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Foundations and framework of superhypersoft sets

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ABSTRACT: Superhypersoft sets (SHSS) represent an advanced extension of soft sets and hypersoft sets, designed to address complex multi-parameterized decision-making problems in uncertain environments. This study explores the foundational theories of soft set and hypersoft set models, leading to the formulation of SHSS as a more generalized and flexible framework for managing intricate data structures. We define key operations and properties of SHSS and establish their basic algebraic laws. The findings show that SHSS improves the mathematical abilities of existing soft set models by adding more structural complexity, which makes them suitable for complicated classification and decision-support systems. Even with such potential, SHSS is still a relatively new field with many undiscovered research possibilities, such as formalization of basic operations, discussion of basic algebraic laws, algorithmic implementations and real-world applications in different fields. This study inspires more developments in soft set theory and uncertainty modeling by establishing the foundation for future research into the theoretical and applied aspects of SHSS.

Key Words: Soft set, hypersoft set, superhypersoft set, generalized set model.

Contents

1. Introduction and Preliminaries

In traditional mathematics, set theory is a fundamental tool for modeling and studying collections of objects. It provides a structural foundation for various mathematical fields. The computational sciences, logic, algebra and other disciplines have made notable use of classical set theory as well as its probabilistic and fuzzy extensions. Even though traditional set theory is powerful in many aspects, it has significant drawbacks when it comes to handling ambiguity, uncertainty and several attribute levels. These limitations are mostly seen in real-world decision-making scenarios with imprecise nature of data.

To compensate for these shortcomings, Molodtsov proposed soft set theory [1] in 1999, offering a more flexible and responsive approach able to deal with imprecise data better than conventional set-based techniques. Soft set theory gives a systematic method of handling uncertainties by enabling a parameterized family of subsets to express various viewpoints of a problem under consideration. This new methodology has now attracted extensive attention in various scientific research areas, such as artificial intelligence, data mining, economics and medical diagnosis.

In order to address such kinds of issues, Molodtsov introduced soft set theory [1] in 1999, which provides a more flexible approach that can handle imprecise data more effectively compared to traditional set-based approaches. Soft set theory provides an organized method to manage uncertainty by allowing various viewpoints on a topic to be expressed through a parameterized family of subsets. Nowadays, a number of scientific study fields, including artificial intelligence (AI), data mining, economics and medical diagnostics, are paying special attention to this innovative methodology. In contrast to classical sets with strict membership criteria or fuzzy sets [2], which involve membership degrees, soft sets introduce a flexible framework where a different subset of a universal set is defined by each parameter. Soft sets' ability to handle complex issues where various attributes are essential for decision-making is enhanced by

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parameterization. It has been demonstrated that applying soft set theory can help resolve ambiguities that arise in pattern recognition, expert systems and decision analysis. The demand for complex frameworks to deal with higher-dimensional uncertainty and intricate decision-making requirements was recognized as soft set theory research advanced. This requirement gave rise to the formulation of many extensions, including fuzzy soft sets [3], intuitionistic fuzzy soft sets [4], neutrosophic soft sets [5], rough soft sets [6] and multi-soft sets [7]. These extensions were proposed to extend the elementary soft set paradigm by adding extra parameters, hybrid approaches, or advanced computational methods to enhance decision support systems.

Despite these developments, generic soft set models failed to make proper utilization of multi-dimensional attribute dependency. Traditional soft sets are very versatile tools for addressing uncertainties but become ineffective when applications demand the management of attributes with interdependences as well as uncertainty at different levels. In this respect, researchers proposed hypersoft sets [8] and, more recently, superhypersoft sets (SHSS) [9], [10] that further augment the structural depth and usability of the soft sets.

Hypersoft sets [8] generalize the simple soft set model by enabling simultaneous consideration of multiple levels of parameters. The multi-parameterized methodology increases the versatility of decision-making models by making it possible to represent complex relations between various attributes. Hypersoft sets have been successfully applied in different fields involving complex decision matrices, for example, medical diagnosis, supply chain management and risk evaluation.

Superhypersoft sets, based on the foundations of hypersoft sets, add an additional level of structural complexity [9]. It offers a greater representational capability of soft sets with additional fine-grained parameterization. Because of such increased complexity, SHSS can handle multi-dimensional uncertainty more robustly, making them suitable for highly complex decision-making models in operations research, machine learning and artificial intelligence.

Although SHSSs are a fascinating idea, they are still relatively new and very little is known about their mathematical characteristics. The absence of basic definitions and standard procedures are some of the issues preventing SHSS from being widely used. By providing a thorough overview of SHSS, examining its properties along with mathematical structure, this study attempts to close some of the gaps. This should encourage advancement in this area and start the process of using them in other scientific and industrial domains.

Definition 1.1 *[1] Let $\mathcal{U}$ be a universe of discourse, the power set of $\mathcal{U}$ is $\mathcal{P}(\mathcal{U})$ and $\mathcal{E}$ be a set of parameters where $\mathcal{A}$ is the collection of attributes ($\mathcal{A} \subseteq \mathcal{E}$). Then, the pair $(\mathcal{F}, \mathcal{A})$, where $\mathcal{F} : \mathcal{A} \to \mathcal{P}(\mathcal{U})$ is called a Soft Set over $\mathcal{U}$.*

Definition 1.2 *[7] Let $a_1, a_2, a_3, \cdots, a_n$ be the distinct attributes whose attributes values belongs to the sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_n$ respectively, where $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$. A pair $(\mathcal{F}, \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \cdots \times \mathcal{A}_n)$ is called a hypersoft set over the universal set $\mathcal{U}$, where $\mathcal{F}$ is the mapping given by $(\mathcal{F}, \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \cdots \times \mathcal{A}_n) \to \mathcal{P}(\mathcal{U})$.*

Definition 1.3 *[9], [10] Let $\mathcal{U}$ be a universe of discourse and let $\mathcal{P}(\mathcal{U})$ denote the power set of $\mathcal{U}$. Consider n distinct attributes $a_1, a_2, \cdots, a_n$ for $n \geq 1$, where each attribute a_i has an associated attribute value set $\mathcal{A}_i$, satisfying:*

$$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset, \quad \text{for } i \neq j, \quad i, j \in \{1, 2, \cdots, n\}.$$

Let $\mathcal{P}(\mathcal{A}_i)$ be the power set of $\mathcal{A}_i$ for each i. Define the Cartesian product of these power sets as

$$\mathcal{A}^1 = \mathcal{P}(\mathcal{A}_1) * \mathcal{P}(\mathcal{A}_2) * \cdots * \mathcal{P}(\mathcal{A}_n).$$

A Superhypersoft set (SHSS) over $\mathcal{U}$ is an ordered pair $(\mathcal{F}, \mathcal{A}^1)$, where:

$$\mathcal{F} : \mathcal{A}^1 \to \mathcal{P}(\mathcal{U})$$

is a mapping that assigns to each $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) \in \mathcal{A}^1$ a subset of the universe $\mathcal{U}$.

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For simplicity throughout the article, we will consider $\mathcal{P}(\mathcal{A}_1) * \mathcal{P}(\mathcal{A}_2) * \cdots * \mathcal{P}(\mathcal{A}_n)$ as $\mathcal{A}^1$ and $\mathcal{P}(\mathcal{B}_1) * \mathcal{P}(\mathcal{B}_2) * \cdots * \mathcal{P}(\mathcal{B}_n)$ as $\mathcal{B}^1$ and so on.

2. Main results

This section discusses some basic definitions and examples on superhypersoft sets.

Definition 2.1 *Consider ($\mathcal{F}_1$,$\mathcal{A}^1$) and ($\mathcal{F}_2$,$\mathcal{B}^1$) be the two superhypersoft sets over a common universe $\mathcal{U}$, we say that ($\mathcal{F}_1$,$\mathcal{A}^1$) is a superhypersoft subset of ($\mathcal{F}_2$,$\mathcal{B}^1$) if*

(i) $\mathcal{A}^1 \subseteq \mathcal{B}^1$.

(ii) $\forall e \in \mathcal{A}^1$ *such that* $\mathcal{F}_1(e) \subseteq \mathcal{F}_2(e)$

Definition 2.2 *Consider $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be the two superhypersoft sets over a common universe $\mathcal{U}$, then the union between them is denoted by $(\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1)$ is the SHSS $(\mathcal{F}_3, \mathcal{C})$, where $\mathcal{C} = \mathcal{A}^1 \cup \mathcal{B}^1$ and $\forall\ e \in \mathcal{C}$,*

$$\mathcal{F}_3(e) = \begin{cases} \mathcal{F}_1(e), & if\ e \in \mathcal{A}^1 - \mathcal{B}^1 \\ \mathcal{F}_2(e), & if\ e \in \mathcal{B}^1 - \mathcal{A}^1 \\ \mathcal{F}_1(e) \cup \mathcal{F}_2(e), & if\ e \in \mathcal{A}^1 \cap \mathcal{B}^1 \end{cases}$$

Definition 2.3 *Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be the SHSS over the same universal set $\mathcal{U}$ such that $(\mathcal{A}^1 \cap \mathcal{B}^1) \neq \phi$. The restricted union is denoted by $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)$ and is defined as $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{C})$. Where $\mathcal{C} = (\mathcal{A}^1 \cap \mathcal{B}^1)$ and for all $e \in \mathcal{C}$, such that $\mathcal{F}_3(e) = \mathcal{F}_1(e) \cup \mathcal{F}_2(e)$.*

Definition 2.4 *Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be the SHSS over the same universal set $\mathcal{U}$ then the restricted difference of $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{C})$ where $\mathcal{C} = (\mathcal{A}^1 \cap \mathcal{B}^1) \neq \phi$ and $\forall e \in \mathcal{C}, \mathcal{F}_3(e) = \mathcal{F}_1(e) - \mathcal{F}_2(e)$.*

Definition 2.5 *Consider $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be the two superhypersoft sets over a common universe $\mathcal{U}$, then the intersection between them is $(\mathcal{F}_3, \mathcal{C})$.*
where $\mathcal{C} = (\mathcal{A}^1 \cap \mathcal{B}^1)\ \forall\ e \in \mathcal{C}$, then $\mathcal{F}_3(e) = \mathcal{F}_1(e) \cap \mathcal{F}_2(e)$.

Definition 2.6 *Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be two SHSS over the same universal sets $\mathcal{U}$. The extended intersection of this two SHSS is denoted by $(\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{C})$, where $\mathcal{C} = \mathcal{A}^1 \cup \mathcal{B}^1$ for all $e \in \mathcal{C}$ such that*

$$\mathcal{F}_3(e) = \begin{cases} \mathcal{F}_1(e), & if\ e \in (\mathcal{A}^1 - \mathcal{B}^1) \\ \mathcal{F}_2(e), & if\ e \in (\mathcal{B}^1 - \mathcal{A}^1) \\ \mathcal{F}_1(e) \cap \mathcal{F}_2(e), & if\ e \in (\mathcal{A}^1 \cap \mathcal{B}^1) \end{cases}$$

Definition 2.7 *Let $\mathcal{E}=\{\mathcal{P}(\mathcal{A}_1)*\mathcal{P}(\mathcal{A}_2)*\mathcal{P}(\mathcal{A}_3) * \cdots * \mathcal{P}(\mathcal{A}_n)\}$ be a cartesian product of the power sets of parameters. The not set of $\mathcal{E}$ is denoted by $\rceil\mathcal{E}$ and is defined by $\rceil\mathcal{E}=\{\mathcal{P}(\rceil\mathcal{A}_1)*\mathcal{P}(\rceil\mathcal{A}_2)*\mathcal{P}(\rceil\mathcal{A}_3) * \cdots * \mathcal{P}(\rceil\mathcal{A}_n)\}$, where $\rceil\mathcal{A}_i$ =not $\mathcal{A}_i$, $\forall\ i$.*

Definition 2.8 *The complement of superhypersoft set $(\mathcal{F}_1, \mathcal{A}^1)$ is denoted by $(\mathcal{F}_1, \mathcal{A}^1)^c = (\mathcal{F}_1^c, \rceil\mathcal{A}^1)$, where $\mathcal{F}_1^c : \rceil\mathcal{A}^1 \to \mathcal{P}(\mathcal{U})$ is a mapping defined by $\mathcal{F}_1^c(\neg e) = \mathcal{U} - \mathcal{F}_1(e)$, $\forall\ \neg e \in \rceil\mathcal{A}^1$.*

Definition 2.9 *The relative complement of SHSS $(\mathcal{F}_1, \mathcal{A}^1)$ is denoted by $(\mathcal{F}_1, \mathcal{A}^1)^r$ and is defined by $(\mathcal{F}_1, \mathcal{A}^1)^r = (\mathcal{F}_1^r, \mathcal{A}^1)$, where $\mathcal{F}_1^r : \mathcal{A}^1 \to \mathcal{P}(\mathcal{U})$ is a mapping given by $\mathcal{F}_1^r(e) = \mathcal{U} - \mathcal{F}_1(e)$ for all $e \in \mathcal{A}^1$.*

Definition 2.10 *Let $(\mathcal{F}_1, \mathcal{A}^1)$ be a SHSS over a universal set $\mathcal{U}$ is said to be an absolute SHSS, denoted by $\tilde{\mathcal{A}}$, if $\forall e \in \mathcal{A}^1$ then $\mathcal{F}_1(e) = \mathcal{U}$.*

Definition 2.11 *A SHSS $(\mathcal{F}_1, \mathcal{A}^1)$ over a universal set $\mathcal{U}$ is said to be a null SHSS denoted by $\tilde{\phi}$, if for all $e \in \mathcal{A}^1$, then $\mathcal{F}_1(e) = \phi$ (null set).*

Example 2.1 *Let $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of universities in a particular country. Let $\mathcal{E}$ be the set of attributes representing academic programs:*

$$\begin{aligned}
\mathcal{E}_1 &= \textit{Basic Science} = \{\textit{Chemistry } (e_1), \textit{Physics } (e_2), \textit{Mathematics } (e_3)\},\\
\mathcal{E}_2 &= \textit{Engineering} = \{\textit{Civil } (e_4), \textit{Mechanical } (e_5)\},\\
\mathcal{E}_3 &= \textit{Arts} = \{\textit{History } (e_6), \textit{Political Science } (e_7)\}.
\end{aligned}$$

Then, their respective power sets are:

$$\begin{aligned}
\mathcal{P}(\mathcal{E}_1) &= \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\},\\
\mathcal{P}(\mathcal{E}_2) &= \{\emptyset, \{e_4\}, \{e_5\}, \{e_4, e_5\}\},\\
\mathcal{P}(\mathcal{E}_3) &= \{\emptyset, \{e_6\}, \{e_7\}, \{e_6, e_7\}\}.
\end{aligned}$$

Let, $\mathcal{A}_1 = \{e_1, e_2\}, \quad \mathcal{A}_2 = \{e_4\}, \quad \mathcal{A}_3 = \{e_6, e_7\}$. *Then their power sets are:*

$$\begin{aligned}
\mathcal{P}(\mathcal{A}_1) &= \{\emptyset, \{e_1\}, \{e_2\}, \{e_1, e_2\}\},\\
\mathcal{P}(\mathcal{A}_2) &= \{\emptyset, \{e_4\}\},\\
\mathcal{P}(\mathcal{A}_3) &= \{\emptyset, \{e_6\}, \{e_7\}, \{e_6, e_7\}\}.
\end{aligned}$$

and $\mathcal{B}_1 = \{e_1, e_2\}, \quad \mathcal{B}_2 = \{e_4, e_5\}, \quad \mathcal{B}_3 = \{e_6, e_7\}$. *Then their power sets are:*

$$\begin{aligned}
\mathcal{P}(\mathcal{B}_1) &= \{\emptyset, \{e_1\}, \{e_2\}, \{e_1, e_2\}\},\\
\mathcal{P}(\mathcal{B}_2) &= \{\emptyset, \{e_4\}, \{e_5\}, \{e_4, e_5\}\},\\
\mathcal{P}(\mathcal{B}_3) &= \{\emptyset, \{e_6\}, \{e_7\}, \{e_6, e_7\}\}.
\end{aligned}$$

Consider the superhypersoft set

$$(\mathcal{F}_1, \mathcal{A}^1) = \{((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1, u_2\}), ((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2, u_3\})\}$$

[Here we consider, $\mathcal{A}^1 = \mathcal{P}(\mathcal{A}_1) * \mathcal{P}(\mathcal{A}_2) * \mathcal{P}(\mathcal{A}_3)$]
Which means that $\mathcal{F}_1$({Physics} and {Civil} and {History or Political Science})=$\{u_1, u_2\}$ and $\mathcal{F}_1$({Chemistry or Physics} and {Civil} and {History or Political Science})=$\{u_2, u_3\}$.
The superhypersoft set provides multiple selections. So, Universities u_1 and u_2 offer the Physics program but not Mathematics or Chemistry, the Civil program but not Mechanical and may offer either the History or Political Science programs. Similarly, Universities u_2 and u_3 offer either the Physics or Chemistry program but not Mathematics, the Civil program but not Mechanical and may include either the History or Political Science programs.

Consider another superhypersoft set

$$\begin{aligned}
(\mathcal{F}_1, \mathcal{B}^1) = \{&((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2, u_3, u_4\}), ((\{e_2\}, \{e_4, e_5\}, \{e_6\}), \{u_2, u_4\}),\\
&((\{e_1, e_2\}, \{e_4\}, \{e_7\}), \{u_2, u_5\})\}
\end{aligned}$$

[We consider, $\mathcal{B}^1 = \mathcal{P}(\mathcal{B}_1) * \mathcal{P}(\mathcal{B}_2) * \mathcal{P}(\mathcal{B}_3)$]
The universities u_2, u_3 and u_4 offer the Physics program but not Mathematics or Chemistry, the Civil program but not Mechanical and may offer either the History or Political Science programs. Similarly, universities u_2 and u_4 offer the Physics program but not Chemistry or Mathematics, either the Civil or Mechanical program and the History program but not Political Science. Additionally, universities u_2 and u_5 offer either the Physics or Chemistry program but not Mathematics, the Civil program but not Mechanical and the Political Science program but not History.

(i) $(\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1) = \{((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2, u_3\}), ((\{e_2\}, \{e_4, e_5\}, \{e_6\}), \{u_2, u_4\}), ((\{e_1, e_2\}, \{e_4\}, \{e_7\}), \{u_2, u_5\}), ((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1, u_2, u_3, u_4\})\}$.

(ii) $(\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1) = \{(\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2\}\}$.

(iii) $(\mathcal{F}_1, \mathcal{A}^1)^c = \{((\{\neg e_2\}, \{\neg e_4\}, \{\neg e_6, \neg e_7\}), \{u_3, u_4, u_5\}), ((\{\neg e_1, \neg e_2\}, \{\neg e_4\}, \{\neg e_6, \neg e_7\}), \{u_1, u_4, u_5\})\}$.

(iv) $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1) = \{(\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1, u_2, u_3, u_4\}\}$.

(v) $(\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_2, \mathcal{B}^1) = \{((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2, u_3\}), ((\{e_2\}, \{e_4, e_5\}, \{e_6\}), \{u_2, u_4\}), ((\{e_1, e_2\}, \{e_4\}, \{e_7\}), \{u_2, u_5\}), ((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2\})\}$.

(vi) $(\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1) = \{(\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1\}\}$.

(vii) $(\mathcal{F}_1, \mathcal{A}^1)^r = (\mathcal{F}_1^r, \mathcal{A}^1) = \{((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_3, u_4, u_5\}), ((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1, u_4, u_5\})\}$.

3. Theoretical Developments

In this section, the basic properties of operations on Superhypersoft sets, such as union, intersection, restricted union, extended intersection, restricted difference, complement, relative complement and restricted symmetric difference has been discussed.

Proposition 3.1 *Properties for the union($\cup$) and intersection($\cap$) operations:*
Let $(\mathcal{F}_1,\mathcal{A}^1)$, $(\mathcal{F}_2,\mathcal{B}^1)$ *and* $(\mathcal{F}_3,\mathcal{C}^1)$ *be three SHSS then*
$(i)(\mathcal{F}_1,\mathcal{A}^1)\cup\tilde{\mathcal{A}}=\tilde{\mathcal{A}}$
$(ii)(\mathcal{F}_1,\mathcal{A}^1)\cup\phi_{\mathcal{A}^1}=(\mathcal{F}_1,\mathcal{A}^1)$
$(iii)(\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1)=\phi_{\mathcal{A}^1}\iff(\mathcal{F}_1,\mathcal{A}^1)=\phi_{\mathcal{A}^1}$ *and* $(\mathcal{F}_2,\mathcal{B}^1)=\phi_{\mathcal{A}^1}$
$(iv)(\mathcal{F}_1,\mathcal{A}^1)\cap\tilde{\mathcal{A}}=(\mathcal{F}_1,\mathcal{A}^1)$
$(v)(\mathcal{F}_1,\mathcal{A}^1)\cap_E(\mathcal{F}_1,\mathcal{A})^r=(\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_1,\mathcal{A}^1)^r=\phi_{\mathcal{A}^1}$
$(vi)(\mathcal{F}_1,\mathcal{A}^1)\cap\phi_{\mathcal{A}^1}=\phi_{\mathcal{A}^1}$
(vii)***Associative laws:***
$(a)((\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1))\cup(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_1,\mathcal{A}^1)\cup((\mathcal{F}_2,\mathcal{B}^1)\cup(\mathcal{F}_3,\mathcal{C}^1))$
$(b)((\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2,\mathcal{B}^1))\cap(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_1,\mathcal{A}^1)\cap((\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1))$
$(viii)$ ***Commutative laws:***
$(a)(\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_2,\mathcal{B}^1)\cup(\mathcal{F}_1,\mathcal{A}^1)$
$(b)(\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_1,\mathcal{A}^1)$
(ix) ***Distributive laws:***
(a) $(\mathcal{F}_1,\mathcal{A}^1)\cup((\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_3,\mathcal{C}^1))$
(b) $(\mathcal{F}_1,\mathcal{A}^1)\cap((\mathcal{F}_2,\mathcal{B}^1)\cup(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2,\mathcal{B}^1))\cup((\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_3,\mathcal{C}^1))$

Proof: (i) to $(viii)$ is straightforward.

(ix) (a) Let $(\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_4,\mathcal{D}^1)$, where $\mathcal{D}^1=\mathcal{B}^1\cap\mathcal{C}^1,\forall\ e\in\mathcal{D}^1$
$\mathcal{F}_4(e)=\mathcal{F}_2(e)\cap\mathcal{F}_3(e)$
Now, $(\mathcal{F}_1,\mathcal{A}^1)\cup((\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1))=(\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_4,\mathcal{D}^1)=(\mathcal{F}_5,\mathcal{E}^1)$, where $\mathcal{E}^1=\mathcal{A}^1\cup\mathcal{D}^1=\mathcal{A}^1\cup(\mathcal{B}^1\cap\mathcal{C}^1)$ and $\forall\ e\in\mathcal{E}^1$,

$$\mathcal{F}_5(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-\mathcal{D}^1=\mathcal{A}^1-(\mathcal{B}^1\cap\mathcal{C}^1)\\ \mathcal{F}_4(e) & if\ e\in\mathcal{D}^1-\mathcal{A}^1=(\mathcal{B}^1\cap\mathcal{C}^1)-\mathcal{A}^1\\ \mathcal{F}_1(e)\cup\mathcal{F}_4(e) & if\ e\in\mathcal{A}^1\cap\mathcal{D}^1=\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1\\ =\mathcal{F}_1(e)\cup((\mathcal{F}_2(e)\cap\mathcal{F}_3(e)) & \end{cases}$$

Again, $(\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_6,\mathcal{E}^1)$, where $\mathcal{E}^1=(\mathcal{A}^1\cup\mathcal{B}^1)\ \forall\ e\in\mathcal{E}^1$,

$$\mathcal{F}_6(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-\mathcal{B}^1\\ \mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-\mathcal{A}^1\\ \mathcal{F}_1(e)\cup\mathcal{F}_2(e) & if\ e\in\mathcal{A}^1\cap\mathcal{B}^1\end{cases}$$

Now, $(\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{F}^1)$, where $\mathcal{F}^1=(\mathcal{A}^1\cup\mathcal{C}^1)\ \forall\ e\in\mathcal{F}^1$,

$$\mathcal{F}_7(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-\mathcal{C}^1\\ \mathcal{F}_3(e) & if\ e\in\mathcal{C}^1-\mathcal{A}^1\\ \mathcal{F}_1(e)\cup\mathcal{F}_3(e) & if\ e\in\mathcal{A}^1\cap\mathcal{C}^1\end{cases}$$

Finally, $((\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\cup(\mathcal{F}_3,\mathcal{C}^1))=(\mathcal{F}_6,\mathcal{E}^1)\cap(\mathcal{F}_7,\mathcal{F}^1)=(\mathcal{F}_8,\mathcal{G}^1)$
where $\mathcal{G}^1=\mathcal{E}^1\cap\mathcal{F}^1=(\mathcal{A}^1\cup\mathcal{B}^1)\cap(\mathcal{A}^1\cup\mathcal{C}^1)=\mathcal{A}^1\cup(\mathcal{B}^1\cap\mathcal{C}^1)$; $\forall\ e\in\mathcal{G}^1$ such that $\mathcal{F}_8(e)=\mathcal{F}_6(e)\cap\mathcal{F}_7(e)$;

$$\mathcal{F}_8(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in(\mathcal{A}^1-\mathcal{B}^1)\cap(\mathcal{A}^1-\mathcal{C}^1)=\mathcal{A}^1-(\mathcal{B}^1\cup\mathcal{C}^1)\\ \mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in(\mathcal{B}^1-\mathcal{A}^1)\cap(\mathcal{C}^1-\mathcal{A}^1)=(\mathcal{B}^1\cup\mathcal{C}^1)-\mathcal{A}^1\\ (\mathcal{F}_1(e)\cup\mathcal{F}_2(e))\cap(\mathcal{F}_1(e)\cap\mathcal{F}_3(e)) & if\ e\in(\mathcal{A}^1\cap\mathcal{B}^1)\cap(\mathcal{A}^1\cap\mathcal{C}^1)=(\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1)\\ =\mathcal{F}_1(e)\cup(\mathcal{F}_2(e)\cap\mathcal{F}_3(e)) & \end{cases}$$

$= \mathcal{F}_5(e)$

$\therefore (\mathcal{F}_8, \mathcal{E}^1) = (\mathcal{F}_5, \mathcal{G}^1)$. So $(\mathcal{F}_1, \mathcal{A}^1) \cup ((\mathcal{F}_2, \mathcal{B}^1) \cap (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1)) \cap ((\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_3, \mathcal{C}^1))$
Hence proved.

(b) Let $(\mathcal{F}_2, \mathcal{B}^1) \cup (\mathcal{F}_3, \mathcal{C}^1) = (\mathcal{F}_4, \mathcal{D}^1)$ where $\mathcal{D}^1 = \mathcal{B}^1 \cup \mathcal{C}^1$; $\forall\ e \in \mathcal{D}^1$ and

$$\mathcal{F}_4(e) = \begin{cases} \mathcal{F}_2(e) & if\ e \in \mathcal{B}^1 - \mathcal{C}^1 \\ \mathcal{F}_3(e) & if\ e \in \mathcal{C}^1 - \mathcal{B}^1 \\ \mathcal{F}_2(e) \cup \mathcal{F}_3(e) & if\ e \in \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

Now $(\mathcal{F}_1, \mathcal{A}^1) \cap ((\mathcal{F}_2, \mathcal{B}^1) \cup (\mathcal{F}_3, \mathcal{C}^1)) = (\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_4, \mathcal{D}^1) = (\mathcal{F}_5, \mathcal{E}^1)$,
where $\mathcal{E}^1 = \mathcal{A}^1 \cap \mathcal{D}^1 = \mathcal{A}^1 \cap (\mathcal{B}^1 \cup \mathcal{C}^1) = ((\mathcal{A}^1 \cap \mathcal{B}^1) \cup (\mathcal{A}^1 \cap \mathcal{C}^1))$; $\forall\ e \in \mathcal{E}^1$, such that $\mathcal{F}_5(e) = \mathcal{F}_1(e) \cap \mathcal{F}_4(e)$ then

$$\mathcal{F}_5(e) = \begin{cases} \mathcal{F}_1(e) \cap \mathcal{F}_2(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{B}^1) - (\mathcal{A}^1 \cap \mathcal{C}^1) = (\mathcal{A}^1 \cap \mathcal{B}^1) - \mathcal{C}^1 \\ \mathcal{F}_1(e) \cap \mathcal{F}_3(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{C}^1) - (\mathcal{A}^1 \cap \mathcal{B}^1) = \mathcal{C}^1 - (\mathcal{A}^1 \cap \mathcal{B}^1) \\ \mathcal{F}_1(e) \cap (\mathcal{F}_2(e) \cup \mathcal{F}_3(e)) & if\ e \in (\mathcal{A}^1 \cap \mathcal{B}^1) \cap (\mathcal{A}^1 \cap \mathcal{C}^1) = \mathcal{A}^1 \cap \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

Again let, $(\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_6, \mathcal{F}^1)$, where $\mathcal{F}^1 = \mathcal{A}^1 \cap \mathcal{B}^1\ \forall\ e \in \mathcal{D}^1$ then $\mathcal{F}_6(e) = \mathcal{F}_1(e) \cap \mathcal{F}_2(e)$ and $(\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_3, \mathcal{C}^1) = (\mathcal{F}_7, \mathcal{G}^1)$, where $\mathcal{G}^1 = \mathcal{A}^1 \cap \mathcal{C}^1\ \forall\ e \in \mathcal{G}^1$ then $\mathcal{F}_7(e) = \mathcal{F}_1(e) \cap \mathcal{F}_3(e)$
Now $((\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1)) \cup ((\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_3, \mathcal{C}^1)) = (\mathcal{F}_6, \mathcal{F}^1) \cup (\mathcal{F}_7, \mathcal{G}^1) = (\mathcal{F}_8, \mathcal{H}^1)$,
where $\mathcal{H}^1 = \mathcal{F}^1 \cup \mathcal{G}^1 = (\mathcal{A}^1 \cap \mathcal{B}^1) \cup (\mathcal{A}^1 \cap \mathcal{C}^1), \forall\ e \in \mathcal{H}^1$ so, $\mathcal{F}_8(e) = \mathcal{F}_6(e) \cup \mathcal{F}_7(e)$ then

$$\mathcal{F}_8(e) = \begin{cases} \mathcal{F}_6(e) = \mathcal{F}_1(e) \cap \mathcal{F}_2(e) & if\ e \in (\mathcal{F}^1 - \mathcal{G}^1) = (\mathcal{A}^1 \cap \mathcal{B}^1) - (\mathcal{A}^1 \cap \mathcal{C}^1) = \mathcal{A}^1 \cap (\mathcal{B}^1 - \mathcal{C}^1) \\ \mathcal{F}_1(e) \cap \mathcal{F}_3(e) & if\ e \in (\mathcal{G}^1 - \mathcal{F}^1) = (\mathcal{A}^1 \cap \mathcal{C}^1) - (\mathcal{A}^1 \cap \mathcal{B}^1) = \mathcal{A}^1 \cap (\mathcal{C}^1 - \mathcal{B}^1) \\ \mathcal{F}_1(e) \cap (\mathcal{F}_2(e) \cup \mathcal{F}_3(e)) & if\ e \in (\mathcal{F}^1 \cap \mathcal{G}^1) = (\mathcal{A}^1 \cap \mathcal{B}^1) \cap (\mathcal{A}^1 \cap \mathcal{C}^1) = \mathcal{A}^1 \cap \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

$\therefore \mathcal{F}_8(e) = \mathcal{F}_5(e)$

So, $(\mathcal{F}_8, \mathcal{H}^1) = (\mathcal{F}_5, \mathcal{E}^1)$. **(Proved)**

Definition 3.1 *In SHSS De Morgan's laws do not hold for union and intersection. Let us now define two new operators AND operator and OR operator as*

(i) AND Operator: *Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be two SHSS, then $(\mathcal{F}_1, \mathcal{A}^1) AND (\mathcal{F}_2, \mathcal{B}^1)$ denoted by $(\mathcal{F}_1, \mathcal{A}^1) \wedge (\mathcal{F}_2, \mathcal{B}^1)$ is defined by $(\mathcal{F}_1, \mathcal{A}^1) \wedge (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{A}^1 * \mathcal{B}^1)$ where, $\mathcal{F}_3(\alpha, \beta) = \mathcal{F}_1(\alpha) \cap \mathcal{F}_2(\beta)$,$\forall\ (\alpha, \beta) \in \mathcal{A}^1 * \mathcal{B}^1$.*

(ii) OR Operation: *Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be two SHSS, then $(\mathcal{F}_1, \mathcal{A}^1) OR (\mathcal{F}_2, \mathcal{B}^1)$ denoted by $(\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1)$ is defined by $(\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{A}^1 * B^1)$ where, $\mathcal{F}_3(\alpha, \beta) = \mathcal{F}_1(\alpha) \cup \mathcal{F}_2(\beta)$,$\forall\ (\alpha, \beta) \in \mathcal{A}^1 * B^1$.*

Example 3.1 *Let* $(\mathcal{F}_1, \mathcal{A}^1) = \{((\{e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_1, u_2\}), ((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), \{u_2, u_3\})\}$ *and* $(\mathcal{F}_2, \mathcal{B}^1) = \{((\{e_1, e_2\}, \{e_4\}, \{, e_7\}), \{u_2, u_5\})\}$ *then*
$(\mathcal{F}_1, \mathcal{A}^1) \wedge (\mathcal{F}_2, \mathcal{B}^1) = \{(((\{e_2\}, \{e_4\}, \{e_6, e_7\}), (\{e_1, e_2\}, \{e_4\}, \{, e_7\})), \{u_2\}),$
$(((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), (\{e_1, e_2\}, \{e_4\}, \{, e_7\})), \{u_2\})\}$ *and*
$(\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1) = \{(((\{e_2\}, \{e_4\}, \{e_6, e_7\}), (\{e_1, e_2\}, \{e_4\}, \{, e_7\})), \{u_1, u_2, u_5\}),$
$(((\{e_1, e_2\}, \{e_4\}, \{e_6, e_7\}), (\{e_1, e_2\}, \{e_4\}, \{, e_7\})), \{u_2, u_3, u_5\})\}$.

Proposition 3.2 *Properties of the relative complement (^{r}):*
Let $(\mathcal{F}_1, \mathcal{A}^1)$ and $(\mathcal{F}_2, \mathcal{B}^1)$ be two SHSS then
$(i)((\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \wedge (\mathcal{F}_2, \mathcal{B}^1)^r$
$(ii)((\mathcal{F}_1, \mathcal{A}^1) \wedge (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \vee (\mathcal{F}_2, \mathcal{B}^1)^r$
$(iii)((\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cap_E (\mathcal{F}_2, \mathcal{B}^1)^r$
$(iv)((\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cup (\mathcal{F}_2, \mathcal{B}^1)^r$
$(v)((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cap (\mathcal{F}_2, \mathcal{B}^1)^r$
$(vi)((\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cup_R (\mathcal{F}_2, \mathcal{B}^1)^r$
$(vii)(\tilde{\mathcal{A}})^r = \phi_{\mathcal{A}^1}$

Proof: (i) Let$(\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{A}^1 * \mathcal{B}^1)$; where $\mathcal{F}_3(\alpha, \beta) = \mathcal{F}_1(\alpha) \cup \mathcal{F}_2(\beta)\ \forall(\alpha, \beta) \in \mathcal{A}^1 * \mathcal{B}^1$
$\Rightarrow ((\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1))^r = ((\mathcal{F}_3, \mathcal{A}^1 * \mathcal{B}^1))^r = (\mathcal{F}_3^r, \mathcal{A}^1 * \mathcal{B}^1)$
Now, $(\mathcal{F}_1, \mathcal{A}^1)^r \wedge (\mathcal{F}_2, \mathcal{B}^1)^r = (\mathcal{F}_1^r, \mathcal{A}^1) \wedge (\mathcal{F}_2^r, \mathcal{B}^1) = (\mathcal{F}_4^r, \mathcal{A}^1 * \mathcal{B}^1)$
where $\mathcal{F}_4^r(\alpha, \beta) = \mathcal{F}_1^r(\alpha) \cap \mathcal{F}_2^r(\beta)\ \forall(\alpha, \beta) \in \mathcal{A}^1 * \mathcal{B}^1$
then $\mathcal{F}_3^r(\alpha, \beta) = \mathcal{U} - \mathcal{F}_3(\alpha, \beta) = \mathcal{U} - [\mathcal{F}_1(\alpha) \cup \mathcal{F}_2(\beta)] = (\mathcal{U} - \mathcal{F}_1(\alpha)) \cap (\mathcal{U} - \mathcal{F}_2(\beta))$
$\Rightarrow \mathcal{F}_3^r(\alpha, \beta) = \mathcal{F}_1^r(\alpha) \cap \mathcal{F}_2^r(\beta)$
$\therefore ((\mathcal{F}_1, \mathcal{A}^1) \vee (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \wedge (\mathcal{F}_2, \mathcal{B}^1)^r$. **(proved)**

(ii) In a similar manner, we can prove (ii)
(iii) LHS: Let,$(\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_3, \mathcal{C}^1), \quad \forall e \in \mathcal{C}^1 = \mathcal{A}^1 \cup \mathcal{B}^1$

$$\text{Then, } \mathcal{F}_3(e) = \begin{cases} \mathcal{F}_1(e) & \text{if } e \in \mathcal{A}^1 - \mathcal{B}^1 \\ \mathcal{F}_2(e) & \text{if } e \in \mathcal{B}^1 - \mathcal{A}^1 \\ \mathcal{F}_1(e) \cup \mathcal{F}_2(e) & \text{if } e \in \mathcal{A}^1 \cap \mathcal{B}^1 \end{cases}$$

Now, $((\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_3, \mathcal{C}^1)^r = (\mathcal{F}_3^r, \mathcal{C}^1); \quad \forall e \in \mathcal{C}^1 = \mathcal{A}^1 \cup \mathcal{B}^1$

$$\Rightarrow \mathcal{F}_3^r(e) = \begin{cases} \mathcal{F}_1^r(e) & \text{if } e \in \mathcal{A}^1 - \mathcal{B}^1 \\ \mathcal{F}_2^r(e) & \text{if } e \in \mathcal{B}^1 - \mathcal{A}^1 \\ \mathcal{F}_1^r(e) \cap \mathcal{F}_2^r(e) & \text{if } e \in \mathcal{A}^1 \cap \mathcal{B}^1 \end{cases}$$

RHS: Let,$(\mathcal{F}_1, \mathcal{A}^1)^r \cap_E (\mathcal{F}_2, \mathcal{B}^1)^r = (\mathcal{F}_1^r, \mathcal{A}^1) \cap_E (\mathcal{F}_2^r, \mathcal{B}^1) = (\mathcal{F}_4^r, \mathcal{G}^1); \forall e \in \mathcal{G}^1 = \mathcal{A}^1 \cup \mathcal{B}^1$

$$\Rightarrow \mathcal{F}_4^r(e) = \begin{cases} \mathcal{F}_1^r(e) & \text{if } e \in \mathcal{A}^1 - \mathcal{B}^1 \\ \mathcal{F}_2^r(e) & \text{if } e \in \mathcal{B}^1 - \mathcal{A}^1 \\ \mathcal{F}_1^r(e) \cap \mathcal{F}_2^r(e) & \text{if } e \in \mathcal{A}^1 \cap \mathcal{B}^1 \end{cases}$$

$\Rightarrow \mathcal{F}_4^r(e) = \mathcal{F}_3^r(e)$
$\Rightarrow ((\mathcal{F}_1, \mathcal{A}^1) \cup (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cap_E (\mathcal{F}_2, \mathcal{B}^1)^r$ **(proved)**
(iv) In a similar manner, we can prove (iv)
(v) Let $((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) = (\mathcal{F}_3, \mathcal{C}^1)$, where $\mathcal{C}^1 = \mathcal{A}^1 \cap \mathcal{B}^1$; $\forall e \in \mathcal{C}^1$ then $\mathcal{F}_3(e) = \mathcal{F}_1(e) \cup \mathcal{F}_2(e)$
Now, $((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_3, \mathcal{C}^1)^r = (\mathcal{F}_3^r, \mathcal{C}^1)$
where, $\mathcal{F}_3^r(e) = \mathcal{U} - [\mathcal{F}_1(e) \cup \mathcal{F}_2(e)] = [\mathcal{U} - \mathcal{F}_1(e)] \cap [\mathcal{U} - \mathcal{F}_2(e)] = \mathcal{F}_1^r(e) \cap \mathcal{F}_2^r(e)$
Again, $(\mathcal{F}_1, \mathcal{A}^1)^r \cap (\mathcal{F}_2, \mathcal{B}^1)^r = (\mathcal{F}_1^r, \mathcal{A}^1) \cap (\mathcal{F}_2^r, \mathcal{B}^1) = (\mathcal{F}_4^r, \mathcal{D}^1)$ where, $\mathcal{D}^1 = \mathcal{A}^1 \cap \mathcal{B}^1$; $\forall e \in \mathcal{D}^1$ then $\mathcal{F}_4^r(e) = \mathcal{F}_1^r(e) \cap \mathcal{F}_2^r(e)$
$\therefore \mathcal{F}_4^r(e) = \mathcal{F}_3^r(e)$
$\Rightarrow ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1))^r = (\mathcal{F}_1, \mathcal{A}^1)^r \cap (\mathcal{F}_2, \mathcal{B}^1)^r$ **(proved)**
(vi) In a similar manner, we can prove (vi).
(vii) Follows directly.

Proposition 3.3 *Properties of the restricted union* $(\cup_R)$ *and extended intersection* $(\cap_E)$*:*
Let $(\mathcal{F}_1, \mathcal{A}^1)$, $(\mathcal{F}_2, \mathcal{B}^1)$ *and* $(\mathcal{F}_3, \mathcal{C}^1)$ *be three SHSS then*
(i) $(\mathcal{F}_1, \mathcal{A}^1) \cup_R \tilde{\mathcal{A}} = \tilde{\mathcal{A}}$ *and* $(\mathcal{F}_1, \mathcal{A}^1) \cup_R \phi_{\mathcal{A}^1} = (\mathcal{F}_1, \mathcal{A}^1)$
(ii) $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1) = \phi_{\mathcal{A}^1} \iff (\mathcal{F}_1, \mathcal{A}^1) = \phi_{\mathcal{A}^1}$ *and* $(\mathcal{F}_2, \mathcal{B}^1) = \phi_{\mathcal{A}^1}$
(iii) $(\mathcal{F}_1, \mathcal{A}^1) \cap_E \tilde{\mathcal{A}} = (\mathcal{F}_1, \mathcal{A}^1)$ *and* $(\mathcal{F}_1, \mathcal{A}^1) \cap_E \phi_{\mathcal{A}^1} = \phi_{\mathcal{A}^1}$
(iv) $(\mathcal{F}_1, \mathcal{A}^1) \cup_R ((\mathcal{F}_2, \mathcal{B}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) \cup_R (\mathcal{F}_3, \mathcal{C}^1)$
(v) $(\mathcal{F}_1, \mathcal{A}^1) \cap_E ((\mathcal{F}_2, \mathcal{B}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_2, \mathcal{B}^1)) \cap_E (\mathcal{F}_3, \mathcal{C}^1)$
(vi) $(\mathcal{F}_1, \mathcal{A}^1) \cup_R ((\mathcal{F}_2, \mathcal{B}^1) \cap (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) \cap ((\mathcal{F}_1, \mathcal{A}^1) \cup_R ((\mathcal{F}_3, \mathcal{C}^1))$
(vii) $(\mathcal{F}_1, \mathcal{A}^1) \cap ((\mathcal{F}_2, \mathcal{B}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1)) \cup_R ((\mathcal{F}_1, \mathcal{A}^1) \cap ((\mathcal{F}_3, \mathcal{C}^1))$
$(viii)$ $(\mathcal{F}_1, \mathcal{A}^1) \cup_R ((\mathcal{F}_2, \mathcal{B}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) \cap_E ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1))$
(ix) $(\mathcal{F}_1, \mathcal{A}^1) \cap_E ((\mathcal{F}_2, \mathcal{B}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_2, \mathcal{B}^1)) \cup_R ((\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1))$

Proof: $(i) - (iii)$ Follows directly.
(iv) Let $(\mathcal{F}_2, \mathcal{B}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1) = (\mathcal{F}_4, \mathcal{D}^1)$, where $\mathcal{D}^1 = \mathcal{B}^1 \cap \mathcal{C}^1$; $\forall e \in \mathcal{D}^1$ then $\mathcal{F}_4(e) = \mathcal{F}_2(e) \cup \mathcal{F}_3(e)$
now, $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_4, \mathcal{D}^1) = (\mathcal{F}_5, \mathcal{E}^1)$, where $\mathcal{E}^1 = \mathcal{A}^1 \cap \mathcal{D}^1 = \mathcal{A}^1 \cap \mathcal{B}^1 \cap \mathcal{C}^1\ \forall e \in \mathcal{E}^1$
then $\mathcal{F}_5(e) = \mathcal{F}_1(e) \cup \mathcal{F}_4(e) = \mathcal{F}_1(e) \cup (\mathcal{F}_2(e) \cup \mathcal{F}_3(e))$
Again, $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_6, \mathcal{G}^1)$, where $\mathcal{G}^1 = \mathcal{A}^1 \cap \mathcal{B}^1$; $\forall e \in \mathcal{G}^1$
then $\mathcal{F}_6(e) = \mathcal{F}_1(e) \cup \mathcal{F}_2(e)$

now, $(\mathcal{F}_6,\mathcal{G}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{H}^1)$, where $\mathcal{H}^1=\mathcal{G}^1\cap\mathcal{C}^1=\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1$; $\forall e\in\mathcal{H}^1$ then $\mathcal{F}_7(e)=\mathcal{F}_6(e)\cup\mathcal{F}_3(e)$
$\Rightarrow \mathcal{F}_7(e)=\mathcal{F}_1(e)\cup\mathcal{F}_2(e)\cup\mathcal{F}_3(e)$
$\therefore \mathcal{F}_7(e)=\mathcal{F}_5(e)$
$\Rightarrow (\mathcal{F}_1,\mathcal{A}^1)\cup_R((\mathcal{F}_2,\mathcal{B}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_2,\mathcal{B}^1))\cup_R(\mathcal{F}_3,\mathcal{C}^1)$ **(proved)**
(v) Let $(\mathcal{F}_2,\mathcal{B}^1)\cap_E(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_4,\mathcal{D}^1)$, where $\mathcal{D}^1=\mathcal{B}^1\cup\mathcal{C}^1$; $\forall\ e\in\mathcal{D}^1$ and

$$\mathcal{F}_4(e)=\begin{cases}\mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-\mathcal{C}^1\\ \mathcal{F}_3(e) & if\ e\in\mathcal{C}^1-\mathcal{B}^1\\ \mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in\mathcal{B}^1\cap\mathcal{C}^1\end{cases}$$

now, $(\mathcal{F}_1,\mathcal{A}^1)\cap_E(\mathcal{F}_4,\mathcal{D}^1)=(\mathcal{F}_5,\mathcal{E}^1)$, where $\mathcal{E}^1=\mathcal{A}^1\cup\mathcal{D}^1=\mathcal{A}^1\cup\mathcal{B}^1\cup\mathcal{C}^1$; $\forall\ e\in\mathcal{E}^1$ and

$$\mathcal{F}_5(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-(\mathcal{B}^1\cup\mathcal{C}^1)\\ \mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-(\mathcal{A}^1\cup\mathcal{C}^1)\\ \mathcal{F}_3(e) & if\ e\in\mathcal{C}^1-(\mathcal{B}^1\cup\mathcal{A}^1)\\ \mathcal{F}_1(e)\cap\mathcal{F}_2(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{B}^1)-\mathcal{C}^1\\ \mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in(\mathcal{B}^1\cap\mathcal{C}^1)-\mathcal{A}^1\\ \mathcal{F}_3(e)\cap\mathcal{F}_1(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{C}^1)-\mathcal{B}^1\\ \mathcal{F}_1(e)\cap\mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1\end{cases}$$

Again Let, $(\mathcal{F}_1,\mathcal{A}^1)\cap_E(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_6,\mathcal{G}^1)$, where $\mathcal{G}^1=\mathcal{A}^1\cup\mathcal{B}^1$; $\forall\ e\in\mathcal{G}^1$ and

$$\mathcal{F}_6(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-\mathcal{B}^1\\ \mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-\mathcal{A}^1\\ \mathcal{F}_1(e)\cap\mathcal{F}_2(e) & if\ e\in\mathcal{A}^1\cap\mathcal{B}^1\end{cases}$$

now, $(\mathcal{F}_6,\mathcal{G}^1)\cap_E(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{H}^1)$, where $\mathcal{H}^1=\mathcal{G}^1\cup\mathcal{C}^1=\mathcal{A}^1\cup\mathcal{B}^1\cup\mathcal{C}^1$; $\forall\ e\in\mathcal{H}^1$ and

$$\mathcal{F}_7(e)=\begin{cases}\mathcal{F}_1(e) & if\ e\in\mathcal{A}^1-(\mathcal{B}^1\cup\mathcal{C}^1)\\ \mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-(\mathcal{A}^1\cup\mathcal{C}^1)\\ \mathcal{F}_3(e) & if\ e\in\mathcal{C}^1-(\mathcal{B}^1\cup\mathcal{A}^1)\\ \mathcal{F}_1(e)\cap\mathcal{F}_2(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{B}^1)-\mathcal{C}^1\\ \mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in(\mathcal{B}^1\cap\mathcal{C}^1)-\mathcal{A}^1\\ \mathcal{F}_3(e)\cap\mathcal{F}_1(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{C}^1)-\mathcal{B}^1\\ \mathcal{F}_1(e)\cap\mathcal{F}_2(e)\cap\mathcal{F}_3(e) & if\ e\in\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1\end{cases}$$

$\therefore \mathcal{F}_7(e)=\mathcal{F}_5(e)$
$\Rightarrow (\mathcal{F}_7,\mathcal{H}^1)=(\mathcal{F}_5,\mathcal{E}^1)$
$\Rightarrow (\mathcal{F}_1,\mathcal{A}^1)\cap_E((\mathcal{F}_2,\mathcal{B}^1)\cap_E(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\cap_E(\mathcal{F}_2,\mathcal{B}^1))\cap_E(\mathcal{F}_3,\mathcal{C}^1)$**(proved)**
(vi) Let $(\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_4,\mathcal{D}^1)$, where $\mathcal{D}^1=\mathcal{B}^1\cap\mathcal{C}^1$; $\forall e\in\mathcal{D}^1$ then $\mathcal{F}_4(e)=\mathcal{F}_2(e)\cap\mathcal{F}_3(e)$
now, $(\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_4,\mathcal{D}^1)=(\mathcal{F}_5,\mathcal{E}^1)$, where $\mathcal{E}^1=\mathcal{A}^1\cap\mathcal{D}^1=\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1$ $\forall e\in\mathcal{E}^1$
then $\mathcal{F}_5(e)=\mathcal{F}_1(e)\cup\mathcal{F}_4(e)=\mathcal{F}_1(e)\cup(\mathcal{F}_2(e)\cap\mathcal{F}_3(e))$
Again, $(\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_6,\mathcal{F}^1)$, where $\mathcal{F}^1=\mathcal{A}^1\cap\mathcal{B}^1$; $\forall e\in\mathcal{F}^1$
then $\mathcal{F}_6(e)=\mathcal{F}_1(e)\cup\mathcal{F}_2(e)$
and $(\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{G}^1)$, where $\mathcal{G}^1=\mathcal{A}^1\cap\mathcal{C}^1$; $\forall e\in\mathcal{G}^1$ then $\mathcal{F}_7(e)=\mathcal{F}_1(e)\cup\mathcal{F}_3(e)$
now, $((\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1))=(\mathcal{F}_6,\mathcal{F}^1)\cap(\mathcal{F}_7,\mathcal{G}^1)=(\mathcal{F}_8,\mathcal{H}^1)$, where $\mathcal{H}^1=\mathcal{F}^1\cap\mathcal{G}^1$; $\forall e\in\mathcal{H}^1$ then $\mathcal{F}_8(e)=\mathcal{F}_6(e)\cap\mathcal{F}_7(e)=(\mathcal{F}_1(e)\cup\mathcal{F}_2(e))\cap(\mathcal{F}_1(e)\cup\mathcal{F}_3(e))=\mathcal{F}_1(e)\cup(\mathcal{F}_2(e)\cap\mathcal{F}_3(e))$
$\Rightarrow \mathcal{F}_8(e)=\mathcal{F}_5(e)$
$\therefore (\mathcal{F}_1,\mathcal{A}^1)\cup_R((\mathcal{F}_2,\mathcal{B}^1)\cap(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\cup_R(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\cup_R((\mathcal{F}_3,\mathcal{C}^1))$**(proved)**
(vii) In a similar manner, we can prove (vii)

$(viii)$ Let $(\mathcal{F}_2, \mathcal{B}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1) = (\mathcal{F}_4, \mathcal{D}^1)$, where $\mathcal{D}^1 = \mathcal{B}^1 \cup \mathcal{C}^1;\ \forall\ e \in \mathcal{D}^1$ and

$$\mathcal{F}_4(e) = \begin{cases} \mathcal{F}_2(e) & if\ e \in \mathcal{B}^1 - \mathcal{C}^1 \\ \mathcal{F}_3(e) & if\ e \in \mathcal{C}^1 - \mathcal{B}^1 \\ \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

now, $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_4, \mathcal{D}^1) = (\mathcal{F}_5, \mathcal{E}^1)$, where $\mathcal{E}^1 = \mathcal{A}^1 \cap \mathcal{D}^1 = \mathcal{A}^1 \cap (\mathcal{B}^1 \cup \mathcal{C}^1);\ \forall\ e \in \mathcal{E}^1$ and $\mathcal{F}_5(e) = \mathcal{F}_1(e) \cup \mathcal{F}_4(e)$

$$\Rightarrow \mathcal{F}_5(e) = \begin{cases} \mathcal{F}_1(e) & if\ e \in \mathcal{A}^1 - (\mathcal{B}^1 \cup \mathcal{C}^1) \\ \mathcal{F}_2(e) & if\ e \in \mathcal{B}^1 - (\mathcal{A}^1 \cup \mathcal{C}^1) \\ \mathcal{F}_3(e) & if\ e \in \mathcal{C}^1 - (\mathcal{B}^1 \cup \mathcal{A}^1) \\ \mathcal{F}_1(e) \cap \mathcal{F}_2(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{B}^1) - \mathcal{C}^1 \\ \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in (\mathcal{B}^1 \cap \mathcal{C}^1) - \mathcal{A}^1 \\ \mathcal{F}_3(e) \cap \mathcal{F}_1(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{C}^1) - \mathcal{B}^1 \\ \mathcal{F}_1(e) \cap \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in \mathcal{A}^1 \cap \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

Again let, $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_6, \mathcal{G}^1)$, where $\mathcal{G}^1 = \mathcal{A}^1 \cap \mathcal{B}^1;\ \forall\ e \in \mathcal{G}^1$
and $\mathcal{F}_6(e) = \mathcal{F}_1(e) \cup \mathcal{F}_2(e)$
again, $(\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1) = (\mathcal{F}_7, \mathcal{H}^1)$, where $\mathcal{H}^1 = \mathcal{A}^1 \cap \mathcal{C}^1;\ \forall\ e \in \mathcal{H}^1$
and $\mathcal{F}_7(e) = \mathcal{F}_1(e) \cup \mathcal{F}_3(e)$
$\therefore (\mathcal{F}_6, \mathcal{G}^1) \cap_E (\mathcal{F}_7, \mathcal{H}^1) = (\mathcal{F}_8, \mathcal{K}^1)$, where $\mathcal{K}^1 = \mathcal{G}^1 \cup \mathcal{H}^1 = (\mathcal{A}^1 \cap \mathcal{B}^1) \cup (\mathcal{A}^1 \cap \mathcal{C}^1) = \mathcal{A}^1 \cap (\mathcal{B}^1 \cup \mathcal{C}^1);\ \forall\ e \in \mathcal{K}^1$ then

$$\mathcal{F}_4(e) = \begin{cases} \mathcal{F}_2(e) & if\ e \in \mathcal{B}^1 - \mathcal{C}^1 \\ \mathcal{F}_3(e) & if\ e \in \mathcal{C}^1 - \mathcal{B}^1 \\ \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

Now, $(\mathcal{F}_1, \mathcal{A}^1) \cap_E (\mathcal{F}_4, \mathcal{D}^1) = (\mathcal{F}_5, \mathcal{E}^1)$, where $\mathcal{E}^1 = \mathcal{A}^1 \cup \mathcal{D}^1 = \mathcal{A}^1 \cup \mathcal{B}^1 \cup \mathcal{C}^1;\ \forall\ e \in \mathcal{E}^1$ and

$$\mathcal{F}_5(e) = \begin{cases} \mathcal{F}_1(e) & if\ e \in \mathcal{A}^1 - (\mathcal{B}^1 \cup \mathcal{C}^1) \\ \mathcal{F}_2(e) & if\ e \in \mathcal{B}^1 - (\mathcal{A}^1 \cup \mathcal{C}^1) \\ \mathcal{F}_3(e) & if\ e \in \mathcal{C}^1 - (\mathcal{B}^1 \cup \mathcal{A}^1) \\ \mathcal{F}_1(e) \cap \mathcal{F}_2(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{B}^1) - \mathcal{C}^1 \\ \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in (\mathcal{B}^1 \cap \mathcal{C}^1) - \mathcal{A}^1 \\ \mathcal{F}_3(e) \cap \mathcal{F}_1(e) & if\ e \in (\mathcal{A}^1 \cap \mathcal{C}^1) - \mathcal{B}^1 \\ \mathcal{F}_1(e) \cap \mathcal{F}_2(e) \cap \mathcal{F}_3(e) & if\ e \in \mathcal{A}^1 \cap \mathcal{B}^1 \cap \mathcal{C}^1 \end{cases}$$

$\therefore \mathcal{F}_8(e) = \mathcal{F}_5(e)$
$\Rightarrow (\mathcal{F}_8, \mathcal{K}^1) = (\mathcal{F}_5, \mathcal{E}^1)$
$\Rightarrow (\mathcal{F}_1, \mathcal{A}^1) \cup_R ((\mathcal{F}_2, \mathcal{B}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) \cap_E ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1))$**(proved)**
(ix) In a similar manner, we can prove (ix)

Proposition 3.4 *Properties for restricted difference ($\sim_R$):*
Let $(\mathcal{F}_1, \mathcal{A}^1)$, $(\mathcal{F}_2, \mathcal{B}^1)$ and $(\mathcal{F}_3, \mathcal{C}^1)$ be three SHSS then
$(i)(\mathcal{F}_1, \mathcal{A}^1) \sim_R \phi_{\mathcal{A}^1} = (\mathcal{F}_1, \mathcal{A}^1)$
$(ii)(\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_1, \mathcal{A}^1) = \phi_{\mathcal{A}^1}$
$(iii)(\mathcal{F}_1, \mathcal{A}^1) \sim_R ((\mathcal{F}_2, \mathcal{B}^1) \cup (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1)) \cap_E ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_3, \mathcal{C}^1))$
$(iv)(\mathcal{F}_1, \mathcal{A}^1) \sim_R ((\mathcal{F}_2, \mathcal{B}^1) \cap_E (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1)) \cup ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_3, \mathcal{C}^1))$
$(v)(\mathcal{F}_1, \mathcal{A}^1) \sim_R ((\mathcal{F}_2, \mathcal{B}^1) \cup_R (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1)) \cap ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_3, \mathcal{C}^1))$
$(vi)(\mathcal{F}_1, \mathcal{A}^1) \sim_R ((\mathcal{F}_2, \mathcal{B}^1) \cap (\mathcal{F}_3, \mathcal{C}^1)) = ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1)) \cup_R ((\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_3, \mathcal{C}^1))$
$(vii)(\mathcal{F}_1, \mathcal{A}^1) \sim_R (\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1)^r$

Proof:$(i)-(ii)$ Follows directly.
(iii) Let $(\mathcal{F}_2,\mathcal{B}^1)\cup(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_4,\mathcal{D}^1)$, where $\mathcal{D}^1=\mathcal{B}^1\cup\mathcal{C}^1$; $\forall\ e\in\mathcal{D}^1$ and

$$\mathcal{F}_4(e)=\begin{cases}\mathcal{F}_2(e) & if\ e\in\mathcal{B}^1-\mathcal{C}^1\\ \mathcal{F}_3(e) & if\ e\in\mathcal{C}^1-\mathcal{B}^1\\ \mathcal{F}_2(e)\cup\mathcal{F}_3(e) & if\ e\in\mathcal{B}^1\cap\mathcal{C}^1\end{cases}$$

Now, $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_4,\mathcal{D}^1)=(\mathcal{F}_5,\mathcal{E}^1)$ where $\mathcal{E}^1=\mathcal{A}^1\cap\mathcal{D}^1=\mathcal{A}^1\cap(\mathcal{B}^1\cup\mathcal{C}^1)$ $\forall e\in\mathcal{E}^1$ such that $\mathcal{F}_5(e)=\mathcal{F}_1(e)-\mathcal{F}_4(e)$ then,

$$\mathcal{F}_5(e)=\begin{cases}\mathcal{F}_1(e)-\mathcal{F}_2(e) & if\ e\in\mathcal{A}^1\cap(\mathcal{B}^1-\mathcal{C}^1)\\ \mathcal{F}_1(e)-\mathcal{F}_3(e) & if\ e\in\mathcal{A}^1\cap(\mathcal{C}^1-\mathcal{B}^1)\\ \mathcal{F}_1(e)-(\mathcal{F}_2(e)\cup\mathcal{F}_3(e)) & if\ e\in\mathcal{A}^1\cap(\mathcal{B}^1\cap\mathcal{C}^1)\end{cases}$$

Again, $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_6,\mathcal{G}^1)$ where $\mathcal{G}^1=\mathcal{A}^1\cap\mathcal{B}^1$ $\forall e\in\mathcal{G}^1$ such that $\mathcal{F}_6(e)=\mathcal{F}_1(e)-\mathcal{F}_2(e)$ &$(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{H}^1)$ $\forall e\in\mathcal{H}^1=\mathcal{A}^1\cap\mathcal{C}^1$ such that $\mathcal{F}_7(e)=\mathcal{F}_1(e)-\mathcal{F}_3(e)$
now, $((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1))\cap_E((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1))=(\mathcal{F}_6,\mathcal{G}^1)\cap_E(\mathcal{F}_7,\mathcal{H}^1)=(\mathcal{F}_8,\mathcal{K}^1)$ $\forall\ e\in\mathcal{K}^1=\mathcal{G}^1\cup\mathcal{H}^1=(\mathcal{A}^1\cap\mathcal{B}^1)\cup(\mathcal{A}^1\cap\mathcal{C}^1)$

$$\mathcal{F}_8(e)=\begin{cases}\mathcal{F}_6(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{B}^1)-(\mathcal{A}^1\cap\mathcal{C}^1)\\ \mathcal{F}_7(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{C}^1)-(\mathcal{A}^1\cap\mathcal{B}^1)\\ \mathcal{F}_6(e)\cap\mathcal{F}_7(e) & if\ e\in(\mathcal{A}^1\cap\mathcal{B}^1)\cap(\mathcal{A}^1\cap\mathcal{C}^1)=\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1\end{cases}$$

$$\Rightarrow\mathcal{F}_8(e)=\begin{cases}\mathcal{F}_1(e)-\mathcal{F}_2(e) & if\ e\in\mathcal{A}^1\cap(\mathcal{B}^1-\mathcal{C}^1)\\ \mathcal{F}_1(e)-\mathcal{F}_3(e) & if\ e\in\mathcal{A}^1\cap(\mathcal{C}^1-\mathcal{B}^1)\\ (\mathcal{F}_1(e)-\mathcal{F}_2(e))\cap(\mathcal{F}_3(e)-\mathcal{F}_3(e)) & if\ e\in\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1\\ =\mathcal{F}_1(e)-(\mathcal{F}_2(e)\cup\mathcal{F}_3(e)) & \end{cases}$$

$$\Rightarrow\mathcal{F}_8(e)=\mathcal{F}_5(e)$$
$$\text{So,}(\mathcal{F}_8,\mathcal{K}^1)=(\mathcal{F}_5,\mathcal{E}^1)$$
$$\therefore(\mathcal{F}_1,\mathcal{A}^1)\sim_R((\mathcal{F}_2,\mathcal{B}^1)\cup(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1))\cap_E((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1))\textbf{(proved)}$$

(iv) In a similar manner, we can prove (iv).
(v) Let $(\mathcal{F}_2,\mathcal{B}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_4,\mathcal{D}^1)$, $\forall\ e\in\mathcal{D}^1=\mathcal{B}^1\cap\mathcal{C}^1$ such that $\mathcal{F}_4(e)=\mathcal{F}_2(e)\cup\mathcal{F}_3(e)$
now, Let $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_4,\mathcal{D}^1)=(\mathcal{F}_5,\mathcal{E}^1)$, $\forall\ e\in\mathcal{E}^1=\mathcal{A}^1\cap\mathcal{D}^1=\mathcal{A}^1\cap\mathcal{B}^1\cap\mathcal{C}^1$ such that $\mathcal{F}_5(e)=\mathcal{F}_1(e)-\mathcal{F}_4(e)$
$\therefore\mathcal{F}_5(e)=\mathcal{F}_1(e)-(\mathcal{F}_2(e)\cup\mathcal{F}_3(e))$.
Again, Let $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_6,\mathcal{F}^1)$, $\forall\ e\in\mathcal{F}^1=\mathcal{A}^1\cap\mathcal{B}^1$ such that $\mathcal{F}_6(e)=\mathcal{F}_1(e)-\mathcal{F}_2(e)$
and $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1)=(\mathcal{F}_7,\mathcal{G}^1)$, $\forall\ e\in\mathcal{G}^1=\mathcal{A}^1\cap\mathcal{C}^1$ such that $\mathcal{F}_7(e)=\mathcal{F}_1(e)-\mathcal{F}_3(e)$
So, $((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1))=(\mathcal{F}_6,\mathcal{F}^1)\cap(\mathcal{F}_7,\mathcal{G}^1)=(\mathcal{F}_1(e)-\mathcal{F}_2(e))\cap(\mathcal{F}_1(e)-\mathcal{F}_3(e))$
$\Rightarrow((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1))=\mathcal{F}_1(e)-(\mathcal{F}_2(e)\cup\mathcal{F}_3(e))=(\mathcal{F}_5,\mathcal{E}^1)$
$\therefore(\mathcal{F}_1,\mathcal{A}^1)\sim_R((\mathcal{F}_2,\mathcal{B}^1)\cup_R(\mathcal{F}_3,\mathcal{C}^1))=((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1))\cap((\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_3,\mathcal{C}^1))$.**(proved)**
(vi) In a similar manner, we can prove (vi)
(vii) Let $(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_3,\mathcal{C}^1)$,$\forall e\in\mathcal{C}^1=\mathcal{A}^1\cap\mathcal{B}^1$ such that $\mathcal{F}_3(e)=\mathcal{F}_1(e)-\mathcal{F}_2(e)$
now, $(\mathcal{F}_2,\mathcal{B})^r=(\mathcal{F}_2^r,\mathcal{B})$ $\forall e\in\mathcal{B}^1$ such that $\mathcal{F}_2^r(e)=\mathcal{U}-\mathcal{F}_2(e)$
then, $(\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2,\mathcal{B}^1)^r=(\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2^r,\mathcal{B}^1)=(\mathcal{F}_4,\mathcal{C}^1)$ $\forall e\in\mathcal{C}^1=\mathcal{A}^1\cap\mathcal{B}^1$ such that $\mathcal{F}_4(e)=\mathcal{F}_1(e)\cap\mathcal{F}_2^r(e)$
To illustrate $(\mathcal{F}_3,\mathcal{C}^1)$ is SHSS equal to $(\mathcal{F}_4,\mathcal{D}^1)$ let $\forall e\in\mathcal{C}^1,\exists\ h\in\mathcal{F}_3(e)$
now, $h\in\mathcal{F}_3(e)\iff h\in\mathcal{F}_1(e)\wedge\ h\notin\mathcal{F}_2(e)$
$\iff h\in\mathcal{F}_1(e)\wedge h\in\mathcal{F}_2^r(e)$
$\iff h\in\mathcal{F}_1(e)\cap\mathcal{F}_2^r(e)$
$\iff h\in\mathcal{F}_4(e)$
$\therefore(\mathcal{F}_1,\mathcal{A}^1)\sim_R(\mathcal{F}_2,\mathcal{B}^1)=(\mathcal{F}_1,\mathcal{A}^1)\cap(\mathcal{F}_2,\mathcal{B}^1)^r$. **(proved)**

We are now prepared to define restricted symmetric difference and discuss its fundamental characteristics.

Definition 3.2 *The restricted symmetric difference of two SHSS* $(\mathcal{F}_1, \mathcal{A}^1)$ *and* $(\mathcal{F}_2, \mathcal{B}^1)$ *over a common a universe* $\mathcal{U}$ *is defined by* $(\mathcal{F}_1, \mathcal{A}^1)\tilde{\Delta}(\mathcal{F}_2, \mathcal{B}^1) = ((\mathcal{F}_1, \mathcal{A}^1) \cup_R (\mathcal{F}_2, \mathcal{B}^1)) \sim_R ((\mathcal{F}_1, \mathcal{A}^1) \cap (\mathcal{F}_2, \mathcal{B}^1))$.

Proposition 3.5 *Properties for restricted difference* $(\tilde{\Delta})$*:*
Let $(\mathcal{F}_1, \mathcal{A}^1)$ *and* $(\mathcal{F}_2, \mathcal{B}^1)$ *be two SHSS over a common universe* $\mathcal{U}$*, then we have the following properties:*
(i) $(\mathcal{F}_1, \mathcal{A}^1)\tilde{\Delta}(\mathcal{F}_1, \mathcal{A}^1) = \phi_{\mathcal{A}^1}$
(ii) $(\mathcal{F}_1, \mathcal{A}^1)\tilde{\Delta}\ \phi_{\mathcal{A}^1} = (\mathcal{F}_1, \mathcal{A}^1)$
(iii) $(\mathcal{F}_1, \mathcal{A}^1)\tilde{\Delta}(\mathcal{F}_2, \mathcal{B}^1) = (\mathcal{F}_2, \mathcal{B}^1)\tilde{\Delta}(\mathcal{F}_1, \mathcal{A}^1)$

Proof:$(i) - (iii)$ Follows directly.

4. Conclusion

This study investigates superhypersoft sets (SHSS) as an advanced extension of soft sets and hypersoft sets. We established the theoretical foundations of SHSS as a more generalized framework for handling complex data structures. Our study rigorously defines SHSS, explores its fundamental properties and examines key operations such as union, intersection, complement and other mathematical functions. Additionally, we analyzed significant propositions and proofs, demonstrating that SHSS adheres to essential algebraic laws, including associativity, commutativity and De Morgan's laws under specific conditions. The findings confirm that SHSS enhances the flexibility and precision of mathematical modeling beyond existing soft set frameworks. SHSS holds potential for applications in decision-making, data analysis, artificial intelligence and fuzzy logic-based systems. Further research could focus on optimizing computational efficiency, extending SHSS to multi-attribute decision models and integrating it with machine learning for enhanced uncertainty management. In future this concept can be refines as done in [11].

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