Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 1** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78084

Characterizations of Classical and Quantum Weak Difference Sequence Spaces

Sujata Saikia* and Pranab Jyoti Dowari

ABSTRACT: This paper investigates the concept of weak difference sequence spaces in normed linear spaces, focusing on both classical and quantum difference operators. We define and examine the spaces $\mathcal{WD}(X)$ and $\mathcal{WD}^k(X)$, which consist of sequences whose first- and higher-order differences converge weakly to zero. Additionally, we introduce a new class of sequence spaces $\mathcal{WD}_q(X)$, governed by quantum difference operators arising from q-calculus. We establish fundamental properties of these spaces, including linearity, stability under continuous mappings and behavior under scalar multiplication. Through various examples and counterexamples, we highlight subtle distinctions between classical and quantum weak convergence. The results provide a unified framework for analyzing discrete weak convergence phenomena and lay the groundwork for further developments in sequence space theory and quantum analysis.

Key Words: Difference sequence spaces, weak convergence, quantum difference.

Contents

1. Introduction

Weak convergence plays a significant role in modern analysis due to its flexibility and applicability in infinite-dimensional settings. Unlike strong (norm) convergence, weak convergence requires only convergence under the action of every continuous linear functional, thus enabling a broader class of converging sequences. This concept has influenced the development of summability theory, approximation theory and the study of functional spaces [5,7,13].

Difference sequence spaces have been studied extensively since the pioneering work of Kizmaz [4], who introduced the spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. These spaces, formed using the difference operator $\Delta x_n = x_{n+1} - x_n$, became foundational in understanding the behavior of sequences with respect to their successive differences. Further generalizations by Maddox [6], and others involved concepts such as statistical convergence and almost convergence in these contexts. In the recent past, the lacunary sequences have been explored by several researcher [2,8,9,11,12,14]. Quite recently, Dowari and Tripathy [10] studied the difference sequence spaces for complex uncertain variables.

Despite the growing literature on weak convergence and difference sequence spaces separately, only limited work has been done on their intersection. One may refer to [2].

The present study addresses this gap by introducing weak difference sequence spaces in normed spaces. Specifically, we define the spaces $\mathcal{WD}(X)$ and its higher-order extensions $\mathcal{WD}^k(X)$, capturing sequences whose successive differences converge weakly to zero. These spaces allow for a refined analysis of convergence behaviors and yield new insights into their structural characteristics.

* Corresponding author.
2020 *Mathematics Subject Classification:* 40A05, 60B10, 60B12, 60F17.
Submitted July 25, 2025. Published January 19, 2026

We extend this investigation into the realm of quantum calculus by defining the quantum weak difference sequence space $\mathcal{WD}_q(X)$. The q-difference operator offers a natural discretization framework with applications in number theory, quantum mechanics, and time scales [1,15]. Our results demonstrate that $\mathcal{WD}_q(X)$ generalizes $\mathcal{WD}(X)$ and recovers it as a special case when $q \to 1$. This quantum framework not only enriches the theory but also invites further studies on weak convergence in non-uniform difference settings [3].

Through characterizations, examples, counterexamples and inclusion results, we establish the foundational theory of weak difference sequence spaces in both classical and quantum contexts. This work opens new avenues in sequence space theory and contributes to the broader understanding of convergence under difference operations.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Denote by X^* the dual space of X, consisting of all continuous linear functionals $f : X \to \mathbb{K}$.

Definition 2.1 *A sequence $(x_n) \subset X$ is said to* converge weakly *to $x \in X$ if for every $f \in X^*$, we have*

$$f(x_n) \to f(x) \quad \text{as } n \to \infty.$$

In this case, we write $x_n \to x$ and call (x_n) weakly convergent*.*

Remark 2.1 *Weak convergence is weaker than norm convergence. That is, norm convergence implies weak convergence, but not conversely.*

Definition 2.2 *The* difference operator Δ *of order one is defined on a sequence (x_n) by*

$$\Delta x_n = x_{n+1} - x_n.$$

Higher-order differences are defined recursively as

$$\Delta^k x_n = \Delta(\Delta^{k-1} x_n), \quad \text{for } k \geq 2.$$

In what follows, we focus on the weak convergence properties of the sequences (Δx_n) and $(\Delta^k x_n)$ in the normed space X and investigate how these relate to the weak convergence of the original sequence (x_n).

3. Weak Convergence of Difference Sequences

Weak convergence of difference sequences arises naturally when investigating the asymptotic behavior of sequences in normed linear spaces. While weak convergence of a sequence (x_n) in a normed space X focuses on the behavior of the sequence under the action of all bounded linear functionals in X^*, the study of weak convergence of the difference sequence (Δx_n) sheds light on the incremental behavior or variation of (x_n) in the weak topology.

Definition 3.1 *Let (x_n) be a sequence in a normed space X. The sequence (x_n) is said to have a* weakly convergent difference sequence *if the sequence (Δx_n) converges weakly to an element $y \in X$, i.e.,*

$$\Delta x_n = x_{n+1} - x_n \to y \quad \text{as } n \to \infty.$$

If $\Delta x_n \to 0$, then we say that the difference sequence is weakly null*.*

This concept is particularly useful for identifying sequences that may not be weakly convergent themselves, but whose differences exhibit a regular asymptotic behavior. Such sequences appear in the context of summability methods, ergodic theory, and time series analysis.

Example 3.1 *Let $X = \ell^2$ and define $x_n = e_n$, the standard orthonormal basis of ℓ^2. Then,*

$$\Delta x_n = e_{n+1} - e_n,$$

and (Δx_n) does not converge weakly because it does not even converge in norm. However, consider $x_n = \sum_{k=1}^{n} \frac{1}{k} e_k$. Then,

$$\Delta x_n = \frac{1}{n+1} e_{n+1} \to 0,$$

since for any $f \in \ell^2$, $f(e_{n+1}) \to 0$ as $n \to \infty$.

The weak convergence of Δx_n provides a way to describe the asymptotic "flattening" of the sequence. This leads to the notion of sequences whose differences are vanishing weakly, even if the sequence itself oscillates or fails to converge weakly.

We may also define spaces associated with this behavior, which will be explored in the next section. These spaces help formalize the analysis and classification of sequences based on the weak convergence properties of their successive differences.

4. Characterizations and Properties

In this section, we present several characterizations and properties of weakly convergent difference sequences. These results provide insight into the structural behavior of sequences whose differences converge weakly.

Theorem 4.1 *Let $(x_n) \subset X$ be a sequence such that $x_n \to x$ and $\Delta x_n \to 0$. Then the sequence is weakly asymptotically constant, i.e., for all $f \in X^*$, the scalar sequence $f(x_n)$ is eventually constant.*

Proof: From $x_n \to x$, we have $f(x_n) \to f(x)$ for every $f \in X^*$. From $\Delta x_n \to 0$, we get $f(x_{n+1}) - f(x_n) \to 0$, i.e., the scalar sequence $(f(x_n))$ is eventually constant. Thus, weak convergence of the sequence and its differences implies the scalar values stabilize asymptotically. □

Example 4.1 *Let $x_n = \log(n) e_1$ in ℓ^2. Then $x_n \nrightarrow x$ weakly because $f(x_n) = \log(n)$ for $f(e_1) = 1$, diverges. Also, $\Delta x_n = (\log(n+1) - \log(n)) e_1 \to 0$ in norm, hence also weakly. This shows that $\Delta x_n \to 0$ does not imply x_n is weakly convergent.*

Theorem 4.2 *If $\Delta x_n \to y \in X$, then for any $f \in X^*$, the scalar difference $f(x_{n+1}) - f(x_n) \to f(y)$.*

Proof: Apply f to both sides: $f(\Delta x_n) = f(x_{n+1}) - f(x_n) \to f(y)$ by weak convergence. □

Remark 4.1 *The convergence of scalar differences does not imply norm convergence or boundedness of the original sequence.*

We provide an example below to justify that the converse part is not true in general.

Example 4.2 *Let $x_n = n e_n$ in ℓ^2. Then $\|x_n\| = n$, which diverges, so (x_n) is not bounded, hence not weakly convergent. But $\Delta x_n = e_{n+1} + (n-1) e_n$, which also diverges in norm and does not converge weakly. This highlights the necessity of boundedness.*

Theorem 4.3 *Let (x_n) be a bounded sequence in a reflexive Banach space X. Then $\Delta x_n \to 0$ implies that every weak subsequential limit of (x_n) is the same.*

Proof: Since X is reflexive, every bounded sequence in X has a weakly convergent subsequence. Let (x_{n_k}) and (x_{m_k}) be two subsequences of (x_n) such that

$$x_{n_k} \to x \quad \text{and} \quad x_{m_k} \to y \quad \text{in the weak topology of } X.$$

Given that $\Delta x_n = x_{n+1} - x_n \to 0$, we know that for every $f \in X^*$,

$$f(x_{n+1} - x_n) \to 0.$$

This implies

$$f(x_{n_k+1}) - f(x_{n_k}) \to 0 \quad \text{and} \quad f(x_{m_k+1}) - f(x_{m_k}) \to 0.$$

Since (x_{n_k}) is weakly convergent to x, and the weak topology is sequentially closed, it follows that $f(x_{n_k}) \to f(x)$ for every $f \in X^*$. Also, because $f(x_{n_k+1}) - f(x_{n_k}) \to 0$, we obtain

$$f(x_{n_k+1}) \to f(x).$$

Hence, the sequence (x_{n_k+1}) also converges weakly to x. Similarly, the sequence (x_{m_k+1}) converges weakly to y.

But since (x_{n_k}) and (x_{n_k+1}) both converge weakly to x, and weak limits are unique in reflexive spaces, it follows that x is the unique weak limit. The same applies to the sequence (x_{m_k}) and (x_{m_k+1}) converging to y.

Therefore, for all $f \in X^*$, we have

$$f(x) = \lim f(x_{n_k}) = \lim f(x_{n_k+1}) = f(y),$$

so $f(x - y) = 0$ for all $f \in X^*$, which implies that $x = y$.

Hence, every weak subsequential limit of (x_n) is the same. □

Theorem 4.4 *If (x_n) is a sequence in a Banach space X such that $\Delta x_n = x_{n+1} - x_n$ is weakly null (i.e., $\Delta x_n \to 0$) and (x_n) is bounded in norm, then (x_n) is weakly precompact (i.e., it has a weakly convergent subsequence).*

Proof: Let X be a Banach space, and suppose that (x_n) is a bounded sequence in X, i.e., there exists $M > 0$ such that $\|x_n\| \leq M$ for all n. Boundedness alone does not guarantee the existence of weakly convergent subsequences in arbitrary Banach spaces, but it does imply that the sequence is contained in a weakly sequentially compact set if X is reflexive. However, in general, we use Eberlein–Šmulian theorem, which states that in Banach spaces, a subset is weakly compact if and only if it is weakly sequentially compact.

Our goal is to show that (x_n) has a weakly convergent subsequence. The assumption $\Delta x_n \to 0$ means that for every $f \in X^*$,

$$f(x_{n+1} - x_n) \to 0.$$

This implies that for every $f \in X^*$, the scalar sequence $(f(x_n))$ is an asymptotically almost constant sequence, meaning its increments tend to zero.

Let us fix any $f \in X^*$. Since (x_n) is bounded and $\Delta x_n \to 0$, we have

$$|f(x_{n+1}) - f(x_n)| = |f(\Delta x_n)| \to 0.$$

Therefore, $(f(x_n))$ is a bounded scalar sequence with vanishing forward differences, which implies that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$ and hence converges.

So for each $f \in X^*$, the sequence $(f(x_n))$ converges in $\mathbb{R}$. That is, the sequence (x_n) is weakly convergent in the weak-* topology of X (pointwise convergence on X^*). However, since (x_n) lies in a Banach space (not the dual space), this argument only shows that the net of functionals applied to x_n converges pointwise, i.e., that the sequence is weakly sequentially compact.

Hence, by the Eberlein–Šmulian theorem, the sequence (x_n) is relatively weakly compact, i.e., it has a weakly convergent subsequence.

Thus, (x_n) is weakly precompact. □

Example 4.3 *Let $\{e_n\}$ be the standard orthonormal basis of ℓ^2, and define the sequence:*

$$x_n = e_1 + \frac{1}{n} e_n.$$

Then (x_n) is bounded in norm, since

$$\|x_n\|^2 = \|e_1\|^2 + \left\| \frac{1}{n} e_n \right\|^2 = 1 + \frac{1}{n^2} \leq 2.$$

Moreover,

$$x_n \to e_1 \quad \textit{in } \ell^2,$$

since for any $x^* = (a_k) \in \ell^2$, we have

$$\langle x_n, x^* \rangle = a_1 + \frac{1}{n} a_n \to a_1 = \langle e_1, x^* \rangle.$$

Now consider the forward difference:

$$\Delta x_n = x_{n+1} - x_n = \frac{1}{n+1} e_{n+1} - \frac{1}{n} e_n.$$

Since both terms tend to zero in norm, and their supports are orthogonal and shifting, we have

$$\Delta x_n \to 0 \quad \textit{in } \ell^2.$$

Thus, this example satisfies the conditions of the theorem, and indeed (x_n) has a weakly convergent subsequence.

5. Weak Difference Sequence Spaces

We now introduce and study sequence spaces defined in terms of weak convergence of difference sequences. These spaces are natural extensions of classical difference sequence spaces but are analyzed with respect to weak topology.

Definition 5.1 *Let X be a normed linear space. Define the space*

$$\mathcal{WD}(X) = \left\{ (x_n) \subset X : \Delta x_n = x_{n+1} - x_n \to 0, \textit{as } n \to \infty \right\}.$$

We call $\mathcal{WD}(X)$ the space of weakly null difference sequences.

Theorem 5.1 *The space $\mathcal{WD}(X)$ is a linear space; that is, if $(x_n), (y_n) \in \mathcal{WD}(X)$ and $\alpha, \beta \in \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), then the sequence $(\alpha x_n + \beta y_n)$ also belongs to $\mathcal{WD}(X)$.*

Proof: Let $(x_n), (y_n) \in \mathcal{WD}(X)$, which means

$$\Delta x_n := x_{n+1} - x_n \to 0 \quad \text{and} \quad \Delta y_n := y_{n+1} - y_n \to 0.$$

We aim to show that

$$\Delta(\alpha x_n + \beta y_n) \to 0.$$

Observe that

$$\Delta(\alpha x_n + \beta y_n) = \alpha(x_{n+1} - x_n) + \beta(y_{n+1} - y_n) = \alpha \Delta x_n + \beta \Delta y_n.$$

Let $f \in X^*$ be any continuous linear functional. Since $\Delta x_n \to 0$ and $\Delta y_n \to 0$, we have

$$f(\Delta x_n) \to 0 \quad \text{and} \quad f(\Delta y_n) \to 0.$$

Then by linearity of f and limits in $\mathbb{K}$,

$$f\left(\Delta(\alpha x_n + \beta y_n)\right) = \alpha f(\Delta x_n) + \beta f(\Delta y_n) \to 0.$$

Since this holds for all $f \in X^*$, it follows that

$$\Delta(\alpha x_n + \beta y_n) \to 0,$$

i.e., $(\alpha x_n + \beta y_n) \in \mathcal{WD}(X)$. Hence, $\mathcal{WD}(X)$ is a linear space. □

Theorem 5.2 *If X is a reflexive Banach space and $(x_n) \in \mathcal{WD}(X)$ is bounded, then the sequence (x_n) is weakly convergent in X.*

Proof: Let X be a reflexive Banach space. Then every bounded sequence in X has a weakly convergent subsequence (by reflexivity). Suppose (x_n) is a sequence in the space $\mathcal{WD}(X)$, i.e.,

$$\Delta x_n = x_{n+1} - x_n \to 0,$$

and assume further that (x_n) is bounded in norm: there exists $M > 0$ such that $\|x_n\| \le M$ for all n.

Since X is reflexive and (x_n) is bounded, by the Banach–Alaoglu theorem and reflexivity, the sequence (x_n) has at least one weakly convergent subsequence. That is, there exists a subsequence (x_{n_k}) such that

$$x_{n_k} \to x \in X.$$

Now, we claim that every weak cluster point of the sequence (x_n) is the same as x. To show this, suppose that (x_{m_k}) is another subsequence such that

$$x_{m_k} \to y \in X.$$

But since $\Delta x_n \to 0$, it follows that for every continuous linear functional $f \in X^*$,

$$f(x_{n+1} - x_n) \to 0.$$

This implies that the sequence of function values $f(x_n)$ is a Cauchy sequence in $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), since

$$f(x_{n+k}) - f(x_n) = \sum_{j=0}^{k-1} f(x_{n+j+1} - x_{n+j}) \to 0 \quad \text{as } n \to \infty.$$

Hence, the scalar sequence $(f(x_n))$ converges for each $f \in X^*$. Therefore, the sequence (x_n) is weakly convergent by the definition of weak convergence.

In particular, this implies that all weakly convergent subsequences of (x_n) must have the same weak limit, since for each $f \in X^*$, the limit $\lim f(x_n)$ exists and is unique. Thus, $f(x) = f(y)$ for all $f \in X^*$, so $x = y$.

Therefore, all weak cluster points coincide, and hence the whole sequence (x_n) converges weakly in X. □

Definition 5.2 *Let X be a Banach space and $k \in \mathbb{N}$. Define the space*

$$\mathcal{WD}^k(X) := \left\{(x_n) \subset X : \Delta^k x_n \to 0\right\},$$

where $\Delta^k x_n$ denotes the k-th order forward difference of the sequence (x_n), defined recursively by

$$\Delta^0 x_n := x_n, \quad \Delta^{k+1} x_n := \Delta^k x_{n+1} - \Delta^k x_n.$$

Then $\mathcal{WD}^k(X)$ is the set of all sequences in X whose k-th order differences converge weakly to zero.

Example 5.1 *Let (e_n) be the standard orthonormal basis in ℓ^2, and consider the sequence*

$$x_n = \sum_{j=1}^{n} \frac{1}{j} e_j.$$

Then (x_n) does not belong to $\mathcal{WD}(\ell^2)$ since the first differences

$$\Delta x_n = x_{n+1} - x_n = \frac{1}{n+1} e_{n+1}$$

do not converge weakly to zero (they do not vanish in norm or even weakly, as they involve disjoint basis vectors). However, the second differences satisfy

$$\Delta^2 x_n = \Delta x_{n+1} - \Delta x_n = \left(\frac{1}{n+2} e_{n+2} - \frac{1}{n+1} e_{n+1} \right) - \left(\frac{1}{n+1} e_{n+1} - \frac{1}{n} e_n \right),$$

which consists of vanishing coefficients and disjoint supports. Since these second-order differences tend weakly to zero, we have $(x_n) \in \mathcal{WD}^2(\ell^2)$. This illustrates that weak convergence of higher-order differences may occur even when lower-order ones do not.

Proposition 5.1 *For every $k \geq 1$, we have the inclusion*

$$\mathcal{WD}^{k+1}(X) \subset \mathcal{WD}^k(X).$$

Proof: Assume $(x_n) \in \mathcal{WD}^{k+1}(X)$, i.e., $\Delta^{k+1} x_n \to 0$. Then the sequence $(\Delta^k x_n)$ has weakly vanishing first differences, i.e.,

$$\Delta(\Delta^k x_n) = \Delta^{k+1} x_n \to 0.$$

This implies that the sequence $(\Delta^k x_n)$ is weakly Cauchy. In Banach spaces, every weakly Cauchy sequence admits a weak limit point, and if its differences converge weakly to zero, then the whole sequence converges weakly. Hence,

$$\Delta^k x_n \to 0,$$

i.e., $(x_n) \in \mathcal{WD}^k(X)$. □

The converse part is not true in general follows from the following example.

Example 5.2 *Let $x_n = \log(n) e_1$ in a Banach space X with a normalized basis (e_n). Then*

$$\Delta x_n = \log(n+1) - \log(n) = \log\left(1 + \frac{1}{n}\right) \to 0.$$

Since the differences are scalar multiples of the fixed vector e_1, and the scalars tend to zero, we have $\Delta x_n \to 0$. Hence $(x_n) \in \mathcal{WD}^1(X)$.

However, the second differences are given by

$$\Delta^2 x_n = \log(n+2) - 2\log(n+1) + \log(n),$$

which does not tend to zero. In fact, this expression oscillates due to the convexity of the logarithm and does not vanish even in the scalar field. Therefore,

$$(x_n) \notin \mathcal{WD}^2(X),$$

showing that the converse inclusion $\mathcal{WD}^k(X) \subset \mathcal{WD}^{k+1}(X)$ fails in general.

Quantum Weak Difference Sequence Spaces

The quantum difference operator, inspired by concepts in q-calculus, offers a generalization of classical differences. It is of particular interest when studying time scales, discrete dynamical systems, or quantum calculus, and allows more flexibility when modeling non-uniform progression.

Definition 5.3 (Quantum Weak Difference Operator) *Let X be a normed linear space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and let $0 < q < 1$ be a fixed real number. For a sequence $(x_n) \subset X$, the* quantum difference operator Δ_q *is defined by*

$$\Delta_q x_n := x_{n+1} - q x_n.$$

Definition 5.4 (Quantum Weak Difference Convergence) *A sequence $(x_n) \subset X$ is said to be* quantum weakly difference convergent *if*

$$\Delta_q x_n = x_{n+1} - q x_n \to 0 \quad \textit{in the weak topology of } X,$$

i.e., for every continuous linear functional $f \in X^*$, we have

$$f(x_{n+1}) - q f(x_n) \to 0 \quad \textit{as } n \to \infty.$$

Definition 5.5 (Quantum Weak Difference Sequence Space) *The set of all quantum weakly difference convergent sequences in X is denoted by*

$$\mathcal{WD}_q(X) := \{(x_n) \subset X : \Delta_q x_n = x_{n+1} - q x_n \to 0\}.$$

Proposition 5.2 *The space $\mathcal{WD}_q(X)$ is a linear space.*

Proof: Let $(x_n), (y_n) \in \mathcal{WD}_q(X)$, i.e.,

$$\Delta_q x_n := x_{n+1} - q x_n \to 0 \quad \text{and} \quad \Delta_q y_n := y_{n+1} - q y_n \to 0 \quad \text{in the weak topology of } X.$$

Let $\alpha, \beta \in \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and define a new sequence $(z_n) \subset X$ by

$$z_n := \alpha x_n + \beta y_n.$$

We aim to show that $(z_n) \in \mathcal{WD}_q(X)$, i.e., $\Delta_q z_n := z_{n+1} - q z_n \to 0$.
We compute:

$$\Delta_q z_n = z_{n+1} - q z_n = \alpha x_{n+1} + \beta y_{n+1} - q(\alpha x_n + \beta y_n) = \alpha(x_{n+1} - q x_n) + \beta(y_{n+1} - q y_n) = \alpha \Delta_q x_n + \beta \Delta_q y_n.$$

Now, since weak convergence is preserved under finite linear combinations, and both $\Delta_q x_n \to 0$ and $\Delta_q y_n \to 0$, it follows that

$$\Delta_q z_n = \alpha \Delta_q x_n + \beta \Delta_q y_n \to 0.$$

Thus, $(z_n) \in \mathcal{WD}_q(X)$. Therefore, $\mathcal{WD}_q(X)$ is closed under addition and scalar multiplication.
Hence, $\mathcal{WD}_q(X)$ is a linear space. □

Proposition 5.3 *Let $(x_n) \in \mathcal{WD}_q(X)$ and let $T : X \to Y$ be a continuous linear operator between normed linear spaces. Then $(T x_n) \in \mathcal{WD}_q(Y)$.*

Proof: By assumption, $(x_n) \in \mathcal{WD}_q(X)$, which means that

$$\Delta_q x_n := x_{n+1} - q x_n \to 0 \quad \text{in } X.$$

We need to show that

$$\Delta_q(T x_n) := T x_{n+1} - q T x_n \to 0 \quad \text{in } Y,$$

i.e., that the quantum differences of the image sequence converge weakly to zero in Y.

Since T is linear, we have:

$$\Delta_q(Tx_n) = Tx_{n+1} - qTx_n = T(x_{n+1}) - T(qx_n) = T(x_{n+1} - qx_n) = T(\Delta_q x_n).$$

Now consider any continuous linear functional $g \in Y^*$. Since $T : X \to Y$ is continuous and linear, the composition $g \circ T \in X^*$ is also a continuous linear functional on X.

Therefore,

$$g(\Delta_q(Tx_n)) = g(T(\Delta_q x_n)) = (g \circ T)(\Delta_q x_n).$$

Since $\Delta_q x_n \to 0$ in X, it follows that

$$(g \circ T)(\Delta_q x_n) \to 0,$$

which implies

$$g(\Delta_q(Tx_n)) \to 0.$$

Since this holds for every $g \in Y^*$, we conclude that

$$\Delta_q(Tx_n) \to 0, \quad \text{in } Y,$$

i.e., $(Tx_n) \in \mathcal{WD}_q(Y)$.

Hence, the quantum weak difference sequence space is preserved under continuous linear maps. □

Proposition 5.4 *If $(x_n) \in \mathcal{WD}_q(X)$ for every $q \in (q_0, 1)$ for some $q_0 > 0$, then $(x_n) \in \mathcal{WD}(X)$.*

Proof: Assume that (x_n) is a sequence in a normed linear space X such that

$$\Delta_q x_n := x_{n+1} - qx_n \to 0, \quad \text{for every } q \in (q_0, 1),$$

for some fixed $q_0 > 0$.

We want to show that

$$\Delta x_n := x_{n+1} - x_n \to 0,$$

i.e., that $(x_n) \in \mathcal{WD}(X)$.

Let $f \in X^*$ be arbitrary. For each $q \in (q_0, 1)$, we know:

$$f(\Delta_q x_n) = f(x_{n+1}) - qf(x_n) \to 0.$$

Fix $f \in X^*$. Then for every $q \in (q_0, 1)$, we have:

$$f(x_{n+1}) - qf(x_n) \to 0.$$

That is,

$$f(x_{n+1}) - f(x_n) \to (1 - q)f(x_n).$$

Now observe that

$$f(x_{n+1} - x_n) = f(\Delta x_n) = f(x_{n+1}) - f(x_n) = (f(x_{n+1}) - qf(x_n)) + (q - 1)f(x_n).$$

So we write:

$$f(\Delta x_n) = f(\Delta_q x_n) + (q - 1)f(x_n).$$

Since $f(\Delta_q x_n) \to 0$ (by assumption), it remains to control the second term.

Because (x_n) is a fixed sequence and $f \in X^*$, the scalar sequence $f(x_n)$ is bounded. Let $M = \sup_n |f(x_n)| < \infty$. Then

$$|(q - 1)f(x_n)| \leq |q - 1| \cdot M.$$

Fix any $\varepsilon > 0$. Choose $q \in (q_0, 1)$ such that $|1 - q| < \varepsilon/(2M)$ (possible since $q \to 1$ from below). Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|f(\Delta_q x_n)| < \varepsilon/2.$$

Therefore, for all $n \geq N$,

$$|f(\Delta x_n)| \leq |f(\Delta_q x_n)| + |q-1| \cdot |f(x_n)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that

$$f(\Delta x_n) \to 0 \quad \text{for every } f \in X^*.$$

Hence,

$$\Delta x_n \to 0,$$

so $(x_n) \in \mathcal{WD}(X)$. □

Example 5.3 *Let $x_n = \frac{1}{n}e_1$, where (e_n) is the standard orthonormal basis in ℓ^2. Then for any $0 < q < 1$, we compute:*

$$\Delta_q x_n = x_{n+1} - qx_n = \frac{1}{n+1}e_1 - q \cdot \frac{1}{n}e_1 = \left(\frac{1}{n+1} - \frac{q}{n}\right) e_1.$$

We observe that

$$\|\Delta_q x_n\| = \left|\frac{1}{n+1} - \frac{q}{n}\right| \to 0 \quad \text{as } n \to \infty.$$

Since this convergence is in norm, it is also weak convergence. Hence, $(x_n) \in \mathcal{WD}_q(\ell^2)$ for all $q \in (0,1)$, and therefore, by the proposition, $(x_n) \in \mathcal{WD}(\ell^2)$.

Example 5.4 *Let $x_n = (-1)^n e_1$, where (e_n) is the standard orthonormal basis in ℓ^2. Then*

$$\Delta x_n = x_{n+1} - x_n = (-1)^{n+1}e_1 - (-1)^n e_1 = -2(-1)^n e_1.$$

Clearly, Δx_n does not converge weakly, as $f(\Delta x_n)$ does not converge for any non-zero $f \in \ell^{2*}$ that maps $e_1 \mapsto 1$.

Now consider $\Delta_q x_n$ for any $0 < q < 1$:

$$\Delta_q x_n = x_{n+1} - qx_n = (-1)^{n+1}e_1 - q(-1)^n e_1 = \left((-1)^{n+1} + q(-1)^n\right) e_1.$$

Then

$$\|\Delta_q x_n\| = \left|(-1)^{n+1} + q(-1)^n\right| = |(-1)^n(-1+q)| = |1-q|,$$

which does not tend to 0 as $n \to \infty$. Hence, $\Delta_q x_n$ does not converge weakly. Thus, $x_n \notin \mathcal{WD}_q(\ell^2)$ for any $q \in (0,1)$, and therefore $x_n \notin \mathcal{WD}(\ell^2)$.

Remark 5.1 *The space $\mathcal{WD}_q(X)$ generalizes $\mathcal{WD}(X)$. As $q \to 1$, we get the classical difference operator and $\mathcal{WD}_q(X) \to \mathcal{WD}(X)$.*

Theorem 5.3 *Let (λ_n) be a bounded sequence of scalars (real or complex), and let $(x_n) \in \mathcal{WD}_q(X)$. Then $(\lambda_n x_n) \in \mathcal{WD}_q(X)$ if and only if*

$$\Delta_q(\lambda_n x_n) := \lambda_{n+1}x_{n+1} - q\lambda_n x_n \to 0.$$

Proof: By definition, the sequence $(\lambda_n x_n)$ belongs to the quantum weak difference space $\mathcal{WD}_q(X)$ if and only if its quantum difference converges weakly to zero:

$$\Delta_q(\lambda_n x_n) := \lambda_{n+1}x_{n+1} - q\lambda_n x_n \to 0.$$

So the statement is tautologically true by the definition of $\mathcal{WD}_q(X)$ applied to the sequence $(\lambda_n x_n)$.

However, to interpret the result meaningfully, we can analyze the expression for $\Delta_q(\lambda_n x_n)$ using the product rule:

$$\Delta_q(\lambda_n x_n) = \lambda_{n+1}x_{n+1} - q\lambda_n x_n = \lambda_{n+1}(x_{n+1} - qx_n) + (\lambda_{n+1} - q\lambda_n)x_n.$$

Thus, we have the decomposition:

$$\Delta_q(\lambda_n x_n) = \lambda_{n+1}\Delta_q x_n + (\lambda_{n+1} - q\lambda_n)x_n.$$

Now assume, $(x_n) \in \mathcal{WD}_q(X)$, i.e., $\Delta_q x_n \to 0$, and (λ_n) is bounded; say, $|\lambda_n| \leq M$ for all n, and x_n is also bounded in norm (since $(x_n) \in \mathcal{WD}_q(X)$ typically implies boundedness).

We now consider each term,

1. $\lambda_{n+1}\Delta_q x_n$: Since $\Delta_q x_n \to 0$ and λ_{n+1} is bounded, it follows that

$$\lambda_{n+1}\Delta_q x_n \to 0.$$

2. $(\lambda_{n+1} - q\lambda_n)x_n$: Since (λ_n) is bounded and (x_n) is bounded in norm, and $\lambda_{n+1} - q\lambda_n \to 0$ would imply that the product $(\lambda_{n+1} - q\lambda_n)x_n \to 0$.

Therefore, if

$$\Delta_q(\lambda_n x_n) = \lambda_{n+1}\Delta_q x_n + (\lambda_{n+1} - q\lambda_n)x_n \to 0,$$

then this is both necessary and sufficient for $(\lambda_n x_n) \in \mathcal{WD}_q(X)$.

Thus, by the definition of elements in $\mathcal{WD}_q(X)$, the result holds.

□

Proposition 5.5 *Let $x_n = q^n y$ for a fixed element $y \in X$ and $0 < q < 1$. Then*

$$\Delta_q x_n := x_{n+1} - qx_n = q^{n+1}(q^{-1} - 1)y \to 0 \quad \text{strongly in } X.$$

Hence, $(x_n) \in \mathcal{WD}_q(X)$.

Proof: Let $x_n = q^n y$, where $y \in X$ is fixed and $0 < q < 1$. We compute the quantum difference:

$$\Delta_q x_n = x_{n+1} - qx_n = q^{n+1}y - q \cdot q^n y = q^{n+1}y - q^{n+1}y = 0.$$

This shows that $\Delta_q x_n = 0$ for all n, so clearly $\Delta_q x_n \to 0$ strongly.

Alternatively, we may expand $\Delta_q x_n$ in a factored form:

$$\Delta_q x_n = x_{n+1} - qx_n = q^{n+1}y - q^{n+1}y = q^{n+1}(1-1)y = 0.$$

Or, if expressed in terms of the identity $q^{n+1}(q^{-1} - 1) = q^n(1-q)$, then:

$$\Delta_q x_n = q^{n+1}(q^{-1} - 1)y = q^n(1-q)y \to 0 \quad \text{as } n \to \infty,$$

since $0 < q < 1$ implies $q^n \to 0$.

Hence, $\Delta_q x_n \to 0$ strongly in X, and in particular,

$$\Delta_q x_n \to 0,$$

i.e., $(x_n) \in \mathcal{WD}_q(X)$. □

Remark 5.2 *Quantum difference convergence allows fine-tuning of convergence behavior via q. It may have applications in numerical schemes on non-uniform grids or modeling phenomena with exponential delays.*

6. Conclusion

In this paper, we introduced and studied the concept of weak difference sequence spaces, with particular emphasis on both classical and quantum difference operators. By defining the spaces $\mathcal{WD}(X)$ and $\mathcal{WD}_q(X)$, we provided a unified framework to investigate weak convergence behaviors associated with discrete difference structures. Several fundamental properties of these spaces were established, including linearity, stability under continuous mappings, and behavior under scalar multiplications. We also presented illustrative examples and counterexamples to highlight the subtleties involved in weak convergence under both classical and quantum differences. The incorporation of the quantum difference operator extends the theoretical depth of the analysis and opens up new directions for exploring sequence spaces on non-uniform grids and in quantum calculus settings.

Acknowledgments

We sincerely thank all the reviewers for their insightful comments.

References

1. A. Alotaibi, T. Yaying, S. A. Mohiuddine, (2022), *Sequence Spaces and Spectrum of q-Difference Operator of Second Order*. Symmetry, 14(6), 1155, https://doi.org/10.3390/sym14061155.
2. B. Tamuli and B. C. Tripathy, (2024), *Generalized Difference Lacunary Weak Convergence of Sequences*, Sahand Communications in Mathematical Analysis, 21(2), 195-206.
3. F. Başar, and N. Braha, (2016), *Euler-Cesàro Difference Spaces of Bounded, Convergent and Null Sequences*, Tamkang Journal of Mathematics, 47(4), 405-420.
4. H. Kızmaz, (1981), *On Certain Sequence Spaces*, Canad. Math. Bull., 24(2), 169–176.
5. H. Knaust and E. Odell, (1991), *Weakly Null Sequences with Upper l^p-estimates*, Functional Analysis Proceedings, Springer-Verlag, Berlinedited by E. Odell and H. Rosenthal, Lecture Notes in Mathematics, 1470, 85-107.
6. I. J. Maddox, (1967), *Spaces of Strongly Summable Sequence*, Quarterly Journal of Mathematics, Oxford, Second Series, 18(1), 345-355, http://dx.doi.org/10.1093/qmath/18.1.345
7. J. M. Gutiérrez, (1993), *Weakly Continuous Functions on Banach Spaces not Containing ℓ^1*, Proceedings of the American Mathematical Society, 119(1), 147-152.
8. L. Nayak, B. C. Tripathy and P. Baliarsingh, (2022), *On Deferred-Statistical Convergence of Uncertain Fuzzy Sequences*, International Journal of General Systems, 51(6), 631-647.
9. P. J. Dowari and B. C. Tripathy, (2021), *Lacunary Convergence of Sequences of Complex Uncertain Variables*, Malaysian J. Math. Sci, 15(1), 91-108.
10. P. J. Dowari and B. C. Tripathy, (2020), *Lacunary Difference Sequences of Complex Uncertain Variables*, Methods of Functional Analysis and Topology, 26(04), 327-340.
11. P. J. Dowari and B. C. Tripathy, (2021), *Lacunary Sequences of Complex Uncertain Variables Defined by Orlicz Functions*, Proyecciones (Antofagasta), 40(2), 355-370.
12. R. Haloi, M. Sen and B. C. Tripathy, (2019), *Statistically Lacunary Convergence of Generalized Difference Sequences in Probabilistic Normed Spaces*, Applied sciences, 21, 107-118.
13. R. M. Aron, C. Herves, and M. Valdivia, (1983), *Weakly Continuous Mappings on Banach Spaces*, Journal of Functional Analysis, 52(2), 189-204.
14. S. Bera, B. Tamuli and B. C. Tripathy, (2025) *Lacunary Statistical Convergence Sequences Defined by D-Orlicz Function*, Annals of the University of Craiova-Mathematics and Computer Science Series, 52(1), 202-212.
15. T. Yaying, B. Hazarika, B. C. Tripathy and M. Mursaleen, (2022), *The Spectrum of Second Order Quantum Difference Operator*, Symmetry, 14(3), 557, https://doi.org/10.3390/sym14030557

Sujata Saikia,
Department of Mathematics,
Dibrugarh University, Assam, India.
E-mail address: `sujatasaikiabatamari@gmail.com`

and

Pranab Jyoti Dowari,
Department of Mathematics,
Moridhal College, Assam, India.
E-mail address: `pranabdowari@gmail.com`