Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78077

The Minimum Pendant Dominating Randic Graph Energy

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ABSTRACT: In this research paper, we introduce the concept of minimum pendant dominating randic graph energy, denoted by $RE_P^D(\Gamma)$, and compute its value for several well-known graph families, including the complete graph, complete bipartite graph, bi-star graph, cocktail party, and barbell graph. Additionally, we investigate theoretical upper and lower bounds for $RE_P^D(\Gamma)$, offering insights into the behavior and range of this energy measure across various classes of graphs.

Key Words: Pendant dominating set, energy, randic matrix, randic energy.

Contents

1. Introduction

In recent decades, spectral graph theory has progressed rapidly, giving rise to several new sub fields within graph theory. The adjacency matrix provides a means to examine the degrees of a graph's vertices. This work introduced new perspectives on applying matrices to graph analysis. Several established graph-theoretical concepts, such as matching and minimum covering of a graph [1], have been explored in relation to spectral parameters, including the spectral radius, the second-largest eigenvalue, the smallest eigenvalue, and the absolute sum of these eigenvalues [8]. Due to their practical significance, energy and spectral radius have also attracted the attention of chemists and computer scientists [19]. Numerous graph polynomials have been defined in the literature based on various graph matrices, including the adjacency matrix, laplacian matrix, degree sum matrix, distance matrix, skew-adjacency matrix, Randić matrix, minimum covering matrix, and labeled matrix [1,2,3,6,10,12,19].

This study aims to introduce and analyze the minimum pendant dominating randic matrix of a graph, focusing on the relationship between vertex degrees and adjacency. It seeks to identify key properties of minimum pendant dominating randic spectra and randic energy, and to derive analytical expressions for the minimum pendant dominating randic energy of several important graph families. Building on prior research, this approach enhances the understanding of how randic-related metrics impact graph energy and spectral characteristics. We have extended the concept of minimum randic energy of a graph to minimum pendant dominating randic energy [21].

Let Γ be a finite, non-empty, simple and undirected graph of order n, with the vertex set $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(\Gamma)$. A subset $S \subseteq V(\Gamma)$ is called a dominating set, if every vertex not in S is adjacent to at least one vertex in S. The domination number of a graph Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ. A dominating set S in Γ is called a pendant dominating set, if the subgraph induced by S contains at least one pendant vertex. The pendant domination number

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2010 *Mathematics Subject Classification:* 05C07, 05C50, 05C69.
Submitted July 25, 2025. Published October 29, 2025

of Γ, denoted by $\gamma_{pe}(\Gamma)$, is the minimum cardinality of such a pendant dominating set. For more details on pendant domination refer [14,15,16]

The adjacency matrix $A(\Gamma) = [a_{ij}]$ of a graph Γ is a square matrix of order $n \times n$, where each entry a_{ij} is defined as

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between vertices } v_i \text{ and } v_j \\ 0, & \text{otherwise} \end{cases}$$

As $A(\Gamma)$ is a symmetric matrix so, all its eigenvalues $\eta_1, \eta_2, \ldots, \eta_n$ are all real numbers with their sum is zero. The concept of graph energy was introduced by I. Gutman [9]. According to him graph energy $E(\Gamma)$ is represented by

$$E(\Gamma) = \sum_{i=1}^{n} |\eta_i|$$

A brief account of graph energy can be found in R. Balakrishnan [3] and D. Cvetkovic [7]. Let Γ be a graph on vertex set $V(\Gamma) = \{v_1, v_2, v_3, \ldots, v_n\}$ and the edge set $E(\Gamma)$. Let d_i be the of a vertex v_i. Any dominating set with minimum possible cardinality is called minimum dominating set. Let D be a minimum dominating set of a graph Γ. The minimum dominating randic matrix $R^D(\Gamma) = (R^D_{ij})_{n\times n}$ is given by Bozurt et al. [4,5,6,12]

$$R^D_{ij}(\Gamma) = \begin{cases} \frac{1}{\sqrt{d_i d_i}}, & \text{if } v_i \sim v_j \\ 1, & \text{if } \ i = j \text{ and } v_i \in D \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $R^D(\Gamma)$ is denoted by $f_n(\Gamma, \eta) = det(\eta I - R^D(\Gamma))$. Since the minimum dominating randic matrix is real and symmetric. Let these eigenvalues be denoted in non-increasing order as $\eta_1 > \eta_2 > \cdots > \eta_n$. The minimum dominating randic energy [21] is determined by

$$RE^D(\Gamma) = \sum_{i=1}^{n} |\eta_i|$$

The spectra of a graph Γ is the list of distinct eigenvalues $\eta_1 > \eta_2 > \cdots > \eta_n$ with their multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_n$ and is as follows [8]:

$$spec^D(\Gamma) = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}$$

we can see the upper and lower bounds on randic energy in [4,5,12].

2. The Minimum Pendant Dominating Randic Graph Energy

Let Γ be a simple graph of order n with vertex set $V(\Gamma) = \{v_1, v_2, v_3, \ldots, v_n\}$ and the edge set $E(\Gamma)$. A subset S of $V(\Gamma)$ is called a pendant dominating set if every vertex in $V - S$ is adjacent to atleast one pendant vertex of S. A minimum pendant dominating set is a pendant dominating set of smallest cardinality, also referred to as a pendant dominating set of minimum power. The minimum pendant dominating randic matrix $R^D_P(\Gamma) = (R^D_{P\,ij})_{n\times n}$ is given by

$$R^D_{P_{ij}}(\Gamma) = \begin{cases} \frac{1}{\sqrt{d_i d_i}}, & \text{if } v_i \sim v_j \quad \text{for } \ i \neq j \\ 1, & \text{if } \ v_i = v_j \quad \text{for } \ \ i = j \\ 1, & \text{if } \ v_i \in S \quad \text{for } i = j \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of $R^D_P(\Gamma)$ is denoted by $f_n(\Gamma, \eta) = det(\eta I - R^D_P(\Gamma))$. Since the minimum pendant dominating randic matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\eta_1 > \eta_2 > \cdots > \eta_n$. The minimum pendant dominating randic energy is given by

$$RE^D_P(\Gamma) = \alpha_i \sum_{i=1}^{n} |\eta_i|$$

3. Fundamental Properties on Eigenvalues of $RE_P^D(\Gamma)$

Let us consider $Q = \sum_{i<j} \frac{1}{d_i d_j}$ where $d_i d_j$ is the product of degrees of two vertices which are adjacent.

Theorem 3.1 *The first three coefficients of $f_n(\Gamma, \eta)$ are given as follows:*
(i)$q_0 = 1$
(ii)$q_1 = -\gamma_{pe}(\Gamma)$
(iii)$q_2 = \begin{pmatrix} \gamma_{pe}(\Gamma) \\ 2 \end{pmatrix} - Q$

Proof: (i) From the definition $f_n(\Gamma, \eta) = det(\eta I - R_P^D(\Gamma))$, we get $q_0 = 1$.
(ii) The sum of all determinants of all 1×1 principal submatrices of $R_P^D(\Gamma)$ is equal to the trace of $R_P^D(\Gamma)$.
$\Longrightarrow q_1 = (-1)^1$ trace of $R_P^D(\Gamma) = -\gamma_{pe}(\Gamma)$
(iii) The term $(-1)^2 q_2$ corresponds to the sum of determinants of all 2×2 principal submatrices of $\left(R_P^D(\Gamma)\right)^2$, we have

$$\begin{aligned}
(-1)^2 q_2 &= \sum_{1 \le i \le j \le n} \begin{vmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{vmatrix} \\
&= \sum_{1 \le i \le j \le n} \left(q_{ii} q_{jj} - q_{ij} q_{ji}\right) \\
&= \sum_{1 \le i \le j \le n} q_{ii} q_{jj} - \sum_{1 \le i \le j \le n} q_{ij} q_{ij} \\
\therefore q_2 &= \begin{pmatrix} \gamma_{pe}(\Gamma) \\ 2 \end{pmatrix} - Q
\end{aligned}$$

□

Theorem 3.2 *If $\eta_1, \eta_2, \ldots, \eta_n$ are the minimum pendant dominating randic eigenvalues of $R_P^D(\Gamma)$, then*
(i) $\sum_{i=1}^{n} \eta_i = \gamma_{pe}(\Gamma)$
(ii) $\sum_{i=1}^{n} \eta_i^2 = \gamma_{pe}(\Gamma) + 2Q$

Proof: (i) As is well known, the sum of the eigenvalues of $R_P^D(\Gamma)$ equals the trace of $R_P^D(\Gamma)$. Therefore, we have

$$\sum_{i=1}^{n} \eta_i = \sum_{i=1}^{n} q_{ii} = |S| = \gamma_{pe}(\Gamma)$$

(ii) As is well known, the sum of the squares of the eigenvalues of $R_P^D(\Gamma)$ equals the trace of $(R_P^D(\Gamma))^2$. Therefore, we have

$$\sum_{i=1}^{n}\eta_i^2 = \sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}\eta_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\eta_i\eta_j = \sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}q_{ji}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(q_{ij}q_{ji} + q_{ji}q_{ij} + q_{ii}q_{jj}\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(q_{ii}^2 + 2q_{ij}q_{ji}\Big)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(q_{ii}^2 + 2q_{ij}^2\Big) = \sum_{i=1}^{n}q_{ii}^2 + 2\sum_{i<j}q_{ij}^2 = |D| + 2Q$$
$$\therefore \sum_{i=1}^{n}\eta_i^2 = \gamma_{pe}(\Gamma) + 2Q$$

□

Theorem 3.3 *Given a graph Γ, if $\mu = min\{|\eta_1|, |\eta_2|, \dots, |\eta_n|\}$, then $\mu \leq |det(RE_P^D(\Gamma))|^{\frac{1}{n}}$*

Proof: W. K. T.

$$|\eta_1||\eta_2|\dots|\eta_n| = |det(RE_P^D(\Gamma))|$$
$$\mu\mu\mu\dots\mu \leq |det(RE_P^D(\Gamma))|$$
$$\implies \mu \leq |det(RE_P^D(\Gamma))|^{\frac{1}{n}}$$

□

4. Bounds on $RE_P^D(\Gamma)$

Theorem 4.1 *Let Γ be a connected graph, then $\sqrt{2Q + \gamma_{pe}(\Gamma)} \leq RE_P^D(\Gamma) \leq \sqrt{n\big(2Q + \gamma_{pe}(\Gamma)\big)}$ where $Q = \sum_{i<j}\frac{1}{d_id_j}$ for which d_id_j is the product of degrees of two vertices which are adjacent.*

Proof: Let $\eta_1, \eta_2, \dots, \eta_n$ be the eigenvalues of $RE_P^D(\Gamma)$. Now by Cauchy's-Schwartz inequality, we have

$$\Big(\sum_{i=1}^{n}a_ib_i\Big)^2 \leq \Big(\sum_{i=1}^{n}a_i^2\Big)\Big(\sum_{i=1}^{n}b_i^2\Big)$$

Put $a_i = 1$ and $b_i = |\eta_i|$ in the above inequality

$$\left(\sum_{i=1}^{n}|\eta_i|\right)^2 \leq \Big(\sum_{i=1}^{n}1\Big)\Big(\sum_{i=1}^{n}|\eta_i|^2\Big)$$
$$\Big(RE_P^D(\Gamma)\Big)^2 \leq n\sum_{i=1}^{n}|\eta_i|^2 = n\sum_{i=1}^{n}\eta_i^2$$
$$\leq n\Big(2Q + \gamma_{pe}(\Gamma)\Big)$$
$$\implies RE_P^D(\Gamma) \leq \sqrt{n\Big(2Q + \gamma_{pe}(\Gamma)\Big)}$$

Now, consider

$$\Big(RE_P^D(\Gamma)\Big)^2 = \sum_{i=1}^{n} |\eta_i|^2$$
$$\Big(RE_P^D(\Gamma)\Big)^2 \geq 2Q + \gamma_{pe}(\Gamma)$$
$$\implies RE_P^D(\Gamma) \geq \sqrt{2Q + \gamma_{pe}(\Gamma)}$$

$$\therefore \sqrt{2Q + \gamma_{pe}(\Gamma)} \leq RE_P^D(\Gamma) \leq \sqrt{n\big(2Q + \gamma_{pe}(\Gamma)\big)}$$

□

Theorem 4.2 *Let* Γ *be a connected graph, then* $\sqrt{2n - Q - 1} \leq RE_P^D(\Gamma) \leq n\sqrt{n - \frac{\Delta(G)}{n}}$

Proof: W. K. T. $\gamma_{pe}(\Gamma) \leq D(\Gamma) + 1$ and thus we have $D(\Gamma) \leq |V(\Gamma)| - \Delta(\Gamma)$. Therefore,

$$\gamma_{pe}(\Gamma) - 1 \leq D(\Gamma) \leq n - \Delta(\Gamma)$$

For any connected graph Γ, we have $2Q \leq (n^2 - n)$ (upper bound). From above theorem

$$RE_P^D(\Gamma) \leq \sqrt{n\Big(2Q + \gamma_{pe}(\Gamma)\Big)} = \sqrt{n\Big[(n^2 - n) + n - \Delta\Big]}$$
$$\leq \sqrt{n(n^2 - \Delta)} = \sqrt{n^2\Big(n - \frac{\Delta}{n}\Big)}$$
$$\implies RE_P^D(\Gamma) \leq n\sqrt{n - \frac{\Delta}{n}}$$

For any connected graph Γ we have, $n \leq 2Q$ (lower bound).

$$RE_P^D(\Gamma) \geq \sqrt{2Q + \gamma_{pe}(\Gamma)} = \sqrt{n + \gamma_{pe}(\Gamma) - 1}$$
$$\geq \sqrt{n + n - Q - 1}$$
$$\implies RE_P^D(\Gamma) \geq \sqrt{2n - Q - 1}$$

$$\therefore \sqrt{2n - Q - 1} \leq RE_P^D(\Gamma) \leq n\sqrt{n - \frac{\Delta(G)}{n}}$$

□

Theorem 4.3 *Let* Γ *be a simple connected graph, then*

$$RE_P^D(\Gamma) \leq \Big(\tfrac{2Q+\gamma_{pe}(\Gamma)}{n}\Big) + \sqrt{(n-1)\Bigg[2Q + \gamma_{pe}(\Gamma) - \Big(\tfrac{2Q+\gamma_{pe}(\Gamma)}{n}\Big)^2\Bigg]}$$

Proof: The Cauchy's-Schwarz inequality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big)$$

Put $a_i = 1$ and $b_i = |\eta_i|$ in the above inequality

$$\Big(\sum_{i=2}^{n} |\eta_i|\Big)^2 \leq \Big(\sum_{i=2}^{n} 1\Big)\Big(\sum_{i=2}^{n} |\eta_i|^2\Big)$$
$$\big(RE_P^D(\Gamma) - \eta_1\big)^2 \leq (n-1)\big(2Q + \gamma_{pe}(\Gamma) - \eta_1^2\big)$$
$$\big(RE_P^D(\Gamma) - \eta_1\big) \leq \sqrt{(n-1)\big(2Q + \gamma_{pe}(\Gamma) - \eta_1^2\big)}$$
$$\Longrightarrow RE_P^D(\Gamma) \leq \eta_1 + \sqrt{(n-1)\big(2Q + \gamma_{pe}(\Gamma) - \eta_1^2\big)}$$

Let $f(x) = x + \sqrt{(n-1)\big(2Q + \gamma_{pe}(\Gamma) - x^2\big)}$. For decreasing function $f'(x) \leq 0$, we have

$$\Longrightarrow 1 + \frac{1}{2\sqrt{(n-1)\big(2Q + \gamma_{pe}(\Gamma) - x^2\big)}}(n-1)(-2x) \leq 0$$
$$1 \leq \frac{x\sqrt{n-1}}{\sqrt{\big(2Q + \gamma_{pe}(\Gamma) - x^2\big)}} \Longrightarrow x \geq \frac{\sqrt{2Q + \gamma_{pe}(\Gamma)}}{\sqrt{n-1}}$$

Since $2Q + \gamma_{pe}(\Gamma) \geq n$ we have,

$$\sqrt{\frac{2Q + \gamma_{pe}(\Gamma)}{n}} \leq \frac{2Q + \gamma_{pe}(\Gamma)}{n} \leq \eta_1$$
$$\Longrightarrow f(\eta_1) \leq f\left(\frac{2Q + \gamma_{pe}(\Gamma)}{n}\right)$$

$$\therefore RE_P^D(\Gamma) \leq \tfrac{2Q+\gamma_{pe}(\Gamma)}{n} + \sqrt{(n-1)\left[(2Q + \gamma_{pe}(\Gamma)) - \left(\tfrac{2Q+\gamma_{pe}(\Gamma)}{n}\right)^2\right]}$$

□

Theorem 4.4 *Let* Γ *be a connected graph. If* $\Delta = det(RE_P^D(\Gamma))$, *then* $RE_P^D(\Gamma) \geq \sqrt{2Q + \gamma_{pe}(\Gamma) + n(n-1)\Delta^{\frac{2}{n}}}$

Proof: Consider

$$\Big(RE_P^D(\Gamma)\Big)^2 = \Big(\sum_{i=1}^{n} |\eta_i|^2\Big) = \Big(\sum_{i=1}^{n} |\eta_i|\Big)\Big(\sum_{i=1}^{n} |\eta_i|\Big)$$
$$= \sum_{i=1}^{n} |\eta_i|^2 + \sum_{i \neq j} |\eta_i||\eta_j|$$
$$\Longrightarrow \Big(RE_P^D(\Gamma)\Big)^2 - \sum_{i=1}^{n} |\eta_i|^2 = \sum_{i \neq j} |\eta_i||\eta_j|$$

Using the Arithmetic mean and Geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\eta_i||\eta_j| \geq \left(\prod_{i \neq j} |\eta_i||\eta_j|\right)^{\frac{1}{n(n-1)}}$$
$$\sum_{i \neq j} |\eta_i||\eta_j| \geq n(n-1)\left(\prod_{i \neq j} |\eta_i||\eta_j|\right)^{\frac{1}{n(n-1)}}$$

Thus

$$\Big(RE_P^D(\Gamma)\Big)^2 - \sum_{i=1}^{n}|\eta_i|^2 \geq n(n-1)\Bigg(\prod_{i\neq j}|\eta_i||\eta_j|\Bigg)^{\frac{1}{n(n-1)}}$$

$$\Big(RE_P^D(\Gamma)\Big)^2 \geq \sum_{i=1}^{n}|\eta_i|^2 + n(n-1)\Bigg(\prod_{i\neq j}|\eta_i||\eta_j|\Bigg)^{\frac{1}{n(n-1)}}$$

$$= \sum_{i=1}^{n}|\eta_i|^2 + n(n-1)\Bigg(\prod_{i=1}^{n}\eta_i\Bigg)^{\frac{2}{n}}$$

$$\Big(RE_P^D(\Gamma)\Big)^2 \geq 2Q + \gamma_{pe}(\Gamma) + n(n-1)\Delta^{\frac{2}{n}}$$

$$\therefore RE_P^D(\Gamma) \geq \sqrt{2Q + \gamma_{pe}(\Gamma) + n(n-1)\Delta^{\frac{2}{n}}}$$

□

Theorem 4.5 *Let Γ be a graph with the minimum pendant dominating randic set $S(\Gamma)$. If the minimum pendant dominating randic energy $RE_P^D(\Gamma)$ is a rational number, then $RE_P^D(\Gamma) \equiv |S|(mod 2)$*

Proof: Let $\eta_1, \eta_2, \ldots, \eta_n$ be the minimum pendant dominating randic eigenvalues of Γ, where $\eta_1, \eta_2, \ldots, \eta_p$ are positive and the remaining are negative. Then

$$\sum_{i=1}^{n}|\eta_i| = \Big(\eta_1 + \eta_2 + \cdots + \eta_p\Big) - \Big(\eta_{p+1} + \cdots + \eta_n\Big)$$

$$= 2\Big(\eta_1 + \eta_2 + \cdots + \eta_p\Big) - \Big(\eta_1 + \eta_2 + \cdots + \eta_n\Big)$$

$$RE_P^D(\Gamma) = 2\Big(\eta_1 + \eta_2 + \cdots + \eta_p\Big) - |S|$$

The eigenvalues $\eta_1, \eta_2, \ldots, \eta_p$ are algebraic integers implies that their sum is also an algebraic integer. Therefore, $2\big(\eta_1 + \eta_2 + \cdots + \eta_p\big)$ must be an integer if $RE_P^D(\Gamma)$ is rational number. □

5. $RE_P^D(\Gamma)$ for Various Standard Graphs

Theorem 5.1 *For a complete graph K_n, we have $RE_P^D(K_n) = \frac{4n-8}{n-1}$ where $n \geq 3$*

Proof: Let K_n be a complete graph with vertex set $V(\Gamma) = \{v_1, v_2, v_3, \ldots, v_n\}$. The minimum pendant dominating randic set is $S = \{v_1, v_2\}$ and the minimum pendant dominating randic matrix is

$$R_P^D(K_n) = \begin{pmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \end{pmatrix}_{n\times n}$$

The spectral polynomial of $R_P^D(K_n)$ is

$$f_n(K_n, \eta) = \begin{pmatrix} \eta - 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \eta - 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \eta & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \eta & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & \eta \end{pmatrix}_{n\times n}$$

$$f_n(K_n,\eta)=\Big(\eta-(\tfrac{n-2}{n-1})\Big)\Big(\eta+\tfrac{1}{n-1}\Big)^{n-3}\Big(\eta^2-(\tfrac{2n-3}{n-1})\eta+(\tfrac{n-4}{n-1})\Big)=0$$

and the spectrum of K_n is

$$Spec_P^D(K_n)=\begin{pmatrix}\frac{(2n-3)+\sqrt{8n-7}}{2(n-1)} & \frac{n-2}{n-1} & \frac{(2n-3)-\sqrt{8n-7}}{2(n-1)} & -\frac{1}{n-1}\\ 1 & 1 & 1 & (n-3)\end{pmatrix}$$

Hence, the minimum pendant dominating randic energy of a complete graph K_n is

$$RE_P^D(K_n)=\tfrac{4n-8}{n-1}$$

□

Theorem 5.2 *For a complete bipartite graph $K_{n,n}$, we have $RE_P^D(K_{n,n})=2+\frac{2\sqrt{n-1}}{\sqrt{n}}$ where $n\geq 2$*

Proof: Let $K_{n,n}$ be a complete bipartite graph with vertex set $V(\Gamma)=\{u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n\}$. The minimum pendant dominating randic set is $S=\{u_1,v_1\}$ and the minimum pendant dominating randic matrix is

$$R_P^D(K_{n,n})=\begin{pmatrix}1 & 0 & \cdots & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ 0 & 0 & \cdots & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 1 & 0 & \cdots & 0\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 & \cdots & 0\end{pmatrix}_{2n\times 2n}$$

The spectral polynomial of $R_P^D(K_{n,n})$ is

$$f_n(K_{n,n},\eta)=\begin{pmatrix}\eta-1 & 0 & \cdots & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ 0 & \eta & \cdots & 0 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \eta & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \eta-1 & 0 & \cdots & 0\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \eta & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & 0 & \cdots & \eta\end{pmatrix}_{2n\times 2n}$$

$$f_n(K_{n,n},\eta)=\eta^{2n-4}\Big(\eta^2-(\tfrac{n-1}{n})\Big)\Big(\eta^2-2\eta+(\tfrac{n-1}{n})\Big)=0$$

and the spectrum of $K_{n,n}$ is

$$Spec_P^D(K_{n,n})=\begin{pmatrix}1+\frac{1}{\sqrt{n}} & \frac{\sqrt{n-1}}{\sqrt{n}} & 1-\frac{1}{\sqrt{n}} & 0 & -\frac{\sqrt{n-1}}{\sqrt{n}}\\ 1 & 1 & 1 & (2n-4) & 1\end{pmatrix}$$

Hence, the minimum pendant dominating randic energy of a complete bipartite graph $K_{n,n}$ is

$$RE_P^D(K_{n,n})=2+\tfrac{2\sqrt{n-1}}{\sqrt{n}}$$

□

Theorem 5.3 *For a double star graph $S_{n,n}$, we have $RE_P^D(S_{n,n})=\frac{\sqrt{5n^2+8n+4}+\sqrt{5n^2+4n}}{n+1}$ where $n\geq 2$*

Proof: Let $S_{n,n}$ be a double star graph with the vertex set $V(\Gamma) = \{v_0, v_1, v_2, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$, where v_0 and u_0 are the two central vertices. The minimum pendant dominating randic set is $S = \{v_0, u_0\}$ and the minimum pendant dominating randic matrix is

$$R_P^D(S_{n,n}) = \begin{pmatrix} 1 & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} & \frac{1}{n+1} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n+1}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n+1}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{n+1} & 0 & \cdots & 0 & 1 & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

The spectral polynomial of $R_P^D(S_{n,n})$ is

$$f_n(S_{n,n},\eta) = \begin{pmatrix} \eta-1 & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} & \frac{1}{n+1} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n+1}} & \eta & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n+1}} & 0 & \cdots & \eta & 0 & 0 & \cdots & 0 \\ \frac{1}{n+1} & 0 & \cdots & 0 & \eta-1 & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & \eta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & \cdots & \eta \end{pmatrix}_{2n\times 2n}$$

$$f_n(S_{n,n},\eta) = \eta^{2n-2}\Big((n+1)\eta^2 - (n+2)\eta - n\Big)\Big((n+1)\eta^2 - n\eta - n\Big) = 0$$

and the spectrum of $S_{n,n}$ is

$$Spec_P^D(S_{n,n}) = \begin{pmatrix} \frac{(n+2)\pm\sqrt{5n^2+8n+4}}{2(n+1)} & 0 & \frac{n\pm\sqrt{5n^2+4n}}{2(n+1)} \\ 1 & (2n-2) & 1 \end{pmatrix}$$

Therefore, the minimum pendant dominating randic energy of $S_{n,n}$ is

$$RE_P^D(S_{n,n}) = \frac{\sqrt{5n^2+8n+4}+\sqrt{5n^2+4n}}{n+1}$$

□

Theorem 5.4 *For a cocktail party graph $K_{2\times n}$ is $RE_P^D(K_{2\times n}) = \frac{2\sqrt{4n^2-16n+21}+2\sqrt{n^2-2n+2}+(3n-7)}{2(n-1)}$ where $n \geq 2$*

Proof: Let $K_{2\times n}$ be a cocktail party graph having the vertex set $V(\Gamma) = \cup_{i=1}^{n}$. The minimum pendant dominating randic set is $S = \{u_1, u_2\}$ and the minimum pendant dominating randic matrix is

$$R_P^D(K_{2\times n}) = \begin{pmatrix} 1 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 1 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 \\ 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

The spectral polynomial of $R_P^D(K_{2\times n})$ is

$$f_n(K_{2\times n},\phi)=\begin{pmatrix} \eta-1 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \eta-1 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \eta & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 \\ 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \eta & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \eta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \eta \end{pmatrix}_{2n\times 2n}$$

$$f_n(K_{2\times n},\eta)=\eta^{n-2}\big((n-1)\eta+1\big)^{n-3}(2\eta-1)\big((2n-2)\eta^2-(2n-4)\eta-1\big)\big((n-1)\eta^2-(2n-5)\eta-1\big)=0$$

and the spectrum of $K_{2\times n}$ is

$$Spec_P^D(K_{2\times n})=\begin{pmatrix} \frac{(2n-5)\pm\sqrt{4n^2-16n+21}}{2(n-1)} & \frac{(n-2)\pm\sqrt{n^2-2n+2}}{2(n-1)} & \frac{1}{2} & 0 & -\frac{1}{n-1} \\ 1 & 1 & 1 & (n-2) & (n-3) \end{pmatrix}$$

Hence, the minimum pendant dominating randic energy of $K_{2\times n}$ is

$$RE_P^D(K_{2\times n})=\frac{2\sqrt{4n^2-16n+21}+2\sqrt{n^2-2n+2}+(3n-7)}{2(n-1)}$$

□

Theorem 5.5 *For a barbell graph $B_{n,n}$ where $n\geq 3$, we have* $RE_P^D(B_{n,n})=\frac{17n^2-25n+6}{3n^2-3n}$

Proof: Let $B_{n,n}$ be a barbell graph having the vertex set $V(\Gamma)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$. The minimum pendant dominating randic set is $S=\{u_1,v_1\}$ and the minimum pendant dominating randic matrix is

$$R_P^D(B_{n,n})=\begin{pmatrix} 1 & \frac{1}{\sqrt{6(n-2)}} & \cdots & \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6(n-2)}} & 0 & \cdots & \frac{1}{n-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n-1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & 0 & \cdots & 0 & 1 & \frac{1}{\sqrt{6(n-2)}} & \cdots & \frac{1}{\sqrt{6(n-2)}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6(n-2)}} & 0 & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n-1} & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

The spectral polynomial of $R_P^D(B_{n,n})$ is

$$f_n(B_{n,n},\eta)=\begin{pmatrix} \eta-1 & \frac{1}{\sqrt{6(n-2)}} & \cdots & \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6(n-2)}} & \eta & \cdots & \frac{1}{n-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n-1} & \cdots & \eta & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & 0 & \cdots & 0 & \eta-1 & \frac{1}{\sqrt{6(n-2)}} & \cdots & \frac{1}{\sqrt{6(n-2)}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6(n-2)}} & \eta & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6(n-2)}} & \frac{1}{n-1} & \cdots & \eta \end{pmatrix}_{2n\times 2n}$$

$$f_n(B_{n,n},\eta)=\big(n\eta+1\big)^{2n-2}\big((6n-6)\eta^2-(16n-25)\eta+(7n-14)\big)\big((6n-6)\eta^2-(6n-1)\eta+n\big)=0$$

and the spectrum of $B_{n,n}$ is

$$Spec_P^D(B_{n,n}) = \begin{pmatrix} \frac{(16n-25)\pm\sqrt{88n^2-296n+289}}{12(n-1)} & \frac{(6n-1)\pm\sqrt{12n^2+12n+1}}{12(n-1)} & -\frac{1}{n} \\ 1 & 1 & (2n-2) \end{pmatrix}$$

Hence, the minimum pendant dominating randic energy of $B_{n,n}$ is

$$RE_P^D(B_{n,n}) = \frac{17n^2-25n+6}{3n^2-3n}$$

□

6. Conclusion

The domination theory and graph theory are both central topics in modern graph theory, with a wide range of applications in various fields such as network theory, computer science, chemistry and even physics. In recent years many scholars are working in this area and also they are introducing new domination parameters. In this research paper, we have initiated the domination parameter for randic energy and also we have made significant progress in our research by calculating the exact values of the minimum pendant dominating randic energy for standard families of graphs and establishing bounds for this parameter in terms of other graph properties.

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