



Lacunary Difference Sequence Spaces of Complex Uncertain Functions: A Geometric Perspective

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ABSTRACT: In this paper, we introduce a new class of sequence spaces of complex uncertain valued functions based on lacunary sequences and higher-order difference operators. We define an uncertain modular and corresponding Luxemburg norm to study the geometric and topological structure of the space. We investigate properties such as k -nearly uniform convexity (k -NUC), reflexivity, property (β) , and drop property. We also show that the space is not rotund in general.

Key Words: Uncertain normed space, lacunary sequence, difference sequence, Luxemburg norm, k -NUC, property (β) , rotundity.

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1. Introduction

Uncertainty Theory has emerged as a powerful analytical framework to handle imprecise, subjective, or expert-based information, where probability theory is insufficient. Introduced by Liu [4], Uncertainty theory lays the concept of uncertain variables and which in turn leads to sequences of complex uncertain variables. The convergence concepts have been extensively explored by several researchers in recent years [21,22].

Lacunary sequence spaces were first studied by Freedman et al. [2], who introduced the space N_{θ} of lacunary strongly convergent sequences. These spaces study convergence behavior by taking sequences over increasing intervals determined by a lacunary sequence $\theta = (k_r)$, where $k_r - k_{r-1} \rightarrow \infty$. Recently, Dowari and Tripathy have integrated the concepts of complex uncertain sequences with lacunary convergence frameworks [17,16,15,14].

The notion of difference sequences was introduced by Kizmaz [7]. Then Esi, Tripathy and Sarma [1] further the generalized difference sequence spaces. Subsequently, Karakaya [20] and Savaş et al. [6] extended this framework to study geometric properties such as uniform convexity and reflexivity in such spaces. Cesàro-type difference sequence spaces, such as those introduced by Et [10], have also been combined with lacunary concept to construct more general settings for analysis. These constructions often employ modular functionals and Luxemburg norms, enabling the study of reflexivity, nearly uniform convexity

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(NUC), the drop property [18], and property (β) [5] in a unified framework. The concept of difference sequences has been studied by many researcher through the lens of Fuzzy ([9], [3], [23]) as well as in terms of uncertainty theory ([12], [19]).

In spite of recent developments in both lacunary sequence spaces and uncertain analysis, few attempts have been made to study the concept of uncertain valued functions. More recently Dowari et al. [13] introduced lacunary convergence of complex uncertain valued functions. In this paper, we aim to introduce a new class of sequence spaces of complex uncertain-valued functions involving lacunary sequences and difference operators. Specifically, we define the space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$, where each term is an uncertain-valued function defined on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The modular structure is constructed pointwise using the generalized difference operator Δ^m , and the Luxemburg norm is defined via integration with respect to the uncertainty measure \mathcal{M} .

Moreover, we study the geometric properties of this space, including k -nearly uniform convexity (k -NUC), property (β) , reflexivity, and drop property. We also show that the space is generally not rotund. These results generalize earlier findings in the crisp (classical) setting and open a new direction for analyzing uncertain dynamical systems, fuzzy data models, and decision processes under uncertainty.

2. Preliminaries

In this section, we recall some fundamental concepts from uncertainty theory and classical sequence space theory. These preliminaries form the foundation for constructing and analyzing the proposed uncertain lacunary difference sequence space of uncertain-valued functions.

2.1. Uncertainty Space and Uncertain Normed Spaces

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an *uncertainty space*, where:

- Γ is a nonempty set of elementary outcomes;
- \mathcal{L} is a σ -algebra of subsets of Γ ;
- $\mathcal{M} : \mathcal{L} \rightarrow [0, 1]$ is a *normal, subadditive, and self-dual* uncertainty measure (see [4]).

An *uncertain variable* is a measurable function $f : \Gamma \rightarrow \mathbb{C}$ or \mathbb{R} . A family of uncertain variables $\{f_k\}_{k=1}^{\infty}$, where each $f_k : \Gamma \rightarrow \mathbb{C}$, is called a sequence of complex uncertain-valued functions.

A mapping $\|\cdot\|_{\mathcal{M}} : \mathcal{U} \rightarrow \mathcal{F}(\Gamma, \mathbb{R}^+)$ is called an *uncertain norm* on a linear space \mathcal{U} if for all $\xi, \eta \in \mathcal{U}$, $\alpha \in \mathbb{C}$, and all $\gamma \in \Gamma$, the following hold:

1. $\|\xi\|_{\mathcal{M}}(\gamma) = 0$ for all $\gamma \in \Gamma$ if and only if $\xi = 0$;
2. $\|\alpha\xi\|_{\mathcal{M}}(\gamma) = |\alpha| \cdot \|\xi\|_{\mathcal{M}}(\gamma)$;
3. $\|\xi + \eta\|_{\mathcal{M}}(\gamma) \leq \|\xi\|_{\mathcal{M}}(\gamma) + \|\eta\|_{\mathcal{M}}(\gamma)$.

The pair $(\mathcal{U}, \|\cdot\|_{\mathcal{M}})$ is then called an *uncertain normed linear space*.

2.2. Lacunary Sequences and Difference Operators

A sequence $\theta = (k_r)$ of non-negative integers is called a *lacunary sequence* if:

$$k_0 = 0, \quad k_r < k_{r+1}, \quad \text{and} \quad h_r := k_r - k_{r-1} \rightarrow \infty \quad \text{as } r \rightarrow \infty.$$

We define the intervals $I_r = (k_{r-1}, k_r]$ and the ratio $q_r = \frac{k_r}{k_{r-1}}$, which plays a key role in analyzing summability and inclusion relations.

For a sequence (f_k) of complex uncertain functions, the forward difference operator of order m is defined recursively by:

$$\Delta^0 f_k := f_k, \quad \Delta f_k := f_k - f_{k+1}, \quad \Delta^m f_k := \Delta(\Delta^{m-1} f_k).$$

Explicitly, we have:

$$\Delta^m f_k = \sum_{j=0}^m (-1)^j \binom{m}{j} f_{k+j}.$$

2.3. Classical Lacunary Difference Sequence Spaces

In the classical setting (see [2], [20]), the Cesàro-lacunary difference sequence space is defined by:

$$\ell(p, \theta, \Delta^m) := \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m x_k| \right)^{p_r} < \infty \right\},$$

where $p = (p_r)$ is a bounded sequence of real numbers with $p_r \geq 1$. Such spaces have been shown to possess rich geometric structure, including nearly uniform convexity (NUC), reflexivity, and property (β) [5], [6].

3. Definition of the Space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$

In this section, we introduce a new class of sequence spaces of complex uncertain-valued functions, which generalizes lacunary difference sequence spaces into the framework of uncertainty theory. This space incorporates both the higher-order difference operator and the lacunary averaging concept over an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

Modular Functional

Let $f = (f_k)$ be a sequence of complex uncertain-valued functions, where each $f_k : \Gamma \rightarrow \mathbb{C}$. Define the *lacunary difference modular functional* σ_{Δ^m} pointwise for each $\gamma \in \Gamma$ as

$$\sigma_{\Delta^m}(f)(\gamma) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r},$$

where $\Delta^m f_k$ denotes the m -th order forward difference of the uncertain function sequence, and $p = (p_r)$ is a bounded sequence of positive real numbers with $p_r \geq 1$. The intervals $I_r = (k_{r-1}, k_r]$ are determined by a lacunary sequence $\theta = (k_r)$ with $h_r = k_r - k_{r-1} \rightarrow \infty$.

Luxemburg-Type Uncertain Norm

The corresponding Luxemburg-type norm of the sequence $f = (f_k)$ in the uncertain normed linear space \mathcal{U} is defined by

$$\|f\|_{\mathcal{M}} = \inf \left\{ \rho > 0 : \int_{\Gamma} \sigma_{\Delta^m} \left(\frac{f}{\rho} \right) (\gamma) d\mathcal{M}(\gamma) \leq 1 \right\}.$$

This norm quantifies the average modular “size” of the uncertain-valued sequence f , integrated over the uncertainty space.

The Space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$

Using the above modular and norm, we define the lacunary difference sequence space of complex uncertain-valued functions as

$$\ell(p, \theta, \Delta^m)_{\mathcal{M}} := \{ f = (f_k) : \|f\|_{\mathcal{M}} < \infty \}.$$

This space consists of all sequences of complex uncertain functions whose uncertain Luxemburg norm is finite. It extends the classical space $\ell(p, \theta, \Delta^m)$ to the setting of uncertain analysis, thereby facilitating the study of convergence and geometric properties of uncertain function sequences.

4. Geometric Properties of $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$

4.1. Convexity of the Modular Functional

In this section, we show that the modular functional σ_{Δ^m} , defined on the uncertain lacunary difference sequence space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$, is convex pointwise on Γ . This property ensures that the Luxemburg norm derived from it satisfies the triangle inequality and leads to a well-defined modular space structure.

Theorem 4.1 *The functional σ_{Δ^m} defined by*

$$\sigma_{\Delta^m}(f)(\gamma) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r},$$

is a convex modular on \mathcal{U} for each fixed $\gamma \in \Gamma$.

Proof: Let $f = (f_k), g = (g_k) \in \mathcal{U}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. For each $k \in \mathbb{N}$, define the convex combination

$$h_k = \alpha f_k + \beta g_k.$$

Then, by linearity of the difference operator, we have

$$\Delta^m h_k = \alpha \Delta^m f_k + \beta \Delta^m g_k.$$

Now fix $\gamma \in \Gamma$. Using convexity of the real-valued function $t \mapsto |t|^{p_r}$ (which is convex for $p_r \geq 1$), and Jensen's inequality, we obtain:

$$|\Delta^m h_k(\gamma)| \leq \alpha |\Delta^m f_k(\gamma)| + \beta |\Delta^m g_k(\gamma)|.$$

Therefore, for each $r \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} |\Delta^m h_k(\gamma)| &\leq \frac{1}{h_r} \sum_{k \in I_r} (\alpha |\Delta^m f_k(\gamma)| + \beta |\Delta^m g_k(\gamma)|) \\ &= \alpha \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right) + \beta \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m g_k(\gamma)| \right). \end{aligned}$$

Since $p_r \geq 1$, the function $t \mapsto t^{p_r}$ is convex, and applying it to the above inequality gives:

$$\begin{aligned} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m h_k(\gamma)| \right)^{p_r} &\leq \left[\alpha \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right) + \beta \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m g_k(\gamma)| \right) \right]^{p_r} \\ &\leq \alpha \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r} + \beta \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m g_k(\gamma)| \right)^{p_r}. \end{aligned}$$

Summing over $r \in \mathbb{N}$ and $k \in I_r$, we obtain,

$$\sigma_{\Delta^m}(h)(\gamma) \leq \alpha \sigma_{\Delta^m}(f)(\gamma) + \beta \sigma_{\Delta^m}(g)(\gamma).$$

Hence, σ_{Δ^m} is convex for each fixed $\gamma \in \Gamma$. □

This convexity result guarantees that the Luxemburg-type norm $\|f\|_{\mathcal{M}}$ defined via the modular σ_{Δ^m} induces a well-structured modular space. It is a foundational step toward proving completeness and geometric properties such as nearly uniform convexity and property (β) .

Theorem 4.2 *The space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$, equipped with the Luxemburg norm*

$$\|f\|_{\mathcal{M}} := \inf \left\{ \rho > 0 : \int_{\Gamma} \sigma_{\Delta^m} \left(\frac{f}{\rho} \right) (\gamma) d\mathcal{M}(\gamma) \leq 1 \right\},$$

is a complete normed space.

Proof: Let $(f^{(n)}) \subset \ell(p, \theta, \Delta^m)_{\mathcal{M}}$ be a Cauchy sequence with respect to the Luxemburg norm $\|\cdot\|_{\mathcal{M}}$. That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|f^{(n)} - f^{(m)}\|_{\mathcal{M}} < \varepsilon.$$

By the definition of the Luxemburg norm, this implies:

$$\int_{\Gamma} \sigma_{\Delta^m} \left(\frac{f^{(n)} - f^{(m)}}{\varepsilon} \right) (\gamma) d\mathcal{M}(\gamma) \leq 1.$$

Thus, the sequence $(f^{(n)})$ is Cauchy in the modular sense, and hence the sequence $(\sigma_{\Delta^m}(f^{(n)}(\gamma)))$ is Cauchy in $L^1(\mathcal{M})$ (since integration over Γ is finite). Therefore, for each $\gamma \in \Gamma$, the sequence of functions $f^{(n)}(\gamma)$ is Cauchy in the normed space of sequences $\ell(p, \theta, \Delta^m)$.

We define the point-wise limit sequence,

$$f_k(\gamma) := \lim_{n \rightarrow \infty} f_k^{(n)}(\gamma), \quad \text{for all } k \in \mathbb{N}, \gamma \in \Gamma.$$

This defines a new sequence $f = (f_k)$ of complex uncertain-valued functions. Now we aim to show that $f \in \ell(p, \theta, \Delta^m)_{\mathcal{M}}$, i.e., $\|f\|_{\mathcal{M}} < \infty$, and that $f^{(n)} \rightarrow f$ in norm.

By Fatou's lemma and the convexity of σ_{Δ^m} , we get:

$$\int_{\Gamma} \sigma_{\Delta^m}(f - f^{(n)})(\gamma) d\mathcal{M}(\gamma) \leq \liminf_{m \rightarrow \infty} \int_{\Gamma} \sigma_{\Delta^m}(f^{(m)} - f^{(n)})(\gamma) d\mathcal{M}(\gamma).$$

This tends to zero as $n \rightarrow \infty$, hence $\|f - f^{(n)}\|_{\mathcal{M}} \rightarrow 0$.

Therefore, $f \in \ell(p, \theta, \Delta^m)_{\mathcal{M}}$, and $f^{(n)} \rightarrow f$ in norm. Thus, the space is complete. \square

Corollary 4.1 *The modular functional σ_{Δ^m} , together with the Luxemburg norm $\|\cdot\|_{\mathcal{M}}$, generates a complete modular space of complex uncertain-valued functions.*

Theorem 4.3 *Let $1 < \inf p_r \leq \sup p_r < \infty$. Then the space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is k -nearly uniformly convex for each $k \in \mathbb{N}$.*

Proof: Let $(f^{(n)}) \subset \ell(p, \theta, \Delta^m)_{\mathcal{M}}$ be a sequence such that

$$\|f^{(n)}\|_{\mathcal{M}} \leq 1 \quad \text{for all } n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow \infty} \|f^{(n)} - f^{(m)}\|_{\mathcal{M}} \geq \varepsilon > 0 \quad \text{for all } m \neq n.$$

Our goal is to show that

$$\left\| \frac{1}{k} \sum_{j=1}^k f^{(n_j)} \right\|_{\mathcal{M}} < 1 - \delta$$

for some $\delta > 0$ and suitable subsequence $\{f^{(n_j)}\}_{j=1}^k$.

Since $\|f^{(n)}\|_{\mathcal{M}} \leq 1$, by definition of the Luxemburg norm, we have

$$\int_{\Gamma} \sigma_{\Delta^m}(f^{(n)})(\gamma) d\mathcal{M}(\gamma) \leq 1 \quad \text{for all } n.$$

Fix $\gamma \in \Gamma$. For each r , define

$$A_r^{(n)}(\gamma) := \left(\frac{1}{h_r} \sum_{k \in I_r} \left| \Delta^m f_k^{(n)}(\gamma) \right| \right)^{p_r}.$$

Then for all n ,

$$\sigma_{\Delta^m}(f^{(n)})(\gamma) = \sum_{r=1}^{\infty} A_r^{(n)}(\gamma).$$

Since the sequence $(f^{(n)})$ is uniformly bounded in modular norm and mutually separated, by a standard diagonal argument and convexity of $t \mapsto t^{p_r}$, we can extract a subsequence $\{f^{(n_j)}\}_{j=1}^k$ such that the average

$$f^{[k]} := \frac{1}{k} \sum_{j=1}^k f^{(n_j)}$$

satisfies, for all $\gamma \in \Gamma$,

$$\sigma_{\Delta^m}(f^{[k]})(\gamma) < (1 - \delta') \sum_{r=1}^{\infty} \left(\frac{1}{k} \sum_{j=1}^k \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k^{(n_j)}(\gamma)| \right)^{p_r} \right),$$

for some $\delta' > 0$ depending on ε and p_r . Integrating over Γ gives

$$\int_{\Gamma} \sigma_{\Delta^m}(f^{[k]})(\gamma) d\mathcal{M}(\gamma) < 1 - \delta,$$

for some $\delta > 0$, which implies

$$\|f^{[k]}\|_{\mathcal{M}} < 1.$$

Hence the space is k -nearly uniformly convex. \square

Corollary 4.2 *The space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is reflexive.*

Proof: It is well-known that every uniformly convex Banach space is reflexive. The space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is a modular space with a convex modular σ_{Δ^m} , and under the condition $1 < \inf p_r \leq \sup p_r < \infty$, it is k -nearly uniformly convex for each $k \in \mathbb{N}$. Nearly uniform convexity (NUC) implies reflexivity (see [5]). Hence, $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is reflexive. \square

Corollary 4.3 *The space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ has property (β) .*

Proof: Property (β) , introduced by Rolewicz [18], is a geometric property related to fixed point theory and best approximation. It is known that every k -nearly uniformly convex space has property (β) . Since $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is shown to be k -NUC, it follows that it also has property (β) . \square

Example 4.1 *Let $\Gamma = [0, 1]$, equipped with the uncertainty measure $\mathcal{M}(A) = \sup_{\gamma \in A} \gamma$, and let $\theta = (k_r)$ with $k_r = 2^r$. Define the sequence $f = (f_k)$, where each*

$$f_k(\gamma) = \frac{\gamma}{k}, \quad \text{for } \gamma \in \Gamma.$$

Then $\Delta^m f_k(\gamma) = O(k^{-m-1})$, so we estimate:

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| = O \left(\frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{1}{k^{m+1}} \right) \leq C \cdot \frac{1}{h_r} \cdot \frac{1}{k_{r-1}^m}.$$

Since $k_{r-1} = 2^{r-1}$ and $h_r = 2^{r-1}$, we obtain:

$$\left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r} = O \left(\frac{1}{2^{r(m+1)}} \right).$$

Thus,

$$\sigma_{\Delta^m}(f)(\gamma) = \sum_{r=1}^{\infty} O \left(\frac{1}{2^{r(m+1)}} \right) < \infty.$$

Hence,

$$\|f\|_{\mathcal{M}} = \inf \left\{ \rho > 0 : \int_0^1 \sigma_{\Delta^m}(f/\rho)(\gamma) d\gamma \leq 1 \right\} < \infty,$$

and so $f \in \ell(p, \theta, \Delta^m)_{\mathcal{M}}$.

Example 4.2 Let $f_k(\gamma) = \gamma \cdot \log(k+1)$. Then for fixed $\gamma \in \Gamma$, $f_k(\gamma) \rightarrow \infty$ slowly. Now we compute,

$$|\Delta f_k(\gamma)| = \gamma \cdot (\log(k+1) - \log(k+2)) < 0.$$

So,

$$|\Delta f_k(\gamma)| \approx \gamma \cdot \frac{1}{k+1}, \quad \text{and hence } |\Delta^m f_k(\gamma)| \sim \frac{\gamma}{k}.$$

Then for $p_r = 1$, we obtain,

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \approx \frac{\gamma}{h_r} \sum_{k \in I_r} \frac{1}{k} \approx \frac{\gamma \log(k_r)}{h_r}.$$

If θ grows rapidly (say $k_r = r$), then $h_r \gg \log(k_r)$, and the expression tends to zero. However, if $k_r = r^2$, then $h_r = 2r + 1$, and

$$\frac{\log k_r}{h_r} \sim \frac{\log r^2}{2r} \sim \frac{2 \log r}{2r} = \frac{\log r}{r},$$

which does not sum over r . Therefore, for such lacunary sequence θ , $f \notin \ell(p, \theta, \Delta^m)_{\mathcal{M}}$, despite $f_k(\gamma) \rightarrow \infty$ slowly. Hence, $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is a proper subspace of \mathcal{U} .

Proposition 4.1 Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_{r \rightarrow \infty} \frac{h_r}{k_r} > 0$, and let $1 \leq \inf p_r \leq \sup p_r < \infty$. Then we have the inclusion

$$\ell(p, \Delta^m)_{\mathcal{M}} \subseteq \ell(p, \theta, \Delta^m)_{\mathcal{M}}.$$

Proof: Let $f = (f_k) \in \ell(p, \Delta^m)_{\mathcal{M}}$. Then by definition,

$$\|f\|_{\Delta^m, \mathcal{M}} := \inf \left\{ \rho > 0 : \int_{\Gamma} \sum_{k=1}^{\infty} |\Delta^m f_k(\gamma)|^{p_k} d\mathcal{M}(\gamma) \leq \rho \right\} < \infty.$$

Now for fixed $\gamma \in \Gamma$,

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \leq \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)|^{p_k} \right)^{1/p_r},$$

using Jensen's inequality (since $p_k \geq 1$).

Now,

$$\sigma_{\Delta^m}(f)(\gamma) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r} \leq \sum_{r=1}^{\infty} \sum_{k \in I_r} |\Delta^m f_k(\gamma)|^{p_k} = \sum_{k=1}^{\infty} |\Delta^m f_k(\gamma)|^{p_k}.$$

Integrating over Γ , we get:

$$\int_{\Gamma} \sigma_{\Delta^m}(f)(\gamma) d\mathcal{M}(\gamma) \leq \int_{\Gamma} \sum_{k=1}^{\infty} |\Delta^m f_k(\gamma)|^{p_k} d\mathcal{M}(\gamma) < \infty.$$

Thus $f \in \ell(p, \theta, \Delta^m)_{\mathcal{M}}$. □

Example 4.3 The inclusion is strict. i.e.,

$$\ell(p, \Delta^m)_{\mathcal{M}} \subsetneq \ell(p, \theta, \Delta^m)_{\mathcal{M}}.$$

Let $\theta = (k_r)$ with $k_r = 2^r$, and define $f_k(\gamma) = \gamma \cdot \chi_{[k_r]}(k)$, where

$$\chi_{[k_r]}(k) = \begin{cases} 1 & \text{if } k = k_r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_k(\gamma) = 0$ except when $k = k_r$, where it equals γ . Hence, the difference $\Delta^m f_k(\gamma)$ is zero almost everywhere, except at isolated points.

Thus we have,

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta^m f_k(\gamma)| \right)^{p_r} = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \cdot |\gamma| \cdot \chi_{[k_r]}(k_r) \right)^{p_r} = \sum_{r=1}^{\infty} \left(\frac{\gamma}{h_r} \right)^{p_r},$$

which converges for suitable p_r and fast-growing h_r .

But,

$$\sum_{k=1}^{\infty} |\Delta^m f_k(\gamma)|^{p_k} = \sum_{r=1}^{\infty} |\gamma|^{p_{k_r}} = \infty.$$

Hence $f \in \ell(p, \theta, \Delta^m)_{\mathcal{M}} \setminus \ell(p, \Delta^m)_{\mathcal{M}}$, so the inclusion is strict.

Theorem 4.4 Let $\inf p_r > 1$. Then the space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is rotund (strictly convex).

Proof: Let $f = (f_k)$, $g = (g_k) \in \ell(p, \theta, \Delta^m)_{\mathcal{M}}$ such that $f \neq g$ and

$$\left\| \frac{f+g}{2} \right\|_{\mathcal{M}} = \|f\|_{\mathcal{M}} = \|g\|_{\mathcal{M}} = 1.$$

We show this leads to a contradiction.

Since the modular σ_{Δ^m} is convex and strictly convex for $p_r > 1$, we have,

$$\sigma_{\Delta^m} \left(\frac{f+g}{2} \right) (\gamma) < \frac{1}{2} \sigma_{\Delta^m}(f)(\gamma) + \frac{1}{2} \sigma_{\Delta^m}(g)(\gamma), \quad \text{for } \gamma \in \Gamma.$$

Now, integrating over Γ we have,

$$\int_{\Gamma} \sigma_{\Delta^m} \left(\frac{f+g}{2} \right) (\gamma) d\mathcal{M}(\gamma) < \frac{1}{2} \int_{\Gamma} \sigma_{\Delta^m}(f)(\gamma) d\mathcal{M}(\gamma) + \frac{1}{2} \int_{\Gamma} \sigma_{\Delta^m}(g)(\gamma) d\mathcal{M}(\gamma) \leq 1.$$

This implies,

$$\left\| \frac{f+g}{2} \right\|_{\mathcal{M}} < 1,$$

which contradicts our assumption. Hence, equality implies $f = g$, and the norm is strictly convex. \square

Example 4.4 (Counter Example) Let $p_r = 1$ for all $r \in \mathbb{N}$. We define two sequences,

$$f_k(\gamma) = \gamma \cdot \chi_{2\mathbb{N}}(k), \quad g_k(\gamma) = \gamma \cdot \chi_{2\mathbb{N}+1}(k).$$

That is,

$$f_k(\gamma) = \begin{cases} \gamma & \text{if } k \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \quad g_k(\gamma) = \begin{cases} \gamma & \text{if } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\Delta^m f_k(\gamma), \Delta^m g_k(\gamma) \in \{0, \gamma, -\gamma\}, \quad \text{for all } k.$$

It can be obtained that,

$$\sigma_{\Delta^m}(f)(\gamma) = \sigma_{\Delta^m}(g)(\gamma) = \sigma_{\Delta^m} \left(\frac{f+g}{2} \right) (\gamma), \quad \forall \gamma \in \Gamma.$$

So,

$$\left\| \frac{f+g}{2} \right\|_{\mathcal{M}} = \|f\|_{\mathcal{M}} = \|g\|_{\mathcal{M}},$$

but clearly $f \neq g$, and $f+g$ is not equal to either.

Hence, the norm is not strictly convex when $p_r = 1$, and rotundity fails.

Theorem 4.5 *Let $\inf p_r > 1$. Then the space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ has the Drop Property.*

Proof: We know that the Drop Property is implied by the rotundity of the space (see [11], [8]). And we have already shown that $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ is strictly convex (rotund) when $\inf p_r > 1$.

Let $C \subset \ell(p, \theta, \Delta^m)_{\mathcal{M}}$ be a closed convex set, and let $f \notin C$. Then, by rotundity, the metric projection

$$\Pi_C(f) := \left\{ g \in C : \|f - g\|_{\mathcal{M}} = \inf_{h \in C} \|f - h\|_{\mathcal{M}} \right\}$$

is a singleton, say $\{g_0\}$, and this projection is unique.

Now we define the drop,

$$D(f, C) := \{tf + (1 - t)g : t \in (0, 1], g \in C\}.$$

The drop property implies,

$$\overline{D(f, C)} \cap C = \{g_0\}.$$

Since the norm is strictly convex and the projection is unique, no point of C other than g_0 lies in the closure of the drop. Hence, the space satisfies the Drop Property. \square

Corollary 4.4 *Every closed convex subset of $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$ has a unique best approximation point, and the space supports drop convergence methods.*

5. Examples

- **Example 1.** We define uncertain-valued functions $f_k : \Gamma \rightarrow \mathbb{R}$ by

$$f_k(\gamma) = \frac{\gamma}{k}, \quad \text{for all } \gamma \in \Gamma = [0, 1].$$

Let $\theta = (k_r)$ be defined by $k_r = 2^r$, so that $h_r = k_r - k_{r-1} = 2^{r-1}$, and take $p_r = 2$, $m = 1$. Then

$$\Delta f_k(\gamma) = f_k(\gamma) - f_{k+1}(\gamma) = \gamma \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{\gamma}{k(k+1)}.$$

Therefore,

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta f_k(\gamma)| = \frac{\gamma}{h_r} \sum_{k \in I_r} \frac{1}{k(k+1)} \leq \frac{\gamma}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{1}{k^2}.$$

Since the series $\sum \frac{1}{k^2}$ converges and the sum over a lacunary block is finite, we have

$$\left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta f_k(\gamma)| \right)^{p_r} = O\left(\frac{1}{2^{2r}}\right).$$

Hence,

$$\sigma_{\Delta}(f)(\gamma) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |\Delta f_k(\gamma)| \right)^2 < \infty.$$

Thus,

$$\|f\|_{\mathcal{M}} = \inf \left\{ \rho > 0 : \int_0^1 \sigma_{\Delta} \left(\frac{f}{\rho} \right) (\gamma) d\gamma \leq 1 \right\} < \infty.$$

Therefore, $f \in \ell(p, \theta, \Delta^1)_{\mathcal{M}}$.

- **Example 2 (Non-inclusion).** Let

$$f_k(\gamma) = \gamma \cdot \log(k+1), \quad \text{for } \gamma \in \Gamma = [0, 1].$$

Then

$$\Delta f_k(\gamma) = \gamma (\log(k+1) - \log(k+2)) \approx \gamma \cdot \left(-\frac{1}{k+1} \right),$$

so

$$|\Delta f_k(\gamma)| \sim \frac{\gamma}{k+1}.$$

Choose $k_r = r^2$, so $h_r = 2r + 1$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta f_k(\gamma)| \sim \frac{\gamma}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{1}{k} \sim \frac{\gamma \cdot \log(r)}{r}.$$

Taking $p_r = 1$, we find

$$\sigma_{\Delta}(f)(\gamma) = \sum_{r=1}^{\infty} \left(\frac{\gamma \cdot \log r}{r} \right),$$

which diverges for any $\gamma > 0$. Thus,

$$f \notin \ell(p, \theta, \Delta^1)_{\mathcal{M}}.$$

6. Conclusion

In this paper, we have introduced and studied the uncertain modular sequence space $\ell(p, \theta, \Delta^m)_{\mathcal{M}}$, which extends classical lacunary and difference sequence spaces into the framework of uncertain normed modular spaces by incorporating uncertainty through an uncertainty measure \mathcal{M} , with a Luxemburg-type modular norm based on generalized differences. We established core properties such as completeness, k -nearly uniform convexity, reflexivity, and the Drop Property. This study further enhances the geometric and topological understanding of uncertain sequence spaces of uncertain valued functions and offers a foundation for future work in fixed point theory, or extending such spaces to statistical convergence or to Orlicz type spaces.

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