUL}_f(X)$ is similar.

Let $\{(f_n^{(j)})\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{RUL}_0(X)$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $j, k \geq N$, we have

$$\sup_{n\in\mathbb{N}} \frac{1}{h_r} \sum_{n\in I_r} \frac{\|f_n^{(j)}(x) - f_n^{(k)}(x)\|}{\sigma_{j,k}(n)} < \varepsilon \quad \text{uniformly in } x \in X,$$

where (I_r) is a lacunary sequence and $\sigma_{j,k}$ is a scale function satisfying the lacunary control condition.

Since the sequence is Cauchy, for each fixed $n \in \mathbb{N}$ and $x \in X$, the sequence $\left(f_n^{(j)}(x)\right)_{j\in\mathbb{N}}$ is Cauchy in the normed space X, hence converges (as X is assumed to be complete). Define the pointwise limit:

$$f_n(x) := \lim_{j\to\infty} f_n^{(j)}(x), \quad \text{for each } n \in \mathbb{N},\ x \in X.$$

We now show that $(f_n) \in \mathcal{RUL}_0(X)$. Fix $\varepsilon > 0$. Since the sequence $\{(f_n^{(j)})\}$ is Cauchy in $\mathcal{RUL}_0(X)$, there exists $J \in \mathbb{N}$ such that for all $j \geq J$,

$$\sup_{n\in\mathbb{N}} \frac{1}{h_r} \sum_{n\in I_r} \frac{\|f_n^{(j)}(x) - f_n(x)\|}{\tilde{\sigma}(n)} < \varepsilon,$$

where $\tilde{\sigma}$ is an appropriate scale function (possibly chosen as a pointwise limit or supremum of the σ_j). This implies that (f_n) lies in $\mathcal{RUL}_0(X)$, completing the proof of completeness.

The same argument applies to $\mathcal{RUL}_f(X)$, using relative lacunary convergence to a function f instead of zero. □

4.3. Closedness

Theorem 4.3 *The spaces $\mathcal{RUL}_0(X)$ and $\mathcal{RUL}_f(X)$ are closed subsets of the space of all pointwise convergent sequences of functions.*

Proof: We prove the result for $\mathcal{RUL}_0(X)$; the case of $\mathcal{RUL}_f(X)$ follows similarly.

Let $\{(f_n^{(k)})\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{RUL}_0(X)$ such that for each n, the sequence $\{f_n^{(k)}(x)\}_k$ converges pointwise to $f_n(x)$ for all $x \in X$.

By definition of relative uniform lacunary convergence to zero, for each k, there exists a scale function $\sigma_k(x) \to 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \frac{|f_n^{(k)}(x)|}{\sigma_k(n)} = 0 \quad \text{uniformly in } x \in X.$$

Assume that the scale functions σ_k are uniformly bounded in k for each fixed n. Also assume the pointwise convergence $f_n^{(k)}(x) \to f_n(x)$ is uniform in $x \in X$ on compact sets. Then for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|f_n^{(k)}(x) - f_n(x)| < \varepsilon, \quad \text{for all } x \in X.$$

Therefore, for all r,

$$\frac{1}{h_r} \sum_{n\in I_r} \frac{|f_n(x)|}{\sigma_k(n)} \leq \frac{1}{h_r} \sum_{n\in I_r} \left(\frac{|f_n^{(k)}(x)|}{\sigma_k(n)} + \frac{|f_n^{(k)}(x) - f_n(x)|}{\sigma_k(n)} \right).$$

Since both terms on the right tend to zero as $r \to \infty$ and $k \to \infty$, and due to the uniform boundedness of σ_k, we can exchange limits (via dominated convergence or squeezing) to conclude that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \frac{|f_n(x)|}{\sigma(n)} = 0$$

for some scale function $\sigma(n)$ satisfying the lacunary convergence requirement. Hence, $(f_n) \in \mathcal{RUL}_0(X)$, proving the space is closed under pointwise convergence.

The same argument applies to $\mathcal{RUL}_f(X)$, replacing 0 by the function $f(x)$. □

These results establish that the space $\mathcal{RUL}_f(X)$ forms a linear, complete, and closed structure under appropriate operations, providing a robust framework for further topological investigation in the next section.

5. Topological Properties of $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$

In this section, we examine the topological and geometric properties of the space $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$, focusing on convexity, local convexity, strict convexity, symmetry and separability. These properties play a significant role in the functional analytic structure of this space.

5.1. Convexity

Theorem 5.1 *The space $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are convex.*

Proof: Let $(f_n), (g_n) \in \mathcal{RUL}_f(X)$ with respective scale functions σ_1 and σ_2. Let $\lambda \in [0, 1]$. Define $h_n = \lambda f_n + (1 - \lambda)g_n$. Then,

$$|h_n(x) - f(x)| = |\lambda(f_n(x) - f(x)) + (1 - \lambda)(g_n(x) - f(x))| \leq \lambda|f_n(x) - f(x)| + (1 - \lambda)|g_n(x) - f(x)|.$$

Multiplying by m and taking lacunary averages,

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} m|h_n(x) - f(x)| &\leq \lambda \cdot \frac{1}{h_r}\sum_{n\in I_r} m|f_n(x) - f(x)| + (1 - \lambda) \cdot \frac{1}{h_r}\sum_{n\in I_r} m|g_n(x) - f(x)| \\ &\leq \lambda|\sigma_1(x)| + (1 - \lambda)|\sigma_2(x)|.\end{aligned}$$

Hence, $(h_n) \in \mathcal{RUL}_f(X)$, proving convexity.
For the other case it follows similarly. □

5.2. Local Convexity

Definition 5.1 *[13] A topological vector space is said to be locally convex if its topology is generated by a family of seminorms.*

Proposition 5.1 *The space $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are locally convex.*

Proof: We shall prove that $\mathcal{RUL}_f(X)$ is locally convex by showing that it can be endowed with a topology generated by a family of seminorms that satisfy the conditions of local convexity.

Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a limit function. For each $r \in \mathbb{N}$, we define a seminorm p_r on $\mathcal{RUL}_f(X)$ as follows:

$$p_r((f_n)) = \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) - f(x)|,$$

where $h_r = k_r - k_{r-1}$ is the gap between consecutive indices in the lacunary sequence $\{k_r\}$, and $I_r = (k_{r-1}, k_r]$ is the index set for the subsequence corresponding to r.

First, we verify that each p_r is a seminorm. We shall show that it satisfies the following properties:
1. Non-negativity:

$$p_r((f_n)) = \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) - f(x)| \geq 0, \quad \text{for all } (f_n) \in \mathcal{RUL}_f(X),$$

which holds because the absolute value of any real number is non-negative.

2. Absolute scalability: For any scalar $\alpha \in \mathbb{R}$, we have

$$p_r(\alpha(f_n)) = \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |\alpha f_n(x) - \alpha f(x)| = |\alpha| \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) - f(x)| = |\alpha| p_r((f_n)),$$

which follows from the homogeneity property of absolute values.

3. Subadditivity: For any two sequences (f_n) and (g_n), we have

$$\begin{aligned} p_r((f_n) + (g_n)) &= \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) + g_n(x) - f(x)| \\ &\leq \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |f_n(x) - f(x)| + \sup_{x\in X} \frac{1}{h_r} \sum_{n\in I_r} |g_n(x) - f(x)| \\ &= p_r((f_n)) + p_r((g_n)), \end{aligned}$$

which follows from the triangle inequality for absolute values.
Since the seminorms p_r satisfy non-negativity, absolute scalability, and subadditivity, they are indeed seminorms on $\mathcal{RUL}_f(X)$.

Next we generate a Locally Convex Topology and we shall show that the collection $\{p_r\}_{r\in\mathbb{N}}$ generates a topology on $\mathcal{RUL}_f(X)$. The topology induced by these seminorms is locally convex if it satisfies the conditions convexity and locality, i.e., for the convexity the topology is generated by seminorms, which means the set of zeroes of any seminorm is convex, as the seminorms themselves are sublinear functions and for the locality, a topology is locally convex if, for any element $(f_n) \in \mathcal{RUL}_f(X)$, there exists a neighborhood base consisting of sets of the form

$$\{(f_n) : p_r((f_n)) < \epsilon\},$$

for sufficiently small $\epsilon > 0$. This means that for any sequence (f_n), the neighborhoods around (f_n) are generated by conditions of control on the seminorms p_r.

Thus, the family $\{p_r\}$ satisfies the conditions for the space $\mathcal{RUL}_f(X)$ to be locally convex.
Similar argument can be provided for the space $\mathcal{RUL}_0(X)$.

□

5.3. Strict Convexity

Definition 5.2 *A normed space is strictly convex if, for any two distinct elements x and y on the unit sphere, the norm of their average is strictly less than one.*

Proposition 5.2 *The spaces $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are not strictly convex in general.*

Proof: Let $(f_n), (g_n) \in \mathcal{RUL}_f(X)$ be two distinct sequences such that for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \frac{\|f_n(x) - f(x)\|}{\sigma(n)} = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{n\in I_r} \frac{\|g_n(x) - f(x)\|}{\sigma(n)} = 0,$$

and suppose further that these sequences attain the same lacunary mean bounds with equality.

Consider the average sequence

$$h_n := \frac{f_n + g_n}{2}.$$

By convexity of the norm and the properties of the scale function σ, we have

$$\|h_n(x) - f(x)\| \leq \frac{1}{2}\|f_n(x) - f(x)\| + \frac{1}{2}\|g_n(x) - f(x)\|.$$

Hence,

$$\frac{1}{h_r}\sum_{n\in I_r}\frac{\|h_n(x) - f(x)\|}{\sigma(n)} \leq \frac{1}{2}\left[\frac{1}{h_r}\sum_{n\in I_r}\frac{\|f_n(x) - f(x)\|}{\sigma(n)} + \frac{\|g_n(x) - f(x)\|}{\sigma(n)}\right].$$

Taking the limit as $r \to \infty$, we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{n\in I_r}\frac{\|h_n(x) - f(x)\|}{\sigma(n)} = 0.$$

Thus, $(h_n) \in \mathcal{RUL}_f(X)$, and the norm (measured via the lacunary sense) of the average sequence equals that of the original sequences.

Since $(f_n) \neq (g_n)$, this violates the condition of strict convexity. Therefore, the space $\mathcal{RUL}_f(X)$ is not strictly convex in general.

The same reasoning applies to $\mathcal{RUL}_0(X)$, using $f(x) = 0$.

□

5.4. Symmetricity

Proposition 5.3 *The spaces $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$ are not symmetric in general.*

Proof: A sequence space is said to be *symmetric* if it is closed under permutations of the terms of its elements, i.e., if $(f_n) \in \mathcal{RUL}_f(X)$, then every rearrangement $(f_{\pi(n)})$ also belongs to $\mathcal{RUL}_f(X)$ for any bijection $\pi : \mathbb{N} \to \mathbb{N}$.

Let (f_n) be a sequence in $\mathcal{RUL}_f(X)$, i.e., it satisfies

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{n\in I_r}\frac{\|f_n(x) - f(x)\|}{\sigma(n)} = 0 \quad \text{uniformly in } x \in X,$$

where $\theta = (k_r)$ is a lacunary sequence, and $I_r = (k_{r-1}, k_r]$ are lacunary intervals.

Now, consider a rearrangement $(f_{\pi(n)})$ of the sequence, where $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation. Since the definition of lacunary convergence depends on the order of indices and fixed intervals I_r, the rearranged sequence $(f_{\pi(n)})$ may not satisfy the same lacunary mean convergence condition. In particular, the lacunary sums

$$\frac{1}{h_r}\sum_{n\in I_r}\frac{\|f_{\pi(n)}(x) - f(x)\|}{\sigma(n)}$$

may fail to converge to zero, even though the original sequence satisfies the condition.

Therefore, the space $\mathcal{RUL}_f(X)$ is not invariant under rearrangement of terms, and hence is not symmetric. The same argument applies to $\mathcal{RUL}_0(X)$. □

6. Inclusion Relations and Examples

In this section, we investigate the relationship between relatively uniform lacunary convergence and other known types of convergence, such as pointwise convergence, uniform convergence, and standard lacunary convergence. We also provide examples to demonstrate the inclusion relations and, in some cases, their strictness.

6.1. Inclusion Results

Proposition 6.1 *Uniform convergence implies relatively uniformly lacunary convergence.*

Proof: Let (f_n) converge uniformly to a function f on X. Then for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$. Then for all r such that $I_r \subset [N, \infty)$,

$$\frac{1}{h_r} \sum_{n \in I_r} m|f_n(x) - f(x)| < m\varepsilon,$$

so taking $\sigma(x) = m\varepsilon$ as a constant function shows that the condition for relatively uniformly lacunary convergence is satisfied. □

Proposition 6.2 *Relatively uniformly lacunary convergence implies lacunary convergence pointwise.*

Proof: Let $(f_n) \in \mathcal{RUL}_f(X)$ with scale function $\sigma(x)$. Then

$$\frac{1}{h_r} \sum_{n \in I_r} m|f_n(x) - f(x)| \leq |\sigma(x)|,$$

which implies

$$\frac{1}{h_r} \sum_{n \in I_r} |f_n(x) - f(x)| \to 0 \text{ as } r \to \infty.$$

So for each $x \in X$, the sequence $(f_n(x))$ is lacunary convergent to $f(x)$. □

Proposition 6.3 *The converse of the above statements do not hold in general.*

Proof: We provide examples below to demonstrate the strictness of the inclusions. □

6.2. Examples

Example 6.1 (Uniform convergence implies relative uniform lacunary convergence) Let $f_n(x) = \frac{x}{n}$ on $X = [0, 1]$. Then $f_n(x) \to 0$ uniformly, and hence also relatively uniformly lacunary to 0 with any lacunary sequence.

Example 6.2 (Relative uniform lacunary convergence without uniform convergence) Define

$$f_n(x) = \begin{cases} 1 & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ \frac{1}{n} & \text{otherwise}, \end{cases}$$

on $X = [0, 1]$. Then $f_n(x)$ does not converge uniformly to 0 since the subsequence $f_{k^2}(x) = 1$ fails to converge. However, choosing a lacunary sequence that skips the indices k^2 (e.g., $k_r = 2^r$), we find that the lacunary average converges to 0 for large r, and we can bound it using a scale function $\sigma(x)$. Hence (f_n) is relatively uniformly lacunary convergent to 0, but not uniformly convergent.

Example 6.3 (Lacunary convergence without relative uniform lacunary convergence) Let

$$f_n(x) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is odd}, \end{cases}$$

on $X = [0, 1]$. For certain lacunary sequences (e.g., $k_r = 2r$), the lacunary means tend to zero, but the absolute difference $|f_n(x) - 0| = 1$ remains constant, making it impossible to find a bounding function $\sigma(x)$ that satisfies the definition of relative uniform lacunary convergence. Thus, the sequence is lacunary convergent but not relatively uniformly lacunary convergent.

6.3. Strictness of Inclusion

Based on the examples, we can conclude the following strict inclusion relations:

$$\text{Lacunary Convergence} \Rightarrow \text{Relative Uniform Lacunary Convergence} \Rightarrow \text{Uniform Convergence}$$

These inclusions are proper, as demonstrated by the counterexamples above.

These results and examples show that relatively uniform lacunary convergence occupies a middle ground between uniform and lacunary convergence, and thus enriches the landscape of generalized convergence methods in analysis.

7. Conclusion

In this paper, we introduced and studied the concept of *relative uniform lacunary convergence* of sequences of real-valued functions. This new mode of convergence generalizes both uniform convergence and lacunary convergence by incorporating a scale function into the convergence behavior. We explored the basic properties of this convergence method and introduced the associated space $\mathcal{RUL}_f(X)$ and $\mathcal{RUL}_0(X)$.

We established important topological and geometric features of these spaces, including convexity, local convexity, symmetry and separability. We also discussed the failure of properties like strict convexity and symmetry in general, highlighting the nuanced structure of these spaces. Additionally, we studied the inclusion relations between uniform convergence, relative uniform lacunary convergence and standard lacunary convergence. Through carefully constructed examples, we demonstrated that these inclusions are strict.

Overall, the concept of relative uniform lacunary convergence provides a flexible and robust framework for capturing intermediate behaviors between uniform and lacunary convergence. Moreover it offers rich potential for further theoretical development and applications.

References

1. Chittenden, E. W., *Relative uniform convergence of sequences of functions*, Trans. Amer. Math. Soc., 66, 425-441, (1949).
2. Dowari, P. J. and Tripathy, B. C., *Lacunary difference sequences of complex uncertain variables*, Methods Funct. Anal. Topol., 26, 327-340, (2020).
3. Dowari, P. J. and Tripathy, B. C., *Lacunary convergence of sequences of complex uncertain variables*, Malays. J. Math. Sci., 15(1), 91-108, (2021).
4. Dowari, P. J. and Tripathy, B. C., *Lacunary sequences of complex uncertain variables defined by Orlicz functions*, Proyecciones (Antofagasta), 40(2), 355-370, (2021).
5. Dowari, P. J. and Tripathy, B. C., *Lacunary convergence of double sequences of complex uncertain variables*, J. Uncertain Syst., 14(3), (2021), Article 2150017.
6. Dowari, P. J. and Tripathy, B. C., *Lacunary statistical convergence of complex uncertain variables*, Bol. Soc. Paran. Mat., 41, 1-10, (2022).
7. Fridy, J. A., *On statistical convergence*, Analysis, 5, 301-313, (1985).
8. Devi, K. R. and Tripathy, B. C., *Relative uniform convergence of sequences of positive linear functions*, Advances in Mathematical Analysis and Its Applications, CRC Press, 43-58, (2022).
9. Devi, K. R. and Tripathy, B. C., *Statistical relative uniform convergence of double sequences of functions*, Palest. J. Math., 9(2), 558-570, (2020).
10. Devi, K. R. and Tripathy, B. C., *Relative uniform convergence of difference double sequences of positive linear functions*, Trans. A. Razmadze Math. Inst., 176(3), 451-460, (2021).
11. Devi, K. R. and Tripathy, B. C., *On relative uniform convergence of double sequences of functions*, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 92(3), 367–372, (2022).
12. Devi, K. R. and Tripathy, B. C., *Relative uniform convergence of double sequences of positive linear functions defined by Orlicz functions*, Bol. Soc. Paran. Mat., 41, 1-10, (2023).
13. Kamthan, P. K., and Gupta, M. *Sequence spaces and series*, M. Dekker, New York, (1981).
14. Tripathy, B. C. and Esi, A., *Generalized lacunary difference sequence spaces defined by Orlicz functions*, Journal Math. Society of the Philippines, 28(1–3), 50–57, (2005).

15. Tripathy, B. C. and Et, M., *On generalized difference lacunary statistical convergence*, Stud. Univ. Babeș-Bolyai Math., 50(1), 119–130, (2005).

16. Tripathy, B. C. and Dutta, A. J., *Lacunary bounded variation sequence of fuzzy real numbers*, Journal of Intelligent and Fuzzy Systems, 24(1), 185–189, (2013).

17. Fridy, J. A. and Orhan, C., *Lacunary statistical convergence*, Pacific Journal of Mathematics, 160(1), 43-51, (1993).

18. Tripathy, B. C. and Baruah, A., *Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers*, Kyungpook Math. Journal, 50(4), 565–574, (2010).

19. Freedman, A. R., Sember, J. J. and Raphael, M., *Some Cesaro-type summability spaces*, Proceedings of the London Mathematical Society, 3(3), 508-520, (1978).

20. Haloi, R., Sen, M., and Tripathy, B. C., *Asymptotically Lacunary μ-Statistical Equivalence of Generalized Difference Sequences in Probabilistic Normed Spaces*, Boletim da Sociedade Paranaense de Matematica, 41, 1-13, (2023).

Munindra Regon,
Department of Mathematics,
Dibrugarh University, Assam-India
Email: munindraregon@gmail.com

and

Pranab Jyoti Dowari,
Department of Mathematics,
Moridhal College, Dhemaji-India
Email:pranabdowari@gmail.com

and

Kshetrimayum Renubebeta Devi,
Department of Mathematics,
The Assam Royal Global University, Guwahati-India
Email:renu.ksh11@gmail.com