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Reflected Generalized Backward Doubly Sdes with Jumps Under Stochastic Conditions: Beyond Right-Continuity

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ABSTRACT: This work addresses reflected generalized backward doubly stochastic differential equations whose obstacle may be irregular, in particular not necessarily right-continuous. The noise is driven by two mutually independent Brownian motions and an independent integer-valued random measure. Under stochastic monotonicity, stochastic Lipschitz, and stochastic linear growth conditions on the generators, we establish the existence and uniqueness of an adapted solution using the Yosida approximation method. We also prove a comparison principle showing that the ordering of solutions is preserved under perturbations of the terminal data and the generator.

Key Words: Reflected backward doubly SDEs, irregular barrier, Mertens decomposition, stochastic monotone condition, stochastic linear growth condition.

Contents

1. Introduction

The notion of generalized backward stochastic differential equations (GBSDEs) was introduced by Pardoux and Zhang [43]. A solution to this equation, associated with a terminal condition ξ and generators (drivers) $f(\omega,t,y,z)$ and $h(\omega,t,y)$, is a pair of stochastic processes $(Y_t,Z_t)_{t\leq T}$ that satisfies

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T h(s,Y_s)dA_s-\int_t^T Z_s dW_s, \tag{1.1}$$

a.s. for all $t\in[0,T]$, where $(W_t)_{t\leq T}$ is a Brownian motion and $(A_t)_{t\leq T}$ is a continuous increasing process representing the local times of a diffusion process at the boundary. Pardoux and Zhang [43] demonstrated existence and uniqueness of the solution in the filtration generated by $(W_t)_{t\leq T}$ under deterministic monotonicity and linear growth assumptions on the driver, together with appropriate square integrability conditions on ξ, $(f(t,0,0,0))_{t\leq T}$, and $(h(t,0))_{t\leq T}$. Owing to the effectiveness of GBSDEs

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in providing probabilistic representation formulas for systems of parabolic or elliptic semilinear partial differential equations (PDEs) with Neumann boundary conditions, several researchers have extended the original results in [43] to more general filtrations, including Brownian-Lévy and Brownian-Poisson frameworks (see, e.g., [19,20,23,42]).

Reflected GBSDEs (RGBSDEs) are a class of GBSDEs in which the first component of the solution, $(Y_t)_{t\leq T}$, is constrained to remain above a given continuous stochastic process $(\xi_t)_{t\leq T}$, referred to as the obstacle or barrier. To enforce this constraint, a non-decreasing process $(K_t)_{t\leq T}$ is introduced, which pushes Y upward toward ξ with minimal energy. In contrast to classical GBSDEs (1.1), a solution to an RGBSDE associated with an obstacle ξ and generators f and h is a triplet of stochastic processes $(Y_t, Z_t, K_t)_{t\leq T}$ that satisfies the following equation:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T h(s,Y_s)dA_s + (K_T - K_t) - \int_t^T Z_s dW_s,\ t \in [0,T]; \\ \forall t \in [0,T],\ Y_t \geq \xi_t \ \text{ and } \ \int_0^T (Y_s - \xi_s)\, dK_s = 0 \text{ a.s.} \end{cases} \tag{1.2}$$

Here, the minimality condition means that the process K increases only when $Y = \xi$, as described by the last requirement in (1.2). Ren and Xia [47] introduced this type of equation (1.2), proving the existence and uniqueness of a solution and providing a probabilistic formula for the viscosity solution of an obstacle problem for PDEs with a nonlinear Neumann boundary condition. Motivated by these obstacle problems, several works have explored (1.2) in contexts that incorporate different jump models, where the barrier ξ is no longer continuous but right-continuous with left limits (RCLL). Ren and El Otmani [48] studied RGBSDEs driven by a Lévy process, establishing its link with an obstacle problem for a class of partial differential–integral equations with a nonlinear Neumann boundary condition. Similarly, Elhachemy and El Otmani [13,14] examined RGBSDEs in the context of noise driven by a Brownian motion and an independent Poisson random measure.

A new extension of GBSDEs was explored by Boufoussi et al. [7], where an additional stochastic driver term is added to (1.1). This term is represented by a backward integral with respect to another independent Brownian motion $(B_t)_{t\leq T}$. In other words, the formulation includes two distinct directions of stochastic integrals with respect to two independent Brownian motions. More specifically, these equations are represented as follows:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T h(s,Y_s)dA_s + \int_t^T g(s,Y_s,Z_s)\overleftarrow{dB}_s - \int_t^T Z_s dW_s,\ t \in [0,T]. \tag{1.3}$$

Here, the integral driven by dW_s is the standard forward stochastic Itô integral, and the integral driven by $\overleftarrow{dB}$ is the backward stochastic Kunita-Itô integral with respect to B. Both integrals are special cases of the Itô-Skorohod integral, as detailed by Nualart and Pardoux [39]. These equations (1.3) are referred to as generalized backward *doubly* SDEs, first introduced by Pardoux and Peng [41] in the classical (non-generalized) case (i.e., without the inclusion of an integral with respect to $(A_t)_{t\leq T}$). In both papers [7,41], the authors proved the existence and uniqueness of a solution under uniform Lipschitz conditions. They applied their findings to provide a probabilistic representation for viscosity solutions of a large class of quasilinear *stochastic* PDEs, including those with Neumann boundary conditions, using (1.3). For other related results, we refer the reader to [11,24,40], among others.

Reflected backward doubly SDEs (RBDSDEs) with one continuous barrier were introduced by Bahlali et al. [5], who investigated the case in which the coefficient f is continuous. They established the existence of minimal and maximal solutions within a Brownian setting. Subsequently, several researchers expanded the theory of RBDSDEs to include scenarios with RCLL barriers, larger filtrations than the Brownian filtration, or relaxed assumptions on the coefficients (see, e.g., [1,2,11,32,36,40,46]). Beyond the right-continuous framework, when the reflecting barrier ξ is not necessarily right-continuous, Berrhazi et al. [6] studied RBDSDEs with non-right-continuous barriers, drawing inspiration from Grigorova et al. [26], who were the first to study RBSDEs with a right upper semicontinuous barrier. Recently, Marzougue and Sagna [38] extended the work of Berrhazi et al. [6] to include jumps driven by an independent Poisson random measure and a stochastic Lipschitz condition on the drivers.

The aim of this paper is to establish existence and uniqueness of a solution for a class of reflected generalized backward doubly stochastic differential equations with jumps (RGBDSDEJs). The equation is driven by two mutually independent Brownian motions and by an independent integer-valued random measure with a general dual predictable projection (compensator). The obstacle is not necessarily right-continuous; it is only right-upper-semicontinuous with left limits. The driver satisfies stochastic Lipschitz and stochastic monotonicity conditions, together with a stochastic linear-growth bound. In this way, our results extend and unify several strands of the literature:

(i) reflected BSDEs with neither a boundary local-time term nor a backward integral, as in [26];

(ii) models with classical (deterministic-constant) Lipschitz or monotone assumptions, as in [5,6,7,11, 24,41,44];

(iii) the stochastic-Lipschitz setting of [38], since a stochastic Lipschitz bound implies a monotonicity inequality for the drivers.

Our framework allows irregular obstacles and noise with both continuous and jump parts, and it uses coefficients whose bounds depend on (ω, t) rather than fixed constants.

The main challenges in our setting are twofold. First, the generator satisfies only weak regularity, namely stochastic monotonicity and stochastic Lipschitz conditions, together with a stochastic linear growth bound. Such stochastic Lipschitz coefficients are standard in mathematical finance, for example in Black–Scholes-type models with stochastic (not necessarily bounded) parameters such as the risk-free rate, volatility, and risk premium. In these cases, pricing American or game options via reflected BSDEs typically requires a stochastic Lipschitz constraint on the coefficient; see [15,16,17,18,21,22] for applications in more general market models with jumps. Stochastic monotone drivers have also been used in the BSDE literature to obtain probabilistic representations of solutions to nonlinear PDEs. In the Brownian case, this approach was developed by Bahlali et al. [4], who proved existence and uniqueness and linked the BSDE framework to viscosity solutions of elliptic PDEs. Obstacle problems with nonlinear Neumann boundary conditions for parabolic semilinear integro-PDEs were studied by Elhachemy and El Otmani [14], who used reflected generalized BSDEs with right-continuous obstacles and stochastic monotone drivers to characterize the viscosity solution. Second, because the reflecting barrier is not necessarily right-continuous, the first component of the solution $(Y_t)_{t\leq T}$ need not be right-continuous either. This places us in a setting where classical tools such as Itô's formula, the Doob–Meyer decomposition for RCLL supermartingales, and Tanaka-type formulas do not apply in their standard forms. We therefore rely on techniques from the theory of optional semimartingales, including the Gal'chouk–Lenglart formula, the Mertens decomposition for strong optional supermartingales, Tanaka-type results adapted to the optional framework, and tools from optimal stopping. Allowing irregular obstacles also covers optimal stopping problems arising in risk-minimizing hedging [26] (see also [27]) and more general control models that admit a large class of stopping strategies, such as split stopping times [3,37].

To obtain the existence of a solution under these stochastic monotonicity conditions on the drivers, we use a Yosida approximation method, where we approximate our RGBDSDEJ with a sequence of equations with stochastic Lipschitz coefficients. Since there is no existence result for this type of equation with jumps driven by an integer-valued random measure under stochastic Lipschitz drivers, we address this problem using the Picard iteration method. Next, using the result for the stochastic Lipschitz case, we derive the existence result in the stochastic monotone case. We note that Yosida's approximation technique has been employed by several authors in this context. Notably, Hu [31] applied it to a class of forward-backward stochastic differential equations defined over an arbitrary time horizon under specific monotonicity conditions on the coefficients (see also [30]). Elmansouri and El Otmani [20] also employed it while dealing with a GBSDE driven by a square-integrable RCLL martingale under stochastic Lipschitz coefficients.

The article is structured as follows. Section 2 covers the notation and assumptions. In Section 3, we formulate the problem, introduce our RGBDSDEJ, provide some characterizations related to the state process, and present the main result of the paper, which is the existence and uniqueness result under stochastic monotone and Lipschitz coefficients. Finally, in Section 4, we provide a comparison principle for such equations.

2. Preliminaries

Let $T \in (0, \infty)$ be a deterministic time horizon. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\mathcal{F}$ containing all $\mathbb{P}$-null sets; hence $(\Omega, \mathcal{F}, \mathbb{P})$ is complete. We assume the existence of three mutually independent objects:

- a one-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$;
- a one-dimensional Brownian motion $B = (B_t)_{t\in[0,T]}$;
- an integer-valued random measure N on $[0,T] \times E$, where $E = \mathbb{R}^\ell \setminus \{0\}$ for some $\ell \in \mathbb{N}^+$, endowed with its Borel σ-field $\mathcal{E}$. We assume that N admits a compensator
$$\upsilon(\omega; dt, de) = Q(\omega, t; de)\, \eta(\omega, t)\, dt,$$
where $\eta : \Omega \times [0,T] \to [0, \infty)$ is a predictable process and Q is a kernel from $\big(\Omega \times [0,T], \mathcal{P}^N\big)$ into $(E, \mathcal{E})$. Here, $\mathcal{P}^N$ denotes the predictable σ-field associated with the filtration generated by N. We impose the integrability condition
$$\int_0^T \int_E |e|^2\, Q(\omega, t; de)\, \eta(\omega, t)\, dt < \infty \quad \text{a.s.}$$
We also set $N(\{0\} \times E) = N\big((0,T] \times \{0\}\big) = \upsilon\big((0,T] \times \{0\}\big) = 0$.
Given the compensator υ of the random measure N, we define then the *compensated* random measure
$$\tilde{N}(\omega; dt, de) := N(\omega; dt, de) - \upsilon(\omega; dt, de).$$

For any process $(\eta_t)_{t\leq T}$, we define the following σ-algebra:
$$\mathcal{F}^{\eta}_{s,t} \triangleq \sigma\{\eta_u - \eta_s : s \leq u \leq t\} \vee \mathcal{N}, \qquad \mathcal{F}^{\eta}_t \triangleq \mathcal{F}^{\eta}_{0,t}, \quad 0 \leq s \leq t \leq T.$$
We define the following collection of σ-algebras
$$\mathcal{F}_t \triangleq \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T} \vee \mathcal{F}^N_t.$$
Here $\mathcal{N}$ denotes the class of $\mathbb{P}$-null sets of $\mathcal{F}$ and we assume that $\mathcal{F}_T = \mathcal{F}$. Note that, the collection $\mathbb{F} := (\mathcal{F}_t)_{t\leq T}$ is neither increasing nor decreasing. In particular, it does not constitute a filtration. In this case, the usual techniques that are used in classical reflected BSDEs (see [12]) do not work. In fact, the section theorem (see [29, Theorem 4.7]) cannot easily be used to deduce that the solution will remain above the obstacle for all time. Then, in order to avoid this problem and to be able to use classical notions and results of stochastic analysis and control theory, we introduce the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\leq T}$ given by:
$$\mathcal{G}_t \triangleq \mathcal{F}^W_t \vee \mathcal{F}^B_T \vee \mathcal{F}^N_t, \quad 0 \leq t \leq T. \tag{2.1}$$
The filtration $\mathbb{G}_t$ is assumed to be right continuous and quasi-left continuous. The last condition means that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{G}$-stopping time such that $\tau_n \nearrow \tau$ for some stopping time τ we have $\bigvee_{n\in\mathbb{N}} \mathcal{G}_{\tau_n} = \mathcal{G}_\tau$.

Remark 2.1
- *The quasi-left continuity of the filtration (2.1) implies, in particular, that the jumps of all $\mathbb{G}$-martingales are exhausted by a countable set of graphs of totally inaccessible stopping times (see, for example, [29, Theorem 3.32] and [33, Proposition 10.19]).*
- *Note that $\mathcal{F}^B_{t,T} \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t$ for any $t < T$, as $\mathcal{F}^B_{t,T} \subseteq \mathcal{F}^B_T$ for any $t < T$, $\mathcal{F}_T = \mathcal{F}^W_T \vee \mathcal{F}^N_T \subseteq \mathcal{G}_T$, $\mathcal{F}_t \vee \mathcal{F}^B_t = \mathcal{G}_t$ for any $t \leq T$, and $\mathcal{G}_0 = \mathcal{F}^B_T = \mathcal{F}_0$.*

The basic assumptions on the data (ξ, f, h, g, A):
We denote by $\mathbb{H} := (\mathcal{H}_t)_{t\leq T}$ the filtration defined by $\mathcal{H}_t = \mathcal{F}^W_t \vee \mathcal{F}^N_t \vee \mathcal{N}$ for $t \in [0,T]$.

Measurability of the data and trajectory properties of the process $(A_t)_{t\leq T}$:

- The process $(A_t)_{t\leq T}$ is a continuous increasing process such that $A_0=0$ and A_t is $\mathcal{F}_t$-measurable for any $t\in[0,T]$;
- f, h and g are jointly measurable;
- $\forall t\in[0,T]$, y, $z\in\mathbb{R}$, $u\in\mathbb{L}^2_Q$, the processes $f(\cdot,t,y,z,u):\Omega\to\mathbb{R}$, $g(\cdot,t,y,z,u):\Omega\to\mathbb{R}$, and $g(\cdot,t,y):\Omega\to\mathbb{R}$ are $\mathcal{F}_t$-measurable.

Stochastic Monotonicity of f and g in y:
There exists two $\mathbb{H}$-adapted processes $\lambda:\Omega\times[0,T]\to\mathbb{R}$ and $\varrho:\Omega\times[0,T]\to\mathbb{R}^{-*}$ such that

(i) For all y, y', $z\in\mathbb{R}$, $u\in\mathbb{L}^2_Q$, $d\mathbb{P}\otimes dt$-a.e.,

$$(y-y')\left(f(t,y,z,u)-f(t,y',z,u)\right)\leq\lambda_t\left|y-y'\right|^2.$$

(ii) For all y, $y'\in\mathbb{R}$, $d\mathbb{P}\otimes dA_t$-a.e.,

$$(y-y')\left(h(t,y)-h(t,y')\right)\leq\varrho_t\left|y-y'\right|^2.$$

Stochastic Lipschitz condition on f in (z,u):
There two $\mathbb{H}$-adapted processes $\gamma,\kappa:\Omega\times[0,T]\to\mathbb{R}^{+*}$ such that

(iii) For all y, z, $z'\in\mathbb{R}$, u, $u'\in\mathbb{L}^2_Q$, $d\mathbb{P}\otimes dt$-a.e.,

$$\left|f(t,y,z,u)-f(t,y,z',u')\right|\leq\gamma_t\left|z-z'\right|+\kappa_t\left\|u-u'\right\|_Q.$$

Stochastic Lipschitz condition on g in y and a Lipschitz condition in (z,u): There exists an $\mathbb{H}$-adapted process $\rho:\Omega\times[0,T]\to\mathbb{R}^{+*}$ and a constant $\alpha\in(0,1)$ such that

(iv) For all y, y', z, $z'\in\mathbb{R}$, u, $u'\in\mathbb{L}^2_Q$, $d\mathbb{P}\otimes dt$-a.e.,

$$\left|g(t,y,z,u)-g(t,y,z',u')\right|^2\leq\rho_t\left|y-y'\right|^2+\alpha\left(\left|z-z'\right|^2+\left\|u-u'\right\|_Q^2\right).$$

Linear growth of f and h:
For some constant $\zeta>0$, and some $[1,\infty)$-valued processes $\{\varphi_t,\psi_t;0\leq t\leq T\}$, such that for all $(t,y)\in[0,T]\times\mathbb{R}$, we have

(v) φ_t and ψ_t are $\mathcal{F}_t$-measurable for any $t\in[0,T]$;

(vi) $|f(t,y,0,0)|\leq\varphi_t+\zeta|y|$ and $|h(t,y)|\leq\psi_t+\zeta|y|$.

Irregular obstacle and integrability assumption:
We consider a $\mathbb{G}$-optional process $\xi:=(\xi_t)_{t\leq T}$, which is assumed to be right-upper semi-continuous and limited from the left. Throughout the paper, this process will be referred to as the irregular barrier (or obstacle).

Let $(V_t)_{t\leq T}$ and $(Q_t)_{t\leq T}$ be the two continuous increasing stochastic processes defined by

$$V_t:=\int_0^t a_s^2ds+\int_0^t\varrho_s^2dA_s,\quad \mathcal{Q}_t:=t+A_t,$$

with $a_s^2:=|\lambda_s|+\lambda_s^2+\rho_s+\gamma_s^2+\kappa_s^2$. We assume that, there exists some $\epsilon>0$, such that $a_s^2\geq\epsilon$. Since $\varrho_s<0$, we can also consider $|\varrho_s|\wedge\varrho_s^2\geq\epsilon$.

Let us set

$$\phi_t^{\beta,\mu}=\beta V_t+\mu A_t \text{ and } \Phi^{\beta,\mu}:=e^{\phi^{\beta,\mu}} \text{ for } \beta,\mu>0.$$

We assume that, for any β, $\mu>0$

(vii) $\mathbb{E}\left[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\Phi_\tau^{2\beta,2\mu}\left|\xi_\tau\right|^2\right]<\infty$.

(viii) $\mathbb{E}\left[\int_0^T\Phi_t^{\beta,\mu}\left(|\varphi_t|^2+|g(t,0,0,0)|^2\right)dt+\int_0^T\Phi_t^{\beta,\mu}\psi_t^2dA_t\right]<\infty$.

Continuity condition on f and g:
For all $y,\, z \in \mathbb{R}$, $u \in \mathbb{L}^2_Q$;

(ix) $d\mathbb{P} \otimes dt$-a.e., the mapping $y \mapsto f(t, y, z, u) : \mathbb{R} \to \mathbb{R}$ is continuous.

(x) $d\mathbb{P} \otimes dA_t$-a.e., the mapping $y \mapsto h(t, y) : \mathbb{R} \to \mathbb{R}$ is continuous.

In the rest of this paper, the previous assumptions will be denoted by **(H-M)**. Additionally, we will work under the filtration $\mathbb{G}$, so all classical concepts of stochastic analysis are considered within this filtration defined by (2.1).

Let τ and σ be two $[0, T]$-valued stopping times such that $\tau \le \sigma$ a.s. By $\mathcal{T}_{\tau,\sigma}$, we mean the set of $[0, T]$-valued stopping times ν such that $\tau \le \nu \le \sigma$ a.s. $\mathcal{T}^p_{\tau,\sigma}$ denotes the subclass of $\mathcal{T}^p_{\tau,\sigma}$ of predictable stopping times (see Definition 3.25 in [29]).
For a process $\mathcal{X} : \Omega \times [0, T] \to \mathbb{R}$ such that, for $\mathbb{P}$-almost all $\omega \in \Omega$, the function $t \mapsto \mathcal{X}_t(\omega)$ has finite right limits at each point $t \in [0, T)$, and finite left limits at each point of $(0, T]$, we set:

- $\mathcal{X}_{s-} = \lim\limits_{u \nearrow s} \mathcal{X}_u$ the left limit of $\mathcal{X}$ at $s \in]0, T]$ and $\Delta\mathcal{X}_s = \mathcal{X}_s - \mathcal{X}_{s-}$ with the convention $\Delta\mathcal{X}_0 = 0$.
- $\mathcal{X}_{s+} = \lim\limits_{s \swarrow u} \mathcal{X}_u$ the right limit of $\mathcal{X}$ at $s \in [0, T[$ and $\Delta_+\mathcal{X}_s = \mathcal{X}_{s+} - \mathcal{X}_s$ with the convention $\Delta_+\mathcal{X}_T = 0$.

Let $\mathcal{V}$ be an RCLL process with finite variation on $[0, T]$. We denote he total variation of $\mathcal{V}$ by $\|\mathcal{V}\| := (\|\mathcal{V}\|_t)_{t \le T}$. The notation $\mathcal{P}$ is reserved for the predictable σ-algebra on $\Omega \times [0, T]$. For progressively measurable RCLL processes $\{Y^n\}_{n\ge1}$ and Y, we say that $Y^n \to Y$ in Uniformly on Compacts in Probability (UCP) if $\sup_{s\in[0,t]} |Y^n_s - Y_s| \to 0$ in probability $\mathbb{P}$ for every $t \in (0, T]$. Additionally, for $x \in \mathbb{R}$, we remember that $x^+ = \max(x, 0)$.
To simplify the notation, in the majority of the paper, we omit the dependence on ω of a given process or random function. Also, all stochastic processes run on the fixed time interval $[0, T]$.

Remark 2.2 *Note that from Remark 2.1, the above $\mathcal{F}_t$-measurability assumptions on the data imply, in particular, $\mathcal{G}_t$-measurability, where the usual notions of adaptedness, progressive measurability, optionality, and predictability are defined. Specifically, the processes $(V_t)_{t\le T}$ and $(\mathcal{Q}_t)_{t\le T}$ are $\mathbb{G}$-progressively measurable.*

Remark 2.3 *From the integrability conditions, we can easily deduce that $\mathbb{E}\int_0^T \Phi^{\mu,\gamma}_s \left|\frac{\varphi_s}{a_s}\right|^2 ds < \infty$, which will be used several times subsequently without specific mention.*

For $\beta, \mu > 0$, we consider the following spaces:

- $\mathbb{L}^2_Q$: The space of $\mathcal{E}$-measurable functions $\phi : E \to \mathbb{R}^k$ such that
$$\|\phi\|^2_Q = \int_E |\phi(e)|^2 \, Q(t, de)\eta(t) < \infty, \ (t, \omega) \text{ a.e.}$$
- $\mathcal{S}^2_{\beta,\mu}$: The space of $\mathbb{R}$-valued, optional processes $(Y_t)_{t\le T}$ such that
$$\|Y\|^2_{\mathcal{S}^2_{\beta,\mu}} = \mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \Phi^{\beta,\mu}_\tau |Y_\tau|^2\Big] < \infty,$$
with the convention $\mathcal{S}^2 := \mathcal{S}^2_{0,0}$.
- $\mathcal{C}^{p,\mathcal{V}}_{\beta,\mu}$: The space of $\mathbb{R}$-valued, optional processes $(Y_t)_{t\le T}$ such that
$$\|Y\|^2_{\mathcal{C}^{p,\mathcal{V}}_{\beta,\mu}} = \mathbb{E}\int_0^T \Phi^{\beta,\mu}_t \, |Y_t|^2 \, d\mathcal{V}_t < \infty$$
for $p \in \{1, 2\}$, where $\mathcal{V}$ is a general RCLL optional non-decreasing process such that $\mathcal{V}_0 = 0$, with the convention $\mathcal{C}^{p,\mathcal{V}} := \mathcal{C}^{p,\mathcal{V}}_{0,0}$.

- $\mathcal{H}^p_{\beta,\mu}$: The space of $\mathbb{R}$-valued, $\mathcal{P}$-measurable processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|^2_{\mathcal{H}^p_{\beta,\mu}} = \mathbb{E}\int_0^T \Phi_t^{\beta,\mu}\, |Z_t|^p\, dt < \infty$$

  for $p \in \{1,2\}$, with the convention $\mathcal{H}^p := \mathcal{H}^p_{0,0}$.

- $\mathcal{L}^2_{\beta,\mu}$: The space of $\mathcal{P}\otimes\mathcal{E}$-measurable processes $U : \Omega\times[0,T]\times E \to \mathbb{R}$ such that

$$\|U\|^2_{\mathcal{L}^2_{\beta,\mu}} = \mathbb{E}\int_0^T \Phi_t^{\beta,\mu}\, \|U_t\|^2_Q\, dt < \infty.$$

- $\mathfrak{L}^2_{\beta,\mu} := \left(\mathcal{S}^2_{\beta,\mu}\cap\mathcal{C}^{2,V}_{\beta,\mu}\right)\times\mathcal{H}^2_{\beta,\mu}\times\mathcal{L}^2_{\beta,\mu}$ and $\mathfrak{B}^2_{\beta,\mu} := \mathfrak{L}^2_{\beta,\mu}\times\mathcal{S}^2\times\mathcal{S}^2$.

Remark 2.4 *Throughout this work, $\mathfrak{c} > 0$ will be used to represent a constant that could change from one line to the. Furthermore, to highlight the dependence of the constant $\mathfrak{c}$ on a particular set of parameters θ, the notation $\mathfrak{c}_\theta$ will be used.*

3. Reflected generalized backward doubly SDEs with jumps and irregular obstacle

3.1. Formulation and properties

We consider the following reflected generalized backward doubly stochastic differential equation with jumps (RGBDSDEJs):

$$\begin{cases}
\text{(i) } Y_\tau = \xi_T + \displaystyle\int_\tau^T f(s,Y_s,Z_s,U_s)ds + \int_\tau^T h(s,Y_s)dA_s + \int_\tau^T g(s,Y_s,Z_s,U_s)\overleftarrow{dB}_s \\
\qquad - \displaystyle\int_\tau^T Z_s dW_s - \int_\tau^T\int_E U_s(e)\tilde{N}(ds,de) + (K_T - K_\tau) + (C_{T-} - C_{\tau-}), \quad \text{a.s. } \forall\tau\in\mathcal{T}_{0,T}; \\
\text{(ii) } Y_\tau \geq \xi_\tau, \ \text{a.s. } \forall\tau\in\mathcal{T}_{0,T}; \\
\text{(iii) } K \text{ is a non-decreasing right-continuous predictable process with} \\
\qquad K = K^c + K^d \text{ (Continuous part + Discontinuous part);} \\
\qquad K_0 = 0, \quad \displaystyle\int_0^T \mathbb{1}_{\{Y_s>\xi_s\}} dK_s^c = 0 \text{ a.s., and } (Y_{\tau-}-\xi_{\tau-})\,\Delta K^d_\tau = 0 \text{ a.s. } \forall\tau\in\mathcal{T}^p_{0,T}; \\
\text{(iv) C is a non-decreasing right-continuous predictable purely discontinuous process} \\
\qquad \text{with } C_{0-} = 0 \ \text{ such that } \ (Y_\tau - \xi_\tau)\,\Delta C_\tau = 0 \text{ a.s. } \forall\tau\in\mathcal{T}_{0,T}.
\end{cases} \tag{3.1}$$

Conditions (iii) and (iv) are referred to as *minimality conditions* or *Skorokhod conditions*.
For simplicity, we will denote $\Theta_t := (Y_t, Z_t, U_t)$ for $t\in[0,T]$, where the triplet (Y,Z,U) corresponds to the three first components of the solution to the BSDE (3.1)-(i).

Remark 3.1 *We note that according to Theorem IV.84 in [8], a process $(Y,Z,U,K,C)\in\mathfrak{B}^2_{\beta,\mu}$ satisfies the equation (3.1)-(i) if and only if a.s. for all $t\in[0,T]$,*

$$\begin{aligned} Y_t = \xi_T &+ \int_t^T f(s,Y_s,Z_s,U_s)ds + \int_t^T h(s,Y_s)dA_s + \int_t^T g(s,Y_s,Z_s,U_s)\overleftarrow{dB}_s \\ &- \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\tilde{N}(ds,de) + (K_T-K_t) + (C_{T-}-C_{t-}). \end{aligned}$$

While conditions (ii)-(iv) can be writing in a similar manner using optional and predictable section theorems (see, e.g., [29, Theorems 4.7] and [29, Theorems 4.8]). We also note that equality (3.1)-(i) still holds with $f(s,Y_s,Z_s,U_s)$, $h(s,Y_s)$, and $g(s,Y_s,Z_s,U_s)$ replaced by $f(s,Y_{s-},Z_s,U_s)$, $h(s,Y_{s-})$, and $g(s,Y_{s-},Z_s,U_s)$, respectively. This, without explicit mention, will be used several times in what follows.

Remark 3.2 *From (3.1)-(i) and Remark 3.1, we deduce that if a process (Y,Z,U,K,C) constitute a solution, then $\Delta_+ Y_t = -\Delta C_t$ for all $t\in[0,T]$ a.s. Therefore, $Y_{t+}\leq Y_t$ for all $t\in[0,T]$ a.s., implying in particular the right-upper semi-continuity of the state process $(Y_t)_{t\leq T}$. On the other hand, we have $\Delta K^d_t = -\Delta Y_t = Y_{t-}-Y_t$ for all $t\in[0,T]$ a.s., and from the Skorokhod condition (iii), we deduce that $K^d_t=\sum_{0<s\leq t}(\xi_{s-}-Y_s)^+\mathbb{1}_{\{\Delta\xi_s<0\}\cap\{Y_{s-}=\xi_{s-}\}}$ (see Remark 2.1 in [21]). Similarly, we can obtain $C_t=\sum_{0<s\leq t}(\xi_s-Y_{s+})^+\mathbb{1}_{\{\Delta_+\xi_s<0\}\cap\{Y_s=\xi_s\}}$.*

Remark 3.3 *Note that since $\Delta_+Y_T=\Delta_+\xi_T=0$ and $Y_T=\xi_T$ (from (3.1)-(i)), it follows from Remark 3.2 that $\Delta C_T=0$, i.e., $C_T=C_{T-}$ a.s.*

Remark 3.4 *From Remarks 2.1 and 3.2, the state process $(Y_t)_{t\leq T}$ of the RGBDSDEJ (3.1) has two type of left-jumps:*

- *Totally inaccessible jumps stemming from the stochastic integral with respect to the compensated integer-valued random measure $\tilde{N}$, due to the quasi-left-continuity of the filtration;*
- *Predictable jumps arising from the negative left predictable jumps of the barrier ξ, which are controller by the purely discontinuous part of the predictable process K.*

Additionally, the process Y has also right-jumps sides, which arise from the negative right-jumps of the barrier ξ.

Definition 3.1 *Let ξ be a given barrier. A quintuplet of processes (Y,Z,U,K,C) is called a solution to the RGBDSDEJ associated with parameters (f,h,g,ξ) if it satisfies the system (3.1) and belongs to $\mathfrak{B}^2_{\beta,\mu}$.*

We begin to give a classical characterization of the state process and the stochastic-drift part associated with (3.1)-(i).

Proposition 3.1 *Let (Y,Z,U,K,C) be a solution of the RGBDSDEJ (3.1). Then, the process $\Xi := \Big(Y_t+\int_0^t h(s,Y_s)dA_s+\int_0^t f(s,\Theta_s)ds+\int_0^t g(s,\Theta_s)\overleftarrow{dB}_s\Big)_{t\leq T}$ is $\mathbb{L}^2$-integrable strong optional supermartingale.*

Proof: For any $\tau\in\mathcal{T}_{0,T}$, using **(H-M)**, we have, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\tau |f(s,\Theta_s)|\,ds\right)^2\right] &\leq \mathbb{E}\left[\left(\int_0^\tau \Phi_s^{-\beta,-\mu}dV_s\right)\left(\int_0^\tau \Phi_s^{\beta,\mu}\left|\frac{f(s,\Theta_s)}{a_s}\right|^2 ds\right)\right]\\
&\leq \mathbb{E}\left[\left(\int_0^\tau e^{-\beta V_s}dV_s\right)\left(\int_0^\tau \Phi_s^{\beta,\mu}\left|\frac{f(s,\Theta_s)}{a_s}\right|^2 ds\right)\right]\\
&\leq \frac{4}{\beta}\left(\frac{\zeta^2}{\epsilon^2}\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}|Y_s|^2dV_s+\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}\left\{\left|\frac{\varphi_s}{a_s}\right|^2+|Z_s|^2+\|U_s\|_Q^2\right\}ds\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\tau g(s,\Theta_s)\overleftarrow{dB}_s\right)^2\right] &= \mathbb{E}\int_0^\tau |g(s,\Theta_s)|^2ds\\
&\leq 2\left(\mathbb{E}\int_0^\tau |Y_s|^2dV_s+\mathbb{E}\int_0^\tau\left(\alpha\left\{|Z_s|^2+\|U_s\|_Q^2\right\}+|g(s,0,0,0)|^2\right)ds\right)\\
&\leq 2\left(\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}|Y_s|^2dV_s+\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}\left(|Z_s|^2+\|U_s\|_Q^2+|g(s,0,0,0)|^2\right)ds\right),
\end{aligned}$$

and

$$\mathbb{E}\left[\left(\int_0^\tau h(s,Y_s)dA_s\right)^2\right] \leq \mathbb{E}\left[\left(\int_0^\tau \Phi_s^{-\beta,-\mu}dA_s\right)\left(\int_0^\tau \Phi_s^{\beta,\mu}\left|h(s,Y_s)\right|^2 dA_s\right)\right]$$
$$\leq \mathbb{E}\left[\left(\int_0^\tau e^{-\mu A_s}dA_s\right)\left(\int_0^\tau \Phi_s^{\beta,\mu}\left\{\left|\psi_s\right|^2 dA_s + \frac{\zeta^2}{\varrho_s^2}\left|Y_s\right|^2 dV_s\right\}\right)\right]$$
$$\leq \frac{2}{\mu}\left(\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}\left|\psi_s\right|^2 dA_s + \frac{\zeta^2}{\epsilon}\mathbb{E}\int_0^\tau \Phi_s^{\beta,\mu}\left|Y_s\right|^2 dV_s\right).$$

Therefore, we deduce the following

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}\left|Y_\tau + \int_0^\tau f(s,\Theta_s)ds + \int_0^\tau g(s,\Theta_s)\overleftarrow{dB}_s + \int_0^\tau h(s,Y_s)dA_s\right|^2$$
$$\leq \mathfrak{c}_{\beta,\mu,\zeta,\epsilon}\left(\left\|\frac{\varphi}{a.}\right\|^2_{\mathcal{H}^2_{\beta,\mu}} + \|g(\cdot,0,0,0)\|^2_{\mathcal{H}^2_{\beta,\mu}} + \|\psi\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}} + \|Y\|^2_{\mathcal{S}^2_{\beta,\mu}} + \|Y\|^2_{\mathcal{C}^{2,V}_{\beta,\mu}} + \|Z\|^2_{\mathcal{H}^2_{\beta,\mu}} + \|U\|^2_{\mathcal{L}^2_{\beta,\mu}}\right) < \infty.$$

On the other hand, from (3.1)-(i), for any $\sigma \in \mathcal{T}_{0,T}$ and any $\tau \in \mathcal{T}_{\sigma,T}$, we have

$$\mathbb{E}\left[Y_\tau - Y_\sigma + \int_\sigma^\tau f(s,\Theta_s)ds + \int_\sigma^\tau g(s,\Theta_s)\overleftarrow{dB}_s + \int_\sigma^\tau h(s,Y_s)dA_s \mid \mathcal{G}_\sigma\right]$$
$$= -\mathbb{E}\left[(K_\tau - K_\sigma) + (C_{\tau-} - C_{\sigma-}) \mid \mathcal{G}_\sigma\right] \leq 0.$$

The last inequality follows from the non-deceasing property of the processes K and C. Finally, since Ξ is an optional process, the claim follows. □

Proposition 3.2 *Let $\beta,\mu > 0$ and $(Y,Z,U) \in \mathfrak{D}^2_{\beta,\mu}$. Then*

$$\left(\int_0^t \Phi_s^{\beta,\mu}Y_{s-}Z_s dW_s + \int_0^t \Phi_s^{\beta,\mu}\int_E Y_{s-}U_s(e)\tilde{N}(ds,de)\right)_{t\leq T}$$

is an RCLL uniformly integrable martingale.

3.2. A priori estimates

In this part, we provide some a priori estimates for the solutions of the RGBDSDEJ (3.1), which will be useful for further applications.

Proposition 3.3 *Let $\beta,\mu > 0$ with $\beta > 4$. Assume that $\alpha \neq \frac{1}{2}$. Let (Y^1,Z^1,U^1,K^1,C^1) and (Y^2_t,Z^2,U^2,K^2,C^2) be two solutions to RGBDSDEJ (3.1) with parameters (ξ^1,f^1,h^1,g^1,A^1) and (ξ^2,f^2,h^2,g^2,A^2), respectively. We denote $\bar{\mathfrak{S}} := \mathfrak{S}^1 - \mathfrak{S}^2$ for $\mathfrak{S} \in \{Y,Z,U,K,C,\xi,f,h,g,A\}$. Then there exists a constant $\mathfrak{c}_{\beta,\mu,\alpha} > 0$ such that*

$$\left\|\bar{Y}\right\|^2_{\bar{\mathcal{S}}^2_{\beta,\mu}} + \left\|\bar{Y}\right\|^2_{\bar{\mathcal{C}}^{2,V}_{\beta,\mu}} + \left\|\bar{Y}\right\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|+A^2}_{\beta,\mu}} + \left\|\bar{Z}\right\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}} + \left\|\bar{U}\right\|^2_{\bar{\mathcal{L}}^2_{\beta,\mu}}$$
$$\leq \mathfrak{c}_{\beta,\mu,\alpha}\left(\|\bar{\xi}\|^2_{\bar{\mathcal{S}}^2_{2\beta,2\mu}} + \left\|\frac{\bar{f}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)}{a.}\right\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}} + \left\|h^1(\cdot,Y^2_-)\right\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|}_{\beta,\mu}} + \left\|\bar{h}(\cdot,Y^2_-)\right\|^2_{\bar{\mathcal{C}}^{2,A^2}_{\beta,\mu}}\right.$$
$$\left. + \left\|\bar{g}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)\right\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}} + \|\bar{\xi}\|_{\bar{\mathcal{S}}^2_{2\beta,2\mu}}\left(\mathbb{E}\left[\left|\bar{K}_T\right|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\left|\bar{C}_T\right|^2\right]^{\frac{1}{2}}\right)\right),$$

where the spaces $\bar{\mathcal{S}}^2_{\beta,\mu}$ is the same space $\mathcal{S}^2_{\beta,\mu}$ with the process $\Phi^{\beta,\mu}$ replaced by $\bar{\Phi}^{\beta,\mu}$ defined by $\bar{\Phi}^{\beta,\mu} := e^{\bar{\phi}^{\beta,\mu}}$ and $\bar{\phi}^{\beta,\mu} := \beta V + \mu\left(\|\bar{A}\| + A^2\right)$. A similar definition holds for other spaces.

Proof: First, note that the process $(\bar{Y},\bar{Z},\bar{U},\bar{K},\bar{C})$ satisfies the following BSDE:

$$\begin{aligned}\bar{Y}_t =&\bar{\xi}_T+\int_t^T\left\{f^1(s,\Theta^1_s)-f^2(s,\Theta^2_s)\right\}ds+\int_t^T\left\{h^1(s,Y^1_s)-f^2(s,Y^2_s)\right\}dA_s+\int_t^T\left\{g^1(s,\Theta^1_s)-g^2(s,\Theta^2_s)\right\}\overleftarrow{dB}_s\\ &+\left(\bar{K}_T-\bar{K}_t\right)+\left(\bar{C}_{T-}-\bar{C}_{t-}\right)-\int_t^T\bar{Z}_sdW_s-\int_t^T\int_E\bar{U}_s(e)\tilde{N}(ds,de). \end{aligned}\tag{3.2}$$

It suffices to prove the result in the case where $\left\|\bar{A}\right\|_T+A^2_T$ is a bounded random variable, and then apply Fatou's Lemma. Using Gal'chouk-Lenglart formula (see Theorem A.2) to the dynamic (3.2), we have

$$\begin{aligned}&\bar{\Phi}^{\beta,\mu}_t\left|\bar{Y}_t\right|^2+\beta\int_t^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Y}_s\right|^2dV_s+\mu\int_t^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Y}_s\right|^2\left\{d\left\|\bar{A}\right\|_s+dA^2_s\right\}+\int_t^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Z}_s\right|^2ds\\ &=\Phi^{\beta,\mu}_T\left|\bar{\xi}_T\right|^2+2\int_t^T\Phi^{\beta,\mu}_s\bar{Y}_{s-}\left(f^1(s,Y^1_{s-},Z^1_s,U^1_s)-f^2(s,Y^2_{s-},Z^2_s,U^2_s)\right)ds\\ &+2\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{Y}_{s-}\left(h^1(s,Y^1_{s-})dA^1_s-h^2(s,Y^2_{s-})dA^2_s\right)+2\int_t^T\Phi^{\beta,\mu}_s\bar{Y}_{s-}d\bar{K}_s\\ &+2\int_t^T\Phi^{\beta,\mu}_s\;\bar{Y}_sd\bar{C}_s+2\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{Y}_{s-}\left(g^1(s,Y^1_{s-},Z^1_s,U^1_s)-g^2(s,Y^2_{s-},Z^2_s,U^2_s)\right)\overleftarrow{dB}_s\\ &+\int_t^T\bar{\Phi}^{\beta,\mu}_s\left|g^1(s,Y^1_{s-},Z^1_s,U^1_s)-g^2(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2ds-\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta\bar{Y}_s\right)^2\\ &-\sum_{t\leq s<T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta_+\bar{Y}_s\right)^2-2\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{Y}_{s-}\bar{Z}_sdW_s-2\int_t^T\bar{\Phi}^{\beta,\mu}_s\int_E\bar{Y}_{s-}\bar{U}_s(e)\tilde{N}(ds,de).\end{aligned}\tag{3.3}$$

Now, as $\bar{K}$ and $N(\cdot,de)$ does not have common jumps, since $\bar{K}$ have only predictable jump times and $N(\cdot,de)$ jumps only at totally inaccessible stopping times, the left jumps term in (3.3) can be expressed as follows:

$$\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta\bar{Y}_s\right)^2=\int_t^T\bar{\Phi}^{\beta,\mu}_s\int_E\left|\bar{U}_s(e)\right|^2N(ds,de)+\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta\bar{K}_s\right)^2\quad\text{a.s.}$$

Then, we have

$$\int_t^T\bar{\Phi}^{\beta,\mu}_s\left\|\bar{U}_s\right\|^2_Q ds-\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta\bar{Y}_s\right)^2\leq-\int_t^T\bar{\Phi}^{\beta,\mu}_s\int_E\left|\bar{U}_s(e)\right|^2\tilde{N}(ds,de)\quad\text{a.s.}$$

Thus,

$$\mathbb{E}\left[\int_t^T\bar{\Phi}^{\beta,\mu}_s\left\|\bar{U}_s\right\|^2_Q ds-\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\left(\Delta\bar{Y}_s\right)^2\right]\leq 0,\tag{3.4}$$

as $\bar{U}\in\mathcal{L}^2_{\beta,\mu}$.
Now, using the minimality condition on the reflecting processes K and C (see Remark 3.2), and the basic inequality $2ab\leq\phi a^2+\frac{1}{\phi}b^2$, $\forall\imath>0$, we can easily derive that, for any $\theta_0>0$,

$$\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{Y}_{s-}d\bar{K}_s\leq 2\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{\xi}_{s-}d\bar{K}_s\leq\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}\left|\bar{\xi}_\tau\right|^2\right)\left(\bar{K}_T-\bar{K}_t\right)\quad\text{a.s.,}\tag{3.5}$$

and

$$\int_t^T\bar{\Phi}^{\beta,\mu}_s\;\bar{Y}_sd\bar{C}_s=2\sum_{t<s\leq T}\bar{\Phi}^{\beta,\mu}_s\;\bar{Y}_s\Delta\bar{C}_s\leq 2\int_t^T\bar{\Phi}^{\beta,\mu}_s\bar{\xi}_sd\bar{C}_s\leq\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}\left|\bar{\xi}_\tau\right|^2\right)\left(\bar{C}_T-\bar{C}_t\right)\quad\text{a.s.}\tag{3.6}$$

Moreover, using **(H-M)**-(i)–(iv), we have

$$\begin{aligned}
&2\bar{Y}_{s-}\left(f^1(s,Y^1_{s-},Z^1_s,U^1_s)-f^2(s,Y^2_{s-},Z^2_s,U^2_s)\right)ds\\
&\leq 2\left|\lambda_s\right|\left|\bar{Y}_{s-}\right|^2 ds+2\bar{Y}_{s-}\left(\gamma_s\left|\bar{Z}_s\right|+\kappa_s\left\|\bar{U}_s\right\|_Q\right)ds+2\bar{Y}_{s-}\bar{f}(s,Y^2_{s-},Z^2_s,U^2_s)ds\\
&\leq\left(2+\theta_1+\frac{1}{\theta_2}\right)\left|\bar{Y}_{s-}\right|^2 dV_s+\theta_2\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|^2_Q\right)ds+\frac{1}{\theta_1}\left|\frac{\bar{f}(s,Y^2_{s-},Z^2_s,U^2_s)}{a_s}\right|^2 ds,
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&2\bar{Y}_{s-}\left(h^1(s,Y^1_{s-})dA^1_s-h^2(s,Y^2_{s-})dA^2_s\right)\\
&\leq 2\bar{Y}_{s-}\left(h^1(s,Y^2_{s-})\{dA^1_s-dA^2_s\}+\bar{h}(s,Y^2_{s-})dA^2_s\right)\\
&\leq\theta_3\left|\bar{Y}_{s-}\right|^2\left\{d\left\|\bar{A}\right\|_s+dA^2_s\right\}+\frac{1}{\theta_3}\left|h^1(s,Y^2_{s-})\right|^2 d\left\|\bar{A}\right\|_s+\frac{1}{\theta_3}\left|\bar{h}(s,Y^2_{s-})\right|^2 dA^2_s,
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&\left|g^1(s,Y^1_{s-},Z^1_s,U^1_s)-g^2(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds\\
&\leq 2\left(\left|g^1(s,Y^1_{s-},Z^1_s,U^1_s)-g^1(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds+\left|\bar{g}(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds\right)\\
&\leq 2\left|\bar{Y}_{s-}\right|^2 dV_s+2\alpha\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|^2_Q\right)ds+2\left|\bar{g}(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds,
\end{aligned}\tag{3.9}$$

for some arbitrary constants θ_1, θ_2, $\theta_3>0$.
Now, by choosing the constant θ_2 to satisfy $\frac{1}{\beta-4}<\theta_2<\beta_\alpha$ with $\beta_\alpha:=(1-2\alpha)\mathbb{1}_{\{\alpha<\frac{1}{2}\}}+(2\alpha-1)\mathbb{1}_{\{\alpha>\frac{1}{2}\}}$, and $\theta_1<\beta-4-\frac{1}{\theta_2}$ in (3.7), $\theta_3<\mu$ in (3.8), then plugging this with (3.9) into (3.3), then taking the expectation on both sides while considering (3.4), (3.5), the result from Proposition 3.2, and Cauchy-Schwartz inequality, we get the existence of a constant $\mathfrak{c}_{\beta,\mu,\alpha}>0$ such that

$$\begin{aligned}
&\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Y}_s\right|^2 dV_s+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Y}_s\right|^2\left\{d\left\|\bar{A}\right\|_s+dA^2_s\right\}+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{Z}_s\right|^2 ds+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left\|\bar{U}_s\right\|^2_Q ds\\
&\leq\mathfrak{c}_{\beta,\mu,\alpha}\left(\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}_\tau\left|\bar{\xi}_\tau\right|^2\right]+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\frac{\bar{f}(s,Y^2_{s-},Z^2_s,U^2_s)}{a_s}\right|^2 ds\right.\\
&+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left\{\left|h^1(s,Y^2_{s-})\right|^2 d\left\|\bar{A}\right\|_s+\left|\bar{h}(s,Y^2_{s-})\right|^2 dA^2_s\right\}\\
&\left.+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{g}(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds+\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}_\tau\left|\bar{\xi}_\tau\right|^2\right]^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\bar{K}_T\right|^2\right]^{\frac{1}{2}}+\mathbb{E}\left[\left|\bar{C}_T\right|^2\right]^{\frac{1}{2}}\right)\right).
\end{aligned}\tag{3.10}$$

Finally, following a similar a approach as above with the same computations done in the proof of Proposition 3.2, taking the essential supremum over $\tau\in\mathcal{T}_{0,T}$, and using (3.10), we derive that

$$\begin{aligned}
\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{\beta,\mu}_\tau\left|\bar{Y}_\tau\right|^2\right]\leq\mathfrak{c}_{\beta,\mu,\alpha}&\left(\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}_\tau\left|\bar{\xi}_\tau\right|^2\right]+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\frac{\bar{f}(s,Y^2_{s-},Z^2_s,U^2_s)}{a_s}\right|^2 ds\right.\\
&+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left\{\left|h^1(s,Y^2_{s-})\right|^2 d\left\|\bar{A}\right\|_s+\left|\bar{h}(s,Y^2_{s-})\right|^2 dA^2_s\right\}\\
&+\mathbb{E}\int_0^T\bar{\Phi}^{\beta,\mu}_s\left|\bar{g}(s,Y^2_{s-},Z^2_s,U^2_s)\right|^2 ds\\
&\left.+\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\bar{\Phi}^{2\beta,2\mu}_\tau\left|\bar{\xi}_\tau\right|^2\right]^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\bar{K}_T\right|^2\right]^{\frac{1}{2}}+\mathbb{E}\left[\left|\bar{C}_T\right|^2\right]^{\frac{1}{2}}\right)\right).
\end{aligned}$$

Completing the proof. □

Using the same notations as in Proportion 3.3, we have the following useful corollaries

Corollary 3.1 *Let $\beta,\mu>0$ with $\beta>3$. Let (Y^1,Z^1,U^1,K^1,C^1) and (Y^2_t,Z^2,U^2,K^2,C^2) be two solutions to RGBDSDEJ (3.1) with parameters (ξ^1,f^1,h^1,g,A^1) and (ξ^2,f^2,h^2,g,A^2), respectively. Then there exists a constant $\mathfrak{c}_{\beta,\mu}>0$ such that*

$$\begin{aligned}
&\|\bar{Y}\|^2_{\bar{\mathcal{S}}^2_{\beta,\mu}}+\|\bar{Y}\|^2_{\bar{\mathcal{C}}^{2,V}_{\beta,\mu}}+\|\bar{Y}\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|+A^2}_{\beta,\mu}}+\|\bar{Z}\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|\bar{U}\|^2_{\bar{\mathcal{L}}^2_{\beta,\mu}}\\
&\le \mathfrak{c}_{\beta,\mu}\Bigg(\|\bar{\xi}\|^2_{\bar{\mathcal{S}}^2_{2\beta,2\mu}}+\left\|\frac{\bar{f}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)}{a_\cdot}\right\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|h^1(\cdot,Y^2_-)\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|}_{\beta,\mu}}+\|\bar{h}(\cdot,Y^2_-)\|^2_{\bar{\mathcal{C}}^{2,A^2}_{\beta,\mu}}\\
&+\|\bar{g}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|\bar{\xi}\|_{\bar{\mathcal{S}}^2_{2\beta,2\mu}}\left(\mathbb{E}\left[\left|\bar{K}_T\right|^2\right]^{\frac12}+\mathbb{E}\left[\left|\bar{C}_T\right|^2\right]^{\frac12}\right)\Bigg),
\end{aligned}$$

Proof: The proof is analogous to the one presented in Proposition 3.3, with the constant $\theta_2>0$ chosen as $\frac{1}{\beta-3}<\theta_2<1-\alpha$, and $\theta_1>0$ satisfying $\theta_1<\beta-3-\frac{1}{\theta_2}$, along with the same choice of $0<\theta_3<\mu$. □

In the case where the coefficient g is independent of (z,u), the following results can be derived directly from Proposition 3.3.

Corollary 3.2 *Let $\beta,\mu>0$ with $\beta>4$. Let (Y^1,Z^1,U^1,K^1,C^1) and (Y^2_t,Z^2,U^2,K^2,C^2) be two solutions to RGBDSDEJ (3.1) with parameters (ξ^1,f^1,h^1,g^1,A^1) and (ξ^2,f^2,h^2,g^2,A^2), respectively, where g^1 and g^2 are independent of the (z,u) variables. Then there exists a constant $\mathfrak{c}_{\beta,\mu}>0$ such that*

$$\begin{aligned}
&\|\bar{Y}\|^2_{\bar{\mathcal{S}}^2_{\beta,\mu}}+\|\bar{Y}\|^2_{\bar{\mathcal{C}}^{2,V}_{\beta,\mu}}+\|\bar{Y}\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|+A^2}_{\beta,\mu}}+\|\bar{Z}\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|\bar{U}\|^2_{\bar{\mathcal{L}}^2_{\beta,\mu}}\\
&\le \mathfrak{c}_{\beta,\mu}\Bigg(\|\bar{\xi}\|^2_{\bar{\mathcal{S}}^2_{2\beta,2\mu}}+\left\|\frac{\bar{f}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)}{a_\cdot}\right\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|h^1(\cdot,Y^2_-)\|^2_{\bar{\mathcal{C}}^{2,\|\bar{A}\|}_{\beta,\mu}}+\|\bar{h}(\cdot,Y^2_-)\|^2_{\bar{\mathcal{C}}^{2,A^2}_{\beta,\mu}}\\
&+\|\bar{g}(s,Y^2_-)\|^2_{\bar{\mathcal{H}}^2_{\beta,\mu}}+\|\bar{\xi}\|_{\bar{\mathcal{S}}^2_{2\beta,2\mu}}\left(\mathbb{E}\left[\left|\bar{K}_T\right|^2\right]^{\frac12}+\mathbb{E}\left[\left|\bar{C}_T\right|^2\right]^{\frac12}\right)\Bigg),
\end{aligned}$$

Corollary 3.3 *Let $\beta,\mu>0$ with $\beta>4$. Let (Y^1,Z^1,U^1,K^1,C^1) and (Y^2_t,Z^2,U^2,K^2,C^2) be two solutions to RGBDSDEJ (3.1) with parameters (ξ,f^1,h^1,g^1,A) and (ξ,f^2,h^2,g^2,A), respectively, where g^1 and g^2 are independent of the (z,u) variables. Then there exists a constant $\mathfrak{c}_{\beta,\mu}>0$ such that*

$$\begin{aligned}
&\|\bar{Y}\|^2_{\mathcal{S}^2_{\beta,\mu}}+\|\bar{Y}\|^2_{\mathcal{C}^{2,V}_{\beta,\mu}}+\|\bar{Y}\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}}+\|\bar{Z}\|^2_{\mathcal{H}^2_{\beta,\mu}}+\|\bar{U}\|^2_{\mathcal{L}^2_{\beta,\mu}}\\
&\le \mathfrak{c}_{\beta,\mu}\left(\left\|\frac{\bar{f}(s,Y^2_-,Z^2_\cdot,U^2_\cdot)}{a_\cdot}\right\|^2_{\mathcal{H}^2_{\beta,\mu}}+\|\bar{h}(\cdot,Y^2_-)\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}}+\|\bar{g}(s,Y^2_-)\|^2_{\mathcal{H}^2_{\beta,\mu}}\right),
\end{aligned}$$

Corollary 3.4 *Let $\beta,\mu>0$ with $\beta>3$. Let (Y,Z,U,K,C) be a solutions to RGBDSDEJ (3.1) with parameters (ξ,f,h,g,A). Then there exists a constant $\mathfrak{c}_{\beta,\mu,\alpha}>0$ such that*

$$\begin{aligned}
&\|Y\|^2_{\mathcal{S}^2_{\beta,\mu}}+\|Y\|^2_{\mathcal{C}^{2,V}_{\beta,\mu}}+\|Y\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}}+\|Z\|^2_{\mathcal{H}^2_{\beta,\mu}}+\|U\|^2_{\mathcal{L}^2_{\beta,\mu}}+\mathbb{E}\left[\left|K_T\right|^2\right]+\mathbb{E}\left[\left|C_T\right|^2\right]\\
&\le \mathfrak{c}_{\beta,\mu,\alpha}\left(\|\xi\|^2_{\mathcal{S}^2_{2\beta,2\mu}}+\left\|\frac{\varphi}{a_\cdot}\right\|^2_{\mathcal{H}^2_{\beta,\mu}}+\|\psi\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}}+\|g(s,0)\|^2_{\mathcal{H}^2_{\beta,\mu}}\right),
\end{aligned}$$

Proof: The proof of this result is similar to the one performed in Proposition 3.3, with a few small modifications. Instead of using inequalities (3.5) and (3.6), we use the following:

$$2\int_t^T \Phi_s^{\beta,\mu} Y_{s-} dK_s \leq 2\int_t^T \bar{\Phi}_s^{\beta,\mu} \bar{\xi}_{s-} d\bar{K}_s \leq \theta_0 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \bar{\Phi}^{2\beta,2\mu} \left|\xi_\tau\right|^2 + \frac{1}{\theta_0}\left(K_T - K_t\right) \quad \text{a.s.},$$

and

$$2\int_t^T \Phi_s^{\beta,\mu}\, \bar{Y}_s d\bar{C}_s = 2\sum_{t<s\leq T} \Phi_s^{\beta,\mu}\, \bar{Y}_s \Delta\bar{C}_s \leq 2\int_t^T \Phi_s^{\beta,\mu} \bar{\xi}_s d\bar{C}_s$$
$$\leq \theta_0 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \Phi^{2\beta,2\mu} \left|\bar{\xi}_\tau\right|^2 + \frac{1}{\theta_0}\left(C_T - C_t\right) \quad \text{a.s.},$$

for all $\theta_0 > 0$. Then, using the BSDE (3.1)-(i), the fact that $K + C_-$ is an increasing process, along with Remark 3.3 (in particular, we derive that $K_T C_{T-} = K_T C_T$ is a positive random variable), and assumptions **(H-M)**-(v)-(vi), we can derive the claim by choosing an appropriate constant θ_0. □

3.3. Existence and uniqueness of a solution

This section provides the existence and uniqueness result for the RGBDSDEJ (3.1) associated with (ξ, f, g, h, A) satisfying the **(H-M)** conditions.

3.3.1. Reasoning scheme. We address two main frameworks:

- *Stochastic Lipschitz case:* We first consider the case where the two generators f and h are stochastic Lipschitz with respect to the y-variable. The existence and uniqueness result is performed in two main steps:
  - The case where f, h, and g are independent of the (y, z, u)-variables.
  - The general case, where we use the Picard iteration method.
- *Stochastic monotone case:* Here, we apply the Yosida approximation method, using the results from the stochastic Lipschitz case. This is also addressed in two stages:
  - The case where f is independent of the (z, u)-variables and g is independent of the (y, z, u)-variables.
  - The general case, where we also employ the Picard iteration method, similar to the approach used in the Stochastic Lipschitz case.

3.3.2. Stochastic Lipschitz coefficients. We assume that f and h are stochastic Lipschitz with respect to y, i.e. there exists an $\mathbb{H}$-adapted process $\chi : \Omega \times [0,T] \to \mathbb{R}^{+*}$ such that for all

$$|f(t,y,z,u) - f(t,y',z,u)| + |h(t,y) - h(t,y')| \leq \chi_t \left|y - y'\right|, \tag{3.11}$$

Toward the end of this part, we define $V_t^\star := \int_0^t \mathfrak{a}_s^2 ds$ and $\mathcal{A}_t := \int_0^t \chi_s dA_s$, where $\mathfrak{a}_s^2 := \rho_s + \chi_s + \gamma_s^2 + \kappa_s^2$ for $s \in [0,T]$. Moreover, we assume that for all $s \in [0,T]$, $\chi_s \geq \epsilon$ for some $\epsilon > 0$.
We denote **(H-M')** the assumption **(H-M)**, where **(H-M)**-(i)-(ii) is replaced by (3.11), and **(H-M)**-(vii)-(viii) by

(vii') $\mathbb{E}\left[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \hat{\Phi}_\tau^{2\beta,2\mu} \left|\xi_\tau\right|^2\right] < \infty$.

(viii') $\mathbb{E}\left[\int_0^T \hat{\Phi}_t^{\beta,\mu}\left(\left|\frac{\varphi_t}{\mathfrak{a}_t}\right|^2 + |g(t,0,0,0)|^2\right) dt + \int_0^T \hat{\Phi}_t^{\beta,\mu} \frac{|\psi_t|^2}{\chi_t} dA_t\right] < \infty$, for any β, $\mu > 0$, where

$$\hat{\phi}_t^{\beta,\mu} = \beta V_t^\star + \mu \mathcal{A}_s \text{ and } \hat{\Phi}^{\beta,\mu} := e^{\hat{\phi}^{\beta,\mu}} \text{ for } \beta, \mu > 0.$$

Finally, we denote by $\mathfrak{R}^2_{\beta,\mu}$ the space $\left(\hat{\mathcal{S}}^2_{\beta,\mu} \cap \hat{\mathcal{C}}^{2,V^\star+\mathcal{A}}_{\beta,\mu}\right) \times \hat{\mathcal{H}}^2_{\beta,\mu} \times \hat{\mathcal{L}}^2_{\beta,\mu}$, which we equip with the norm $\|\cdot\|_{\mathfrak{B}^2_{\beta,\mu}}$ defined by $\|(Y,Z,U)\|_{\mathfrak{R}^2_{\beta,\mu}} := \|Y\|^2_{\hat{\mathcal{S}}^2_{\beta,\mu}} + \|Y\|^2_{\hat{\mathcal{C}}^{2,V^\star+\mathcal{A}}_{\beta,\mu}} + \|Z\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}} + \|U\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}$, for $(Y,Z,U) \in \mathfrak{R}^2_{\beta,\mu}$. Here, the spaces $\hat{\mathcal{S}}^2_{\beta,\mu}$ is the same space $\mathcal{S}^2_{\beta,\mu}$ with the process $\Phi^{\beta,\mu}$ replaced by $\hat{\Phi}^{\beta,\mu}$. A similar definition holds for other spaces. Note that the space $\left(\mathfrak{R}^2_{\beta,\mu}, \|\cdot\|_{\mathfrak{R}^2_{\beta,\mu}}\right)$ is a Banach space (see, e.g., [26, Proposition 2.1]). Similar observation holds for the space $\hat{\mathcal{C}}^{2,V^\star+\mathcal{A}}_{\beta,\mu} \times \hat{\mathcal{H}}^2_{\beta,\mu} \times \hat{\mathcal{L}}^2_{\beta,\mu}$ with the corresponding norm $\|\cdot\|^2_{\hat{\mathcal{C}}^{2,V^\star+\mathcal{A}}_{\beta,\mu}} + \|\cdot\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}} + \|\cdot\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}$.

Case where f, h, and g do not depend on (y,z,u):
In the following proposition, we prove the existence and uniqueness of a solution to the RGBDSDEJ (3.1) with data (ξ, f, h, g, A) in the case where f, h, and g do not depend on (y,z,u).

Proposition 3.4 *Under **(H-M')**, the RGBDSDEJ (3.1) admits a unique solution* $(Y,Z,U,K,C) \in \mathfrak{R}^2_{\beta,\mu} \times \mathcal{S}^2 \times \mathcal{S}^2$ *for all* $(\beta,\mu) \in (4,+\infty) \times (0,+\infty)$*, and for each* $\tau \in \mathcal{T}_{0,T}$*, we have*

$$Y_\tau = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{\tau,T}} \mathbb{E}\left[\xi_\sigma + \int_\tau^\sigma f(s)ds + \int_\tau^\sigma h(s)dA_s + \int_\tau^\sigma g(s)\overleftarrow{dB}_s \mid \mathcal{G}_\tau\right] \quad a.s. \tag{3.12}$$

Proof: For the existence and uniqueness result, we follow a similar approach to that used in [38, Proposition 3.2], based on Mertens decomposition (see Theorem A.1) and various general results from stochastic analysis (see, e.g., [9, Theorem 22, p.432], [9, Remark b, p.435], [26, Remark A.4], and [34, Proposition B.11]). The integrability property follows from Corollary 3.4 in conjunction with assumptions **(H-M)**-(vii')-(viii') and the fact that, by construction, $(K,C) \in \mathcal{S}^2 \times \mathcal{S}^2$. □

The previous proposition allows us to derive the following corollary.

Corollary 3.5 *Let* $(\beta,\mu) \in (2,+\infty) \times (0,+\infty)$ *and* $(y,z,u) \in \mathfrak{R}^2_{\beta,\mu}$*. Under **(H-M')**, there exists a process* $(Y,Z,U,K,M) \in \mathfrak{R}^2_{\beta,\mu} \times \mathcal{S}^2 \times \mathcal{S}^2$*, which is the unique solution of the RGBDSDEJ (3.1) associated with data* $(\xi, f(\cdot, y_\cdot, z_\cdot, u_\cdot), h(\cdot, y_\cdot), g(\cdot, y_\cdot, z_\cdot, u_\cdot), A)$*.*

Proof: Define $\mathfrak{f}(t) := f(t, y_t, z_t, u_t)$, $\mathfrak{h}(t) := h(t, y_t)$, $\mathfrak{g}(t) := g(t, y_t, z_t, u_t)$. Using the Lipschitz property of f, h, and g in (y,z,u), we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left\{\left(\left|\frac{\mathfrak{f}(s)}{\mathfrak{a}_s}\right|^2 + |\mathfrak{g}(s)|^2\right)ds + \frac{|\mathfrak{h}(s)|^2}{\chi_s}dA_s\right\} \\
&\leq \mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left\{8\,|y_s|^2\,dV_s^\star + 2(3+\alpha)\left(|z_s|^2 + \|u_s\|_Q^2\right)ds\right\} \\
&+ 2\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left(\left|\frac{\varphi_s}{\mathfrak{a}_s}\right|^2 + |g(s,0)|^2\right)ds + 2\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left\{|y_s|^2\,d\mathcal{A}_s + \frac{|\psi_s|^2}{\chi_s}dA_s\right\} < \infty.
\end{aligned}$$

Therefore, Proposition 3.4 yields the desired result. □

General stochastic Lipschitz case:
The first main results of the current paper is giving in the following theorem:

Theorem 3.1 *Let* $(\beta,\mu) \in (3,+\infty) \times (0,+\infty)$*. Under **(H-M')**, there exists constants* $\beta_0 > 0$ *and* $\mu_0 > 0$ *such that, for all* $\beta \geq \beta_0$ *and* $\mu \geq \mu_0$*, the RGBDSDEJ (3.1) has a unique solution* $(Y,Z,U,K,C) \in \mathfrak{R}^2_{\beta,\mu} \times \mathcal{S}^2 \times \mathcal{S}^2$*.*

Proof: Using the Picard approximation sequence is our method for proving existence.
In light of this, we examine the sequence $\{\Theta^n\}_{n\geq 0} \subset \mathfrak{R}^2_{\beta,\mu}$ defined in a recurrence ways as follows: $Y^0 = Z^0 = U^0 = 0$, then for any $n \geq 0$, we let $(Y^{n+1}, Z^{n+1}, U^{n+1}, K^{n+1}, C^{n+1})$ to be the unique solution of the RGBDSDEJ (3.1) associated with $(\xi, f(\cdot, \Theta^n_\cdot), h(\cdot, Y^n_\cdot), g(\cdot, \Theta^n_\cdot))$. In other word, the process $(Y^{n+1}, Z^{n+1}, U^{n+1}, K^{n+1}, C^{n+1})$ satisfies

$$\begin{cases} \text{(i) } (Y^{n+1}, Z^{n+1}, U^{n+1}, K^{n+1}, C^{n+1}) \in \mathfrak{R}^2_{\beta,\mu} \times \mathcal{S}^2 \times \mathcal{S}^2; \\ \text{(ii) } Y^{n+1}_\tau = \xi_T + \int_\tau^T f(s, \Theta^n_s) ds + \int_\tau^T h(s, Y^n_s) dA_s + \int_\tau^T g(s, \Theta^n_s) \overleftarrow{dB}_s \\ \qquad - \int_\tau^T Z^{n+1}_s dW_s - \int_\tau^T \int_E U^{n+1}_s(e) \tilde{N}(ds, de) + \left(K^{n+1}_T - K^{n+1}_\tau\right) \\ \qquad + \left(C^{n+1}_{T-} - C^{n+1}_{\tau-}\right), \qquad \forall \tau \in \mathcal{T}_{0,T} \text{ a.s.}; \\ \text{(iii) } Y^{n+1}_\tau \geq \xi_\tau, \text{ a.s. } \forall \tau \in \mathcal{T}_{0,T}; \\ \text{(iv) } K^{n+1} \text{ is a non-decreasing right-continuous predictable process with} \\ \qquad K^{n+1} = K^{n+1,c} + K^{n+1,d} \text{ such that } K_0 = 0, \quad \int_0^T \mathbb{1}_{\{Y^{n+1}_s > \xi_s\}} dK^{n+1,c}_s = 0, \\ \qquad \text{and } \left(Y^{n+1}_{\tau-} - \xi_{\tau-}\right) \Delta K^{n+1,d}_\tau = 0 \text{ a.s. } \forall \tau \in \mathcal{T}^p_{0,T}; \\ \text{(v) } C^{n+1} \text{ is a non-decreasing right-continuous predictable purely discontinuous} \\ \qquad \text{process with } C^{n+1}_{0-} = 0 \text{ such that } \left(Y^{n+1}_\tau - \xi_\tau\right) \Delta C^{n+1}_\tau = 0 \text{ a.s. } \forall \tau \in \mathcal{T}_{0,T}. \end{cases} \tag{3.13}$$

Using Corollary 3.5, we deduce that, for every $n \geq 1$, the RGBDSDEJ (3.13) has a unique solution.
Now, in order to simplify notations, we set $\hat{\mathcal{R}}^{n+1} = \mathcal{R}^{n+1} - \mathcal{R}^n$ for $\mathcal{R}^n \in \{Y^n, Z^n, U^n, K^n, C^n\}$ and $\mathfrak{f}^{\hat{\Theta}^n}(t) = f(t, \Theta^n_t) - f(t, \Theta^{n-1}_t)$, $\mathfrak{h}^{\hat{\Theta}^n}(t) = h(t, Y^n_t) - h(t, Y^{n-1}_t)$, $\mathfrak{g}^{\hat{\Theta}^n}(t) = g(t, \Theta^n_t) - g(t, \Theta^{n-1}_t)$.
From (3.13)-(ii), the state process $\hat{Y}^{n+1}$ satisfies the following BDSDE:

$$\begin{aligned} \hat{Y}^{n+1}_t = & \int_t^T \mathfrak{f}^{\hat{\Theta}^n}(s) ds + \int_t^T \mathfrak{h}^{\hat{\Theta}^n}(s) dA_s + \int_t^T \mathfrak{g}^{\hat{\Theta}^n}(s) \overleftarrow{dB}_s - \int_t^T \hat{Z}^{n+1}_s dW_s \\ & - \int_\tau^T \int_E \hat{U}^{n+1}_s(e) \tilde{N}(ds, de) + \left(\hat{K}^{n+1}_T - \hat{K}^{n+1}_t\right) + \left(\hat{C}^{n+1}_{T-} - \hat{C}^{n+1}_{t-}\right). \end{aligned} \tag{3.14}$$

In order to apply Gal'chouk-Lenglart formula to the Dynamic (3.14), we need the following estimations related to the newly introduced generators:

$$\begin{aligned} 2\hat{Y}^{n+1}_s \mathfrak{f}^{\hat{\Theta}^n}(s) ds & \leq 2\left|\hat{Y}^{n+1}_s\right| \left(\chi_s \left|\hat{Y}^n_s\right| + \gamma_s \left|\hat{Z}^n_s\right| + \kappa_s \left\|\hat{U}^n_s\right\|_Q\right) ds \\ & \leq \left(\chi_s + \frac{1}{\varepsilon_1}\{\gamma^2_s + \kappa^2_s\}\right) \left|\hat{Y}^{n+1}_s\right|^2 ds + \chi_s \left|\hat{Y}^n_s\right| ds + \varepsilon_1 \left(\left|\hat{Z}^n_s\right|^2 + \left\|\hat{U}^n_s\right\|^2_Q\right) ds \\ & \leq \left(1 + \frac{1}{\varepsilon_1}\right) \left|\hat{Y}^{n+1}_s\right|^2 dV^\star_s + \left|\hat{Y}^n_s\right|^2 dV^\star_s + \varepsilon_1 \left(\left|\hat{Z}^n_s\right|^2 + \left\|\hat{U}^n_s\right\|^2_Q\right) ds, \end{aligned} \tag{3.15}$$

and

$$2\hat{Y}^{n+1}_s \mathfrak{h}^{\hat{\Theta}^n}(s) dA_s \leq 2\chi_s \left|\hat{Y}^{n+1}_s\right| \left|\hat{Y}^n_s\right| dA_s \leq \frac{1}{\varepsilon_2} \left|\hat{Y}^{n+1}_s\right|^2 d\mathcal{A}_s + \varepsilon_2 \left|\hat{Y}^n_s\right| d\mathcal{A}_s, \tag{3.16}$$

and

$$\left|\mathfrak{g}^{\hat{\Theta}^n}(s)\right|^2 ds \leq \rho_s \left|\hat{Y}^n_s\right|^2 ds + \alpha \left(\left|\hat{Z}^n_s\right|^2 + \left\|\hat{U}^n_s\right\|^2_Q\right) ds \leq \left|\hat{Y}^n_s\right|^2 dV^\star_s + \alpha \left(\left|\hat{Z}^n_s\right|^2 + \left\|\hat{U}^n_s\right\|^2_Q\right) ds. \tag{3.17}$$

Next, applying the Gal'chouk-Lenglart formula to $\hat{\Phi}^{\beta,\mu} \left|\hat{Y}^{n+1}\right|^2$ with $\hat{Y}^{n+1}$ defined by (3.14), and using the Skorokhod condition of the reflection processes $\hat{K}^{n+1}$ and $\hat{C}^{n+1}$, which implies $\hat{Y}^{n+1}_{s-} d\hat{K}^{n+1}_s \leq 0$

and $\hat{Y}_s^{n+1} d\hat{C}_s^{n+1} \leq 0$, along with inequalities (3.15), (3.16), (3.17), as well as the basic inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ for all $\varepsilon > 0$, and following similar argumentation as in Proposition 3.3, we derive, for any β, μ, ε_1, $\varepsilon_2 > 0$,

$$\begin{aligned}
&\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n+1}\right|^2 \left(\left(\beta - 1 - \frac{1}{\varepsilon_1}\right) dV_s^\star + \left(\mu - \frac{1}{\varepsilon_2} d\mathcal{A}_s\right)\right) + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left(\left|\hat{Z}_s^{n+1}\right|^2 + \left\|\hat{U}_s^{n+1}\right\|_Q^2\right) ds \\
&\leq 2\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 dV_s^\star + \varepsilon_2 \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 d\mathcal{A}_s + (\varepsilon_1 + \alpha)\,\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left\{\left|\hat{Z}_s^{n}\right|^2 + \left\|\hat{U}_s^{n}\right\|_Q^2\right\} ds \\
&\leq (\varepsilon_1 + \alpha)\left(\frac{2}{\varepsilon_1 + \alpha}\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 dV_s^\star + \frac{\varepsilon_2}{\varepsilon_1 + \alpha}\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 d\mathcal{A}_s + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left\{\left|\hat{Z}_s^{n}\right|^2 + \left\|\hat{U}_s^{n}\right\|_Q^2\right\} ds\right).
\end{aligned}$$

Fix $\varepsilon_1 > 0$, choose $\varepsilon_2 = \varepsilon_1 + \alpha$, and define $\bar{\mathfrak{c}} = \frac{2}{\varepsilon_1 + \alpha}$, $\beta_0 = 1 + \frac{1}{\varepsilon} + \bar{\mathfrak{c}}$, and $\mu_0 = 1 + \frac{1}{\varepsilon_2}$. By choosing $\beta \geq \beta_0$ and $\mu \geq \mu_0$, we obtain

$$\begin{aligned}
&\bar{\mathfrak{c}}\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n+1}\right|^2 dV_s^\star + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n+1}\right|^2 d\mathcal{A}_s + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left\{\left|\hat{Z}_s^{n+1}\right|^2 + \left\|\hat{U}_s^{n+1}\right\|_Q^2\right\} ds \\
&\leq (\varepsilon_1 + \alpha)\left(\bar{\mathfrak{c}}\mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 dV_s^\star + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left|\hat{Y}_s^{n}\right|^2 d\mathcal{A}_s + \mathbb{E}\int_t^T \hat{\Phi}_s^{\beta,\mu} \left\{\left|\hat{Z}_s^{n}\right|^2 + \left\|\hat{U}_s^{n}\right\|_Q^2\right\} ds\right).
\end{aligned}$$

Using simple iterations, we establish that

$$\begin{aligned}
&\bar{\mathfrak{c}}\left\|\hat{Y}^{n+1}\right\|_{\hat{\mathcal{C}}_{\beta,\mu}^{2,V^\star}}^2 + \left\|\hat{Y}^{n+1}\right\|_{\hat{\mathcal{C}}_{\beta,\mu}^{2,\mathcal{A}}}^2 + \left\|\hat{Z}^{n+1}\right\|_{\hat{\mathcal{H}}_{\beta,\mu}^2}^2 + \left\|\hat{U}^{n+1}\right\|_{\hat{\mathcal{L}}_{\beta,\mu}^2}^2 \\
&\leq (\varepsilon_1 + \alpha)^n \left(\bar{\mathfrak{c}}\left\|\hat{Y}^{1}\right\|_{\hat{\mathcal{C}}_{\beta,\mu}^{2,V^\star}}^2 + \left\|\hat{Y}^{1}\right\|_{\hat{\mathcal{C}}_{\beta,\mu}^{2,\mathcal{A}}}^2 + \left\|\hat{Z}^{1}\right\|_{\hat{\mathcal{H}}_{\beta,\mu}^2}^2 + \left\|\hat{U}^{1}\right\|_{\hat{\mathcal{L}}_{\beta,\mu}^2}^2\right).
\end{aligned}$$

Choosing $\varepsilon_1 > 0$ such that $\varepsilon_1 < 1 - \alpha$, we deduce that $\{\Theta^n\}_{n\geq 0}$ is a Cauchy sequence in the Banach space $\hat{\mathcal{C}}_{\beta,\mu}^{2,V^\star + \mathcal{A}} \times \hat{\mathcal{H}}_{\beta,\mu}^2 \times \hat{\mathcal{U}}_{\beta,\mu}^2$ for any $\beta \geq \beta_0$ and $\mu \geq \mu_0$. It remains to show that the sequence $\{Y^n\}_{n\geq 0}$ is a Cauchy sequence in the Banach space $\hat{\mathcal{S}}_{\beta,\mu}^2$. To this end, for any two integers $n, m \geq 0$, we denote $\mathcal{R}^{n,m} = \mathcal{R}^n - \mathcal{R}^m$ for $\mathcal{R}^n \in \{Y^n, Z^n, U^n, K^n, C^n\}$ and $\mathfrak{f}^{\Theta^{n,m}}(t) = f(t, \Theta_t^n) - f(t, \Theta_t^m)$, $\mathfrak{h}^{\Theta^{n,m}}(t) = h(t, Y_t^n) - h(t, Y_t^m)$, $\mathfrak{g}^{\Theta^{n,m}}(t) = g(t, \Theta_t^n) - g(t, \Theta_t^m)$.
Similar to (3.14), we have the following dynamic:

$$\begin{aligned}
Y_t^{n+1,m+1} &= \int_t^T \mathfrak{f}^{\Theta^{n,m}}(s) ds + \int_t^T \mathfrak{h}^{\Theta^{n,m}}(s) dA_s + \int_t^T \mathfrak{g}^{\Theta^{n,m}}(s) \overleftarrow{dB}_s - \int_t^T Z_s^{n+1,m+1} dW_s \\
&\quad - \int_\tau^T \int_E U_s^{n+1,m+1}(e) \tilde{N}(ds, de) + \left(K_T^{n+1,m+1} - K_t^{n+1,m+1}\right) + \left(C_{T-}^{n+1,m+1} - C_{t-}^{n+1,m+1}\right).
\end{aligned} \tag{3.18}$$

Now, by applying the Gal'chouk-Lenglart formula to $\hat{\Phi}_\tau^{\beta,\mu}\left|Y_\tau^{n+1,m+1}\right|^2$ on $[\tau, T]$ with $Y^{n+1,m+1}$ defined by (3.18), taking the essential supremum over all $\tau \in \mathcal{T}_{0,T}$, and taking the expectation, and following similar computations as in Proposition 3.3, we obtain

$$\begin{aligned}
\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \hat{\Phi}_\tau^{\beta,\mu}\left|Y_\tau^{n+1,m+1}\right|^2\right] &\leq 2\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left|Y_{s-}^{n+1,m+1}\right|\left|\mathfrak{f}^{\Theta^{n,m}}(s)\right| ds + 2\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left|Y_{s-}^{n+1,m+1}\right|\left|\mathfrak{h}^{\Theta^{n,m}}(s)\right| dA_s \\
&\quad + \mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left|\mathfrak{g}^{\Theta^{n,m}}(s)\right|^2 ds + 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T \hat{\Phi}_s^{\beta,\mu} Y_{s-}^{n+1,m+1}\mathfrak{g}^{\Theta^{n,m}}(s)\overleftarrow{dB}_s\right| \\
&\quad + 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T \hat{\Phi}_s^{\beta,\mu} Y_{s-}^{n+1,m+1} Z_s^{n+1,m+1} dW_s\right| \\
&\quad + 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T \hat{\Phi}_s^{\beta,\mu}\int_E Y_{s-}^{n+1,m+1} U_s^{n+1,m+1}(e)\tilde{N}(ds, de)\right|.
\end{aligned} \tag{3.19}$$

But, we have

$$2\left|Y_s^{n+1,m+1}\right|\left|\mathfrak{f}^{\Theta^{n,m}}(s)\right| ds \leq \left|Y_s^{n+1,m+1}\right|^2 dV_s^\star + \left|\frac{\mathfrak{f}^{\Theta^{n,m}}(s)}{\mathfrak{a}_s}\right|^2 ds, \tag{3.20}$$

and

$$2\left|Y_s^{n+1,m+1}\right|\left|\mathfrak{h}^{\Theta^{n,m}}(s)\right|dA_s\leq\left|Y_s^{n+1,m+1}\right|^2d\mathcal{A}_s+\frac{\left|\mathfrak{h}^{\Theta^{n,m}}(s)\right|^2}{\chi_s}dA_s. \tag{3.21}$$

Moreover, we have

$$\left|\mathfrak{g}^{\Theta^{n,m}}(s)\right|^2ds\leq 3\left|Y_s^{n,m}\right|^2dV_s^\star+3\alpha\left(\left|Z_s^{n,m}\right|^2+\left\|U_s^{n,m}\right\|_Q^2\right)ds, \tag{3.22}$$

and

$$\left|\frac{\mathfrak{f}^{\Theta^{n,m}}(s)}{\mathfrak{a}_s}\right|^2ds\leq 3\left|Y_s^{n,m}\right|^2dV_s^\star+3\left(\left|Z_s^{n,m}\right|^2+\left\|U_s^{n,m}\right\|_Q^2\right)ds,\quad \frac{\left|\mathfrak{h}^{\Theta^{n,m}}(s)\right|^2}{\chi_s}dA_s\leq\left|Y_s^{n,m}\right|^2d\mathcal{A}_s. \tag{3.23}$$

Finally, it remains to use the B-D-G inequality as the one used in Proposition 3.2, as follows:

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T\hat{\Phi}_s^{\beta,\mu}Y_{s-}^{n+1,m+1}Z_s^{n+1,m+1}dW_s\right|\right]\leq\frac{1}{4}\left\|Y^{n+1,m+1}\right\|^2_{\hat{\mathcal{S}}^2_{\beta,\mu}}+4\mathfrak{c}^2\left\|Z^{n+1,m+1}\right\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}. \tag{3.24}$$

Similarly, we can derive that,

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T\hat{\Phi}_s^{\beta,\mu}\int_E Y_{s-}^{n+1,m+1}U_s^{n+1,m+1}(e)\tilde{N}(ds,de)\right|\right]\leq\frac{1}{4}\left\|Y^{n+1,m+1}\right\|^2_{\hat{\mathcal{S}}^2_{\beta,\mu}}+4\mathfrak{c}^2\left\|U^{n+1,m+1}\right\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}, \tag{3.25}$$

and

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\int_\tau^T\hat{\Phi}_s^{\beta,\mu}Y_{s-}^{n+1,m+1}\mathfrak{g}^{\Theta^{n,m}}(s)\overleftarrow{dB}_s\right|\right]\leq\frac{1}{4}\left\|Y^{n+1,m+1}\right\|^2_{\hat{\mathcal{S}}^2_{\beta,\mu}}+4\mathfrak{c}^2\left\|\mathfrak{g}^{\Theta^{n,m}}\right\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}. \tag{3.26}$$

Plugging (3.20), (3.21), (3.22), (3.23), (3.24), (3.25), and (3.26) into (3.19), we derive the existence of a constant $\mathfrak{c}>0$ such that

$$\begin{aligned}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\hat{\Phi}_\tau^{\beta,\mu}\left|Y_\tau^{n+1,m+1}\right|^2\right]\leq\mathfrak{c}\Big(&\left\|Y^{n+1,m+1}\right\|^2_{\hat{\mathcal{C}}^{2,V^\star}_{\beta,\mu}}+\left\|Y^{n+1,m+1}\right\|^2_{\hat{\mathcal{C}}^{2,\mathcal{A}}_{\beta,\mu}}+\left\|Z^{n+1,m+1}\right\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}\\&+\left\|U^{n+1,m+1}\right\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}+\left\|Y^{n,m}\right\|^2_{\hat{\mathcal{C}}^{2,V^\star}_{\beta,\mu}}+\left\|Y^{n,m}\right\|^2_{\hat{\mathcal{C}}^{2,\mathcal{A}}_{\beta,\mu}}+\left\|Z^{n,m}\right\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}+\left\|U^{n,m}\right\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}\Big).\end{aligned}$$

Since $\{Y^n,Z^n,U^n\}_{n\geq 0}$ is a Cauchy sequence in the Banach space $\hat{\mathcal{C}}^{2,V^\star+\mathcal{A}}_{\beta,\mu}\times\hat{\mathcal{H}}^2_{\beta,\mu}\times\hat{\mathcal{U}}^2_{\beta,\mu}$ for any $\beta\geq\beta_0$ and $\mu\geq\mu_0$ (with β_0 and μ_0 defined above), we deduce that $\{Y^n\}_{n\geq 0}$ is also a Cauchy sequence in $\hat{\mathcal{S}}^2_{\beta,\mu}$ for the same value range of β and μ. Therefore, the sequence $\{Y^n,Z^n,U^n\}_{n\geq 0}$ converges to a limit $\Theta:=(Y,Z,U)$ in the space $\mathfrak{R}^2_{\beta,\mu}$ for any $\beta\geq\beta_0$ and $\mu\geq\mu_0$. In other word, the triplet (Y,Z,U) satisfies

$$\lim_{n\to+\infty}\left\|(Y^n-Y,Z^n-Z,U^n-U)\right\|_{\mathfrak{R}^2_{\beta,\mu}}=0,\quad\forall(\beta,\mu)\in[\beta_0,+\infty)\times[\mu_0,+\infty). \tag{3.27}$$

Now, let's focus on the convergence of the driver and martingale parts of (3.13)-(ii).
To this end, based on (3.27), Cauchy-Schwarz inequality, and the stochastic Lipschitz property of f and h, we infer that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T\left(f(s,\Theta_s^n)-f(s,\Theta_s)\right)ds\right|^2\right]&\leq\frac{1}{\beta}\mathbb{E}\int_0^T\hat{\Phi}_s^{\beta,\mu}\left|\frac{f(s,\Theta_s^n)-f(s,\Theta_s)}{\mathfrak{a}_s}\right|^2ds\\&\leq\frac{3}{\beta}\left(\left\|Y^n-Y\right\|^2_{\hat{\mathcal{C}}^{2,V^\star}_{\beta,\mu}}+\left\|Z^n-Z\right\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}+\left\|U^n-U\right\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}\right)\xrightarrow[n\to+\infty]{}0,\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T\left(h(s,Y_s^n)-h(s,Y_s)\right)dA_s\right|^2\right]&\leq\frac{1}{\mu}\mathbb{E}\int_0^T\hat{\Phi}_s^{\beta,\mu}\frac{\left|h(s,Y_s^n)-h(s,Y_s)\right|^2}{\chi_s}ds\\&\leq\frac{1}{\mu}\left\|Y^n-Y\right\|^2_{\hat{\mathcal{C}}^{2,\mathcal{A}}_{\beta,\mu}}\xrightarrow[n\to+\infty]{}0.\end{aligned} \tag{3.29}$$

In the same way, by the B-D-G inequality and (3.27), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T (g(s,\Theta_s^n)-g(s,\Theta_s))\overleftarrow{dB}_s\right|^2\right] &\leq \mathfrak{c}\mathbb{E}\int_0^T \hat{\Phi}_s^{\beta,\mu}\left|g(s,\Theta_s^n)-g(s,\Theta_s)\right|^2 ds\\ &\leq \mathfrak{c}\left(\|Y^n-Y\|^2_{\mathcal{C}^{2,V^\star}_{\beta,\mu}}+\alpha\left\{\|Z^n-Z\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}+\|U^n-U\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}\right\}\right). \end{aligned} \tag{3.30}$$

For the martingale part, we simply use once more (3.27) the B-D-G inequality as in Proposition 3.2, which yields to

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T Z_s^n dW_s-\int_t^T Z_s dW_s\right|^2\right]\leq \mathfrak{c}\,\|Z^n-Z\|^2_{\hat{\mathcal{H}}^2_{\beta,\mu}}, \tag{3.31}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T\int_E U_s^n(e)\tilde{N}(ds,de)-\int_t^T\int_E U_s(e)\tilde{N}(ds,de)\right|^2\right]\leq \mathfrak{c}\,\|U^n-U\|^2_{\hat{\mathcal{L}}^2_{\beta,\mu}}. \tag{3.32}$$

Finally, for any $\tau\in\mathcal{T}_{0,T}$, let us set $\hat{K}^n_\tau := K^n_\tau + C^n_{\tau-}$ and

$$\hat{K}_\tau := Y_0 - Y_\tau - \int_0^\tau f(s,\Theta_s)ds - \int_0^\tau h(s,Y_s)dA_s + \int_0^\tau g(s,\Theta_s)\overleftarrow{dB}_s + \int_0^\tau Z_s dW_s - \int_0^\tau\int_E U_s(e)\tilde{N}(ds,de).$$

From (3.27), (3.28), (3.29), (3.30), (3.31), (3.32), we deduce that

$$\lim_{n\to\infty}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\left|\hat{K}^n_\tau-\hat{K}_\tau\right|^2\right]=0.$$

Clearly, the process $\hat{K}$ is predictable, increasing, and exhibits finite left and right limit. Moreover, from Proposition 3.1 and Mertens decomposition, we derive that $\hat{K}$ has the form $\hat{K}_\tau = K_\tau + C_{\tau-}$ for any $\tau\in\mathcal{T}_{0,T}$, where K is a right-continuous, predictable non-decreasing process, and C is an adapted, right-continuous non-decreasing process. Moreover, we have $\mathbb{E}\left[|K_T|^2+|C_T|^2\right]<\infty$ and then $(K,C)\in\mathcal{S}^2\times\mathcal{S}^2$. Henceforth, we deduce that (Y,Z,U,K,C) is a solution to RGBDSDEJ (3.1) with stochastic Lipschitz coefficients f, h, and g, a lower obstacle ξ.
For the uniqueness result, the claim can be easily derived from Corollary 3.1 and the Mertens decomposition. Indeed, for any two solutions (Y^1,Z^1,U^1,K^1,C^1) and (Y^2,Z^2,U^2,K^2,C^2) of the RGBDSDEJ (3.1) associated with (ξ,f,g,h,A), define $\hat{\mathcal{R}}=\mathcal{R}^1-\mathcal{R}^2$ for $\mathcal{R}\in\{Y,Z,U,K,C\}$. From Corollary 3.1, we conclude that $\left(\hat{Y},\hat{Z},\hat{U}\right)=(0,0,0)$. Finally, by Proposition 3.1 and the uniqueness of the associated Mertens processes, we derive that $(K^1,C^1)=(K^2,C^2)$, thus completing the proof. □

3.3.3. Stochastic monotone coefficients. Now, we deal the main finding of this paper concerning the existence and uniqueness result for the RGBDSDEJ (3.1) with data (ξ,f,h,g,A) satisfying the assumption **(H-M)**.

The reasoning is divided into two main stages:

1. Generator independent of the (z,u)-variables: A Yosida approximation approach. It is well known that Yosida's approximation technique has been used by numerous authors in this context. In particular, Hu [28] (see also [30]) established existence and uniqueness for a class of forward–backward stochastic differential equations on arbitrarily prescribed time horizons under specific monotonicity conditions on the coefficients. In this part, we use this approximation approach to show that equation (3.1) has a solution in the case of stochastically monotone coefficients. First, we assume that the RGBDSDEJ (3.1) driver f is independent of the (z,u) variables and that the coefficient g is independent of the (y,z,u) variables, i.e., $\mathfrak{f}(t,y):=f(,t,y,z,u)$ and $\mathfrak{g}(t):=g(t,y,z,u)$ for any $(y,z,u)\in\mathbb{R}^2\times\mathbb{L}^2_Q$. After that, we approximate the coefficients $(\mathfrak{f}(t,y)+\lambda_t y)_{t\leq T}$ and $(h(t,y)+\varrho_t y)_{t\leq T}$ by a family of stochastic Lipschitz mappings F_ε and H_ε indexed by $\varepsilon\in(0,1]$, which yields a sequence of equations whose solutions converge to the solution of (3.1) in this special case. For classical generalized BSDEs in the Brownian setting, such equations have been studied by Pardoux and Răşcanu [44]. Convergence results for classical GBSDEs driven by RCLL martingales associated with the family $\{F_\varepsilon,H_\varepsilon\}_{\varepsilon\in(0,1]}$ have been employed by Elmansouri and El Otmani [20], and by Elhachemy and El Otmani [14] for the reflected situation in the Brownian–Poisson setting. The idea here is to construct a family of approximating RGBDSDEJs associated with $\{F_\varepsilon,H_\varepsilon\}_{\varepsilon\in(0,1]}$, then prove that the corresponding family of solutions is a Cauchy sequence in an appropriate Banach space. The limiting process of this family is then exactly the desired solution.

The main result of this step is given in the following theorem:

Theorem 3.2 *Let $\beta,\mu>0$. Assume that $\mathfrak{g}(\cdot)\in\mathcal{H}^2_{\beta,\mu}$ for any $\beta,\mu>0$. Under the assumption **(H-M)**, there exists a unique solution (Y,Z,U,K,C) of the RGBDSDEJ (3.1) associated with $(\xi,\mathfrak{f},h,\mathfrak{g},A)$, which belongs to $\mathfrak{B}^2_{\beta,\mu}$ for any $(\beta,\mu)\in(2,+\infty)\times(0,+\infty)$.*

Proof: The proof is done in several steps.

Step 1: Yosida approximation of f and h.
First, let us consider the corresponding BSDE associated with $(\xi,\mathfrak{f},h,\mathfrak{g},A)$, i.e., we have

$$\begin{aligned} Y_t=\xi_T+\int_t^T\mathfrak{f}(s,Y_s)ds+\int_t^T h(s,Y_s)dA_s+\int_t^T\mathfrak{g}(s)\overleftarrow{dB}_s-\int_t^T Z_sdW_s\\ -\int_t^T\int_E U_s(e)\tilde{N}(ds,de)+(K_T-K_t)+(C_{T-}-C_{t-}). \end{aligned} \tag{3.33}$$

The BSDE (3.33) can be written in the following equivalent form

$$\begin{aligned} Y_t=\xi_T+\int_t^T(F(s,Y_s)+\lambda_sY_s)\,ds+\int_t^T(H(s,Y_s)+\varrho_sY_s)\,dA_s+\int_t^T\mathfrak{g}(s)\overleftarrow{dB}_s\\ -\int_t^T Z_sdW_s-\int_\tau^T\int_E U_s(e)\tilde{N}(ds,de)+(K_T-K_t)+(C_{T-}-C_{t-}). \end{aligned}$$

with

$$F(s,y):=\mathfrak{f}(s,y)-\lambda_s y,\qquad\text{and}\qquad H(s,y):=h(s,y)-\varrho_s y.$$

Therefore, studying the existence and uniqueness problem for the date $(\xi,\mathfrak{f},h,\mathfrak{g},A)$ is equivalent of studying the following RGBDSDEJ:

$$\begin{cases} \text{(i) } Y_\tau=\xi_T+\int_\tau^T(F(s,Y_s)+\lambda_sY_s)\,ds+\int_\tau^T(H(s,Y_s)+\varrho_sY_s)\,dA_s+\int_\tau^T\mathfrak{g}(s)\overleftarrow{dB}_s\\ \qquad -\int_\tau^T Z_sdW_s-\int_\tau^T\int_E U_s(e)\tilde{N}(ds,de)+(K_T-K_\tau)+(C_{T-}-C_{\tau-}),\quad \text{a.s. }\forall\tau\in\mathcal{T}_{0,T};\\ \text{(ii) } Y_\tau\geq\xi_\tau,\ \text{a.s. }\forall\tau\in\mathcal{T}_{0,T};\\ \text{(iii) } K \text{ is a non-decreasing right-continuous predictable process with}\\ \qquad K=K^c+K^d \text{ such that } K_0=0,\ \int_0^T\mathbb{1}_{\{Y_s>\xi_s\}}dK^c_s=0\text{ a.s.},\\ \qquad \text{and } (Y_{\tau-}-\xi_{\tau-})\,\Delta K^d_\tau=0\text{ a.s. }\forall\tau\in\mathcal{T}^p_{0,T};\\ \text{(iv) C is a non-decreasing right-continuous predictable purely discontinuous}\\ \qquad \text{process with } C_{0-}=0 \text{ such that } (Y_\tau-\xi_\tau)\,\Delta C_\tau=0\text{ a.s. }\forall\tau\in\mathcal{T}_{0,T}. \end{cases} \tag{3.34}$$

It is evident that the novel drivers F and H meet the subsequent monotonicity property:

- $(y-y')\,(F(s,y)-F(s,y'))\leq 0.$

- $(y-y')(H(s,y)-H(s,y')) \leq 0.$

Using this, it follows from [44, Annex B, p.524], that for every, $(\omega,t,y) \in \Omega \times [0,T] \times \mathbb{R}$ and $\phi > 0$, there exists a unique $J_\phi^F = J_\phi^F(\omega,t,y)$, $J_\phi^H = J_\phi^H(\omega,t,y) \in \mathbb{R}$ such that

$$J_\phi^F - \phi F(\omega,t,J_\phi^F) = y, \quad J_\phi^H - \phi H(\omega,t,J_\phi^H) = y.$$

The Yosida approximation of F and H is defined respectively by $F_\phi = F_\phi(\omega,t,y)$ and $H_\phi = H_\phi(\omega,t,y) \in \mathbb{R}$ such that

$$\begin{aligned} F_\phi(\omega,t,y) &:= \frac{1}{\phi}\left(J_\phi^F(\omega,t,y)-y\right) = F\left(t, y+\phi F_\phi\right), \\ H_\phi(\omega,t,y) &:= \frac{1}{\phi}\left(J_\phi^H(\omega,t,y)-y\right) = H\left(t, y+\phi H_\phi\right). \end{aligned} \tag{3.35}$$

Note that (F_ϕ, H_ϕ) is the unique pair satisfying (3.35).
From [44, Annex B, Proposition 6.7], recall that

(Y1) For every $y \in \mathbb{R}$, the processes $F_\phi(\cdot,\cdot,y)$, $H_\phi(\cdot,\cdot,y) : \Omega \times [0,T] \to \mathbb{R}$ are $\mathcal{F}_t$-progressively measurable and we have

(Y2) $\forall \phi, \delta > 0$, $\forall t \in [0,T]$, $\forall y, y' \in \mathbb{R}$, $\mathbb{P}$-a.s.

(i) $\left|J_\phi^F(t,y) - J_\phi^F(t,y')\right| + \left|J_\phi^H(t,y) - J_\phi^H(t,y')\right| \leq |y-y'|.$

(ii) $(y-y')(F_\phi(t,y) - F_\phi(t,y')) \leq 0.$

(iii) $(y-y')(H_\phi(t,y) - H_\phi(t,y')) \leq 0.$

(iv) $|F_\phi(t,y) - F_\phi(t,y')| + |H_\phi(t,y) - H_\phi(t,y')| \leq \frac{2}{\phi}|y-y'|.$

(v) $|F_\phi(t,y)| \leq |F(t,y)|$ and $|H_\phi(t,y)| \leq |H(t,y)|$.

(vi) $(y-y')(F_\phi(t,y) - F_\delta(t,y')) \leq (\phi+\delta) F_\phi(t,y) F_\delta(t,y'),$
$(y-y')(H_\phi(t,y) - H_\delta(t,y')) \leq (\phi+\delta) H_\phi(t,y) H_\delta(t,y').$

Step 2: Approximating equation
Let $0 < \phi \leq 1$. The *Yosida approximating equations* of the RGBDSDEJ (3.34) is given by

$$\begin{cases} \text{(i) } Y_\tau^\phi = \xi_T + \int_\tau^T \left(F_\phi(s,Y_s^\phi) + \lambda_s Y_s^\varepsilon\right) ds + \int_\tau^T \left(H_\phi(s,Y_s^\phi) + \varrho_s Y_s^\phi\right) dA_s \\ \qquad + \int_\tau^T \mathfrak{g}(s) \overleftarrow{dB}_s - \int_\tau^T Z_s^\phi dW_s - \int_\tau^T \int_E U_s^\phi(e) \tilde{N}(ds,de) + \left(K_T^\phi - K_\tau^\phi\right) \\ \qquad + \left(C_{T-}^\phi - C_{\tau-}^\phi\right), \quad \text{a.s. } \forall \tau \in \mathcal{T}_{0,T}; \\ \text{(ii) } Y_\tau^\phi \geq \xi_\tau, \text{ a.s. } \forall \tau \in \mathcal{T}_{0,T}; \\ \text{(iii) } K^\phi \text{ is a non-decreasing right-continuous predictable process with} \\ \qquad K^\phi = K^{\phi,c} + K^{\phi,d} \text{ suhc that } K_0^\phi = 0, \int_0^T \mathbb{1}_{\{Y_s > \xi_s\}} dK_s^{\phi,c} = 0 \text{ a.s.,} \\ \qquad \text{and } (Y_{\tau-} - \xi_{\tau-}) \Delta K_\tau^{\phi,d} = 0 \text{ a.s. } \forall \tau \in \mathcal{T}_{0,T}^p; \\ \text{(iv) C is a non-decreasing right-continuous predictable purely discontinuous} \\ \qquad \text{process with } C_{0-}^\phi = 0 \text{ such that } \left(Y_\tau^\phi - \xi_\tau\right) \Delta C_\tau^\phi = 0 \text{ a.s. } \forall \tau \in \mathcal{T}_{0,T}. \end{cases} \tag{3.36}$$

Define $\hat{V}^\phi := \int_0^t \left(|\lambda_s| + \frac{2}{\phi}\right) ds$, $\mathcal{A}_t^\phi := \int_0^t (|\varrho_s| + \frac{2}{\phi}) dA_s$, and $\hat{\Phi}_t^{\phi,(\beta,\mu)} := e_t^{\beta \hat{V}_t^\phi + \mu \mathcal{A}^\phi}$ for $t \in [0,T]$.

Note that from **(Y2)**-(iv), we derive that the generators $(F_\phi(t,y)+\lambda_t y)_{t\le T}$ and $(H_\phi(t,y)+\varrho_t y)_{t\le T}$ are stochastic Lipschitz with respect to $\left(|\lambda_t|+\frac{2}{\phi}\right)_{t\le T}$ and $\left(|\varrho_t|+\frac{2}{\phi}\right)_{t\le T}$, respectively, and that $\mathfrak{g}(\cdot)\in\mathcal{H}^2_{\beta,\mu}$. Moreover, since $|\varrho_s|^2=|\varrho_s|\,|\varrho_s|\ge\epsilon\,|\varrho_s|$, we deduce that

$$\begin{aligned}&\mathbb{E}\int_0^T\hat{\Phi}^{\phi,(\beta,\mu)}_s\left(\frac{|\varphi_s|^2}{|\lambda_s|+\frac{2}{\phi}}+|\mathfrak{g}(s)|^2\right)ds+\mathbb{E}\int_0^T\hat{\Phi}^{\phi,(\beta,\mu)}_s\frac{|\psi_s|^2}{|\varrho_s|+\frac{2}{\phi}}dA_s\\&\le\mathfrak{c}_{\phi,\epsilon,T}\mathbb{E}\int_0^T\Phi^{\beta,\mu}_s\left(\left\{|\varphi_s|^2+|\mathfrak{g}(s)|^2\right\}ds+|\psi_s|^2\,dA_s\right)<\infty.\end{aligned}$$

Therefore, using Theorem 3.1, we derive that the RGBDSDEJ (3.36) admits a unique solution $(Y^\phi,Z^\phi,U^\phi,K^\phi,C^\phi)\in\mathfrak{R}^{2,1}_{\beta,\mu}\times\mathcal{S}^2\times\mathcal{S}^2$ for any $(\beta,\mu)\in(\beta_\phi,+\infty)\times(\mu_\phi,+\infty)$, for some constants $\beta_\phi,\mu_\phi>0$, where $\mathfrak{R}^{2,1}_{\beta,\mu}$ is simply the space $\mathfrak{R}^2_{\beta,\mu}$ considered in the subsection 3.3.2 with $\Phi^{\beta,\mu}$ replaced by $\hat{\Phi}^{\phi,(\beta,\mu)}$. Note that here we only need the existence result. The integrability property of the solution for any given parameters $(\beta,\mu)\in(2,+\infty)\times(0,+\infty)$ will be obtained in the following step.

Step 3: Uniform boundedness of $\{(Y^\phi,Z^\phi,U^\phi,K^\phi,C^\phi)\}_{0<\phi\le1}$.
Let $0<\phi\le1$. By applying the Gal'chouk-Lenglart formula, we have, for all $t\in[0,T]$ and each $\varepsilon\in(0,1]$, and any $\beta,\mu>0$,

$$\begin{aligned}&\Phi^{\beta,\mu}_t\left|Y^\phi_t\right|^2+\beta\int_t^T\Phi^{\beta,\mu}_s\left|Y^\phi_s\right|^2dV_s+\mu\int_t^T\Phi^{\beta,\mu}_s\left|Y^\phi_s\right|^2dA_s+\int_t^T\Phi^{\beta,\mu}_s\left|Z^\phi_s\right|^2ds\\&=\Phi^{\beta,\mu}_T|\xi_T|^2+2\int_t^T\Phi^{\beta,\mu}_sY^\phi_{s-}\left(F_\varepsilon(s,Y^\phi_{s-})+\lambda_sY^\phi_{s-}\right)ds\\&+2\int_t^T\Phi^{\beta,\mu}_sY^\phi_{s-}\left(H(s,Y^\phi_{s-})+\varrho_sY^\phi_{s-}\right)dA_s+\int_t^T\Phi^{\beta,\mu}_s|\mathfrak{g}(s)|^2ds+2\int_t^T\Phi^{\beta,\mu}_sY^\phi_{s-}dK^\phi_s\\&+2\int_t^T\Phi^{\beta,\mu}_sY^\phi_sdC^\phi_s+2\int_t^T\Phi^{\beta,\mu}_sY^\varepsilon_{s-}\mathfrak{g}(s)\overleftarrow{dB}_s-\sum_{t<s\le T}\Phi^{\beta,\mu}_s\left(\Delta Y^\phi_s\right)^2\\&-\sum_{t\le s<T}\hat{\Phi}^{\beta,\mu}_s\left(\Delta_+Y^\phi_s\right)^2-2\int_t^T\Phi^{\beta,\mu}_sY^\phi_{s-}Z^\phi_sdW_s-2\int_t^T\Phi^{\beta,\mu}_s\int_EY^\phi_{s-}U^\phi_s(e)\tilde{N}(ds,de).\end{aligned}\tag{3.37}$$

From proprieties **(Y2)**-(ii)-(iii)-(v) and **(H-M)**-(vi), for any $\theta_1,\theta_2>0$, we obtain

$$2Y^\phi_{s-}\left(F_\phi(s,Y^\phi_{s-})+\lambda_sY^\phi_{s-}\right)ds\le2\left|Y^\phi_{s-}\right||F_\phi(s,0)|\,ds+2\,|\lambda_s|\left|Y^\phi_{s-}\right|^2ds\le(2+\theta_1)\left|Y^\phi_s\right|^2dV_s+\frac{1}{\theta_1}\left|\frac{\varphi_s}{a_s}\right|^2ds,\tag{3.38}$$

and

$$2Y^\phi_{s-}\left(H_\varepsilon(s,Y^\phi_{s-})+\varrho_sY^\phi_{s-}\right)dA_s\le2\left|Y^\phi_{s-}\right||H_\phi(s,0)|\,ds\le\theta_2\left|Y^\phi_s\right|^2dA_s+\frac{1}{\theta_2}|\psi_s|^2\,dA_s.\tag{3.39}$$

Now, using the Skorokhod condition, for any $\theta_3>0$, we have

$$\begin{aligned}2\int_t^T\Phi^{\beta,\mu}_sY^\phi_{s-}dK^\phi_s&=2\int_t^T\Phi^{\beta,\mu}_s\xi_{s-}dK^\phi_s\le2\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\Phi^{\beta,\mu}_\tau|\xi_\tau|\right)\left(K^\phi_T-K^\phi_t\right)\\&\le\theta_3\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\Phi^{2\beta,2\mu}_\tau|\xi_\tau|^2+\frac{1}{\theta_3}\left|K^\phi_T-K^\phi_t\right|^2.\end{aligned}\tag{3.40}$$

Similarly, we have

$$2\int_t^T\Phi^{\beta,\mu}_sY^\phi_sdC^\phi_s\le2\int_t^T\Phi^{\beta,\mu}_s\xi_sdC^\phi_s\le\theta_3\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\Phi^{2\beta,2\mu}_\tau|\xi_\tau|^2+\frac{1}{\theta_3}\left|C^\phi_T-C^\phi_t\right|^2.\tag{3.41}$$

Plugging (3.38), (3.39), (3.40), (3.41) into (3.37), taking the expectation, using Propitiation 3.2, and following similar computations as those used in the proof of Proposition 3.3, we have

$$\begin{aligned}
&\mathbb{E}\left[\Phi_t^{\beta,\mu}\left|Y_t^{\phi}\right|^2\right]+(\beta-2-\theta_1)\,\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Y_s^{\phi}\right|^2 dV_s+(\mu-\theta_2)\,\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Y_s^{\phi}\right|^2 dA_s\\
&+\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Z_s^{\phi}\right|^2 ds+\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left\|U_s^{\phi}\right\|_Q^2 ds\\
&\leq (1+2\theta_3)\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\Phi_\tau^{2\beta,2\mu}\left|\xi_\tau\right|^2\right]+\frac{1}{\theta_1}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\frac{\varphi_s}{a_s}\right|^2 ds+\frac{1}{\theta_2}\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|\psi_s\right|^2 dA_s\\
&+\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\mathfrak{g}(s)\right|^2 ds+\frac{1}{\theta_3}\left(\mathbb{E}\left[\left|K_T^{\phi}-K_t^{\phi}\right|^2\right]+\mathbb{E}\left[\left|C_T^{\phi}-C_t^{\phi}\right|^2\right]\right).
\end{aligned}\tag{3.42}$$

On the hand, from **(Y2)**-(iv)-(v) and (H-M)-(v)-(vi), we have

$$\begin{aligned}
\left|F_\phi(s,Y_s^{\phi})+\lambda_s Y_s^{\varepsilon}\right|^2 \leq 2\left|F_\phi(s,Y_s^{\phi})\right|^2+2\left|\lambda_s\right|^2\left|Y_s^{\phi}\right|^2 &\leq 2\left|F(s,Y_s^{\phi})\right|^2+2\left|\lambda_s\right|^2\left|Y_s^{\phi}\right|^2\\
&\leq 4\left|\mathfrak{f}(s,Y_s^{\phi})\right|^2+6\left|\lambda_s\right|^2\left|Y_s^{\phi}\right|^2\\
&\leq 8\left|\varphi_s\right|^2+\left(8\zeta^2+6\left|\lambda_s\right|^2\right)\left|Y_s^{\phi}\right|^2,
\end{aligned}\tag{3.43}$$

and

$$\left|H_\phi(s,Y_s^{\phi})+\varrho_s Y_s^{\phi}\right|^2 \leq 8\left|\psi_s\right|^2+\left(8\zeta^2+6\left|\varrho_s\right|^2\right)\left|Y_s^{\phi}\right|^2.\tag{3.44}$$

This with **(Y2)**-(v) and Cauchy-Schwarz inequality, implies

$$\begin{aligned}
\mathbb{E}\left[\left(\int_t^T \left\{F_\varepsilon(s,Y_s^{\phi})+\lambda_s Y_s^{\phi}\right\}ds\right)^2\right] &\leq \frac{1}{\beta}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\frac{F_\phi(s,Y_s^{\phi})+\lambda_s Y_s^{\phi}}{a_s}\right|^2 ds\\
&\leq \frac{1}{\beta}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left(\frac{1}{\epsilon}\left\{\frac{8}{\epsilon}\zeta^2+6\right\}\left|Y_s^{\phi}\right|^2 dV_s+8\left|\frac{\varphi_s}{a_s}\right|^2 ds\right).
\end{aligned}\tag{3.45}$$

and

$$\mathbb{E}\left[\left(\int_t^T \left\{H_\phi(s,Y_s^{\phi})+\varrho_s Y_s^{\phi}\right\}dA_s\right)^2\right] \leq \frac{1}{\mu}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left(\left|Y_s^{\phi}\right|^2\left\{6dV_s+8\zeta^2 dA_s\right\}+\left|\psi_s\right|^2 dA_s\right).\tag{3.46}$$

Moreover, using isometric formulas, for any $t\in[0,T]$, we have,

$$\mathbb{E}\left[\left(\int_t^T \mathfrak{g}(s)\overleftarrow{dB}_s\right)^2\right] \leq \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\mathfrak{g}(s)\right|^2 ds,\quad \mathbb{E}\left[\left(\int_t^T Z_s^{\phi}dW_s\right)^2\right] \leq \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Z_s^{\phi}\right|^2 ds,\tag{3.47}$$

and

$$\mathbb{E}\left[\left(\int_t^T\int_E U_s^{\phi}(e)\tilde{N}(ds,de)\right)^2\right] \leq \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left\|U_s^{\phi}\right\|_Q^2 ds.\tag{3.48}$$

Using the BSDE (3.36)-(ii), squaring, and employing estimations (3.45), (3.46), (3.47), and (3.48), we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(K_T^\phi - K_t^\phi\right) + \left(C_{T-}^\phi - C_{t-}^\phi\right)\right|^2\right] \\
&\leq 7\left(\mathbb{E}\left[\Phi_t^{\beta,\mu}\left|Y_t^\phi\right|^2\right] + \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \Phi_\tau^{2\beta,2\mu}|\xi_\tau|^2\right] + \left(\frac{1}{\beta}\left\{\frac{8}{\epsilon^2}\zeta^2 + \frac{6}{\epsilon}\right\} + \frac{6}{\mu}\right)\mathbb{E}\int_t^T \left|Y_s^\phi\right|^2 dV_s\right. \\
&+\frac{8}{\beta}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\frac{\varphi_s}{a_s}\right|^2 ds + \frac{8\zeta^2}{\mu}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Y_s^\phi\right|^2 dA_s + \frac{1}{\mu}\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}|\psi_s|^2 dA_s \\
&\left.+\mathbb{E}\int_t^T \Phi_s^{\beta,\mu}|\mathfrak{g}(s)|^2 ds + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Z_s^\phi\right|^2 ds + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left\|U_s^\phi\right\|_Q^2 ds\right).
\end{aligned} \tag{3.49}$$

Going back to (3.42), choosing $\theta_3 > \mathfrak{c}_{\beta,\mu,\epsilon}$ with $\mathfrak{c}_{\beta,\mu,\epsilon} = \max\left(7, \frac{8}{\beta}, \frac{1}{\beta}\left\{\frac{8}{\epsilon^2}\zeta^2 + \frac{6}{\epsilon}\right\} + \frac{6}{\mu}, \frac{8\zeta^2}{\mu}\right)$, $\theta_1 < \beta - 2$, and $\theta_2 < \mu$, we derive the existence of a constant $\mathfrak{c}_{\beta,\mu,\epsilon} > 0$ (independent of ϕ) such that

$$\begin{aligned}
&\mathbb{E}\left[\Phi_t^{\beta,\mu}\left|Y_t^\phi\right|^2\right] + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Y_s^\phi\right|^2 dV_s + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Y_s^\phi\right|^2 dA_s + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|Z_s^\phi\right|^2 ds + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left\|U_s^\phi\right\|_Q^2 ds \\
&\leq \mathfrak{c}_{\beta,\mu,\epsilon}\left(\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \Phi_\tau^{2\beta,2\mu}|\xi_\tau|^2\right] + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}\left|\frac{\varphi_s}{a_s}\right|^2 ds + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}|\psi_s|^2 dA_s + \mathbb{E}\int_t^T \Phi_s^{\beta,\mu}|\mathfrak{g}(s)|^2 ds\right).
\end{aligned} \tag{3.50}$$

By evaluating at $t = 0$ in (3.50) and using (3.49), along with $C_{0-} = 0$ and Remark 3.3, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|Y_s^\phi\right|^2 dV_s + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|Y_s^\phi\right|^2 dA_s + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|Z_s^\phi\right|^2 ds + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left\|U_s^\phi\right\|_Q^2 ds + \mathbb{E}\left[\left|K_T^\phi + C_T^\phi\right|^2\right] \\
&\leq \mathfrak{c}_{\beta,\mu,\epsilon}\left(\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \Phi_\tau^{2\beta,2\mu}|\xi_\tau|^2\right] + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left(\left|\frac{\varphi_s}{a_s}\right|^2 + |\mathfrak{g}(s)|^2\right) ds + \mathbb{E}\int_0^T \Phi_s^{\beta,\mu}|\psi_s|^2 dA_s\right).
\end{aligned} \tag{3.51}$$

Finally, coming back to (3.37) and applying the B-D-G inequality along with the estimation (3.51), we derive that for any $(\beta,\mu) \in (2,+\infty)\times(0,+\infty)$,

$$\begin{aligned}
&\left\|Y^\phi\right\|_{\mathcal{S}^2_{\beta,\mu}}^2 + \left\|Y^\phi\right\|_{\mathcal{C}^{2,V}_{\beta,\mu}}^2 + \left\|Y^\phi\right\|_{\mathcal{C}^{2,A}_{\beta,\mu}}^2 + \left\|Z^\phi\right\|_{\mathcal{H}^2_{\beta,\mu}}^2 + \left\|U^\phi\right\|_{\mathcal{L}^2_{\beta,\mu}}^2 + \left\|K^\phi + C^\phi\right\|_{\mathcal{S}^2}^2 \\
&\leq \mathfrak{c}_{\beta,\mu,\epsilon}\left(\|\xi\|_{\mathcal{S}^2_{2\beta,2\mu}}^2 + \left\|\frac{\varphi}{a_\cdot}\right\|_{\mathcal{H}^2_{\beta,\mu}}^2 + \|\psi\|_{\mathcal{C}^{2,A}_{\beta,\mu}}^2 + \|\mathfrak{g}(\cdot)\|_{\mathcal{H}^2_{\beta,\mu}}^2\right),
\end{aligned} \tag{3.52}$$

Step 4: $\{(Y^\phi, Z^\phi, U^\phi, K^\phi, C^\phi)\}_{0<\phi\leq 1}$ is a Cauchy sequence in $\mathfrak{B}^2_{\beta,\mu}$.
Let $0 < \phi, \delta \leq 1$. Define $\mathcal{R}^{\phi,\delta} := \mathcal{R}^\phi - \mathcal{R}^\delta$ for $\mathcal{R} \in \{Y, Z, U, K, C\}$ and $\mathfrak{H}_{\phi,\delta}(s, \mathcal{R}_s^{\phi,\delta}) := \mathfrak{H}_\phi(s, \mathcal{R}_s^\phi) - \mathfrak{H}_\delta(s, \mathcal{R}_s^\delta)$ for $\mathfrak{H} \in \{F, H\}$. From (3.36)-(i), we have

$$\begin{aligned}
Y_t^{\phi,\delta} &= \int_t^T \left\{F_{\phi,\delta}(s, Y_s^{\phi,\delta}) + \lambda_s Y_s^{\phi,\delta}\right\} ds + \int_t^T \left\{H_{\phi,\delta}(s, Y_s^{\phi,\delta}) + \varrho_s Y_s^{\phi,\delta}\right\} dA_s \\
&+ \left(K_T^{\phi,\delta} - K_t^{\phi,\delta}\right) + \left(C_{T-}^{\phi,\delta} - C_{t-}^{\phi,\delta}\right) - \int_t^T Z_s^{\phi,\delta} dW_s - \int_t^T \int_E U_s^{\phi,\delta}(e)\tilde{N}(ds,de).
\end{aligned} \tag{3.53}$$

In order to apply Gal'chouk-Lenglart formula to the process $\Phi_t^{\beta,\mu}\left|Y_t^{\phi,\delta}\right|^2$, we need first estimate the driver term in the dynamic (3.53). But from **(Y2)**-(vi), we have

$$Y_s^{\phi,\delta}\left\{F_{\phi,\delta}(s, Y_s^{\phi,\delta}) + \lambda_s Y_s^{\phi,\delta}\right\} \leq (1+\delta)F_\phi(s, Y_s^\phi)F_\delta(s, Y_s^\delta) + |\lambda_s|\left|Y_s^{\phi,\delta}\right|^2, \tag{3.54}$$

and

$$Y_s^{\phi,\delta}\left\{H_{\phi,\delta}(s, Y_s^{\phi,\delta}) + \varrho_s Y_s^{\phi,\delta}\right\} \leq (1+\delta)H_\phi(s, Y_s^\phi)H_\delta(s, Y_s^\delta). \tag{3.55}$$

Using the Cauchy-Schwarz inequality with inequalities (3.43), (3.44), and (3.52) , we obtain

$$\mathbb{E}\int_0^T \Phi_s^{\beta,\mu} F_\phi(s,Y_s^\phi)F_\delta(s,Y_s^\delta)ds \le \left(\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|F_\phi(s,Y_s^\phi)\right|^2 ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|F_\delta(s,Y_s^\delta)\right|^2 ds\right)^{\frac{1}{2}} \le \mathfrak{c}_{\beta,\mu,\epsilon}, \tag{3.56}$$

and

$$\mathbb{E}\int_0^T \Phi_s^{\beta,\mu} H_\phi(s,Y_s^\phi)H_\delta(s,Y_s^\delta)dA_s \le \left(\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|H_\phi(s,Y_s^\phi)\right|^2 ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \Phi_s^{\beta,\mu}\left|H_\delta(s,Y_s^\delta)\right|^2 ds\right)^{\frac{1}{2}} \le \mathfrak{c}_{\beta,\mu,\epsilon}. \tag{3.57}$$

Next, following similar computations as those used in **Step 3**, and using the Skorokhod condition, which implies $Y_{s-}^{\phi,\delta}dK_s^{\phi,\delta}+Y_s^{\phi,\delta}dC_s^{\phi,\delta}\le 0$, along with inequalities (3.54), (3.55), (3.56), and (3.57), we derive that for any $(\phi,\delta)\in(0,1]\times(0,1]$, and all $(\beta,\mu)\in(2,+\infty)\times(0,+\infty)$,

$$\left\|Y^{\phi,\delta}\right\|^2_{\mathcal{S}^2_{\beta,\mu}}+\left\|Y^{\phi,\delta}\right\|^2_{\mathcal{C}^{2,V}_{\beta,\mu}}+\left\|Y^{\phi,\delta}\right\|^2_{\mathcal{C}^{2,A}_{\beta,\mu}}+\left\|Z^{\phi,\delta}\right\|^2_{\mathcal{H}^2_{\beta,\mu}}+\left\|U^{\phi,\delta}\right\|^2_{\mathcal{L}^2_{\beta,\mu}}\le \mathfrak{c}_{\beta,\mu,\epsilon}\left(\phi+\delta\right). \tag{3.58}$$

Therefore $\{(Y^\phi,Z^\phi,U^\phi,)\}_{0<\phi\le 1}$ is a Cauchy sequence in $\mathfrak{L}^2_{\beta,\mu}$ for any $(\beta,\mu)\in(2,+\infty)\times(0,+\infty)$. Then, there exists a triplet (Y,Z,U) such that

$$\lim_{\phi\to 0^+}\left(\left\|Y^\phi-Y\right\|^2_{\mathcal{S}^2_{\beta,\mu}}+\left\|Y^\phi-Y\right\|^2_{\mathcal{C}^{2,V+A}_{\beta,\mu}}+\left\|Z^\phi-Z\right\|^2_{\mathcal{H}^2_{\beta,\mu}}+\left\|U^\phi-U\right\|^2_{\mathcal{L}^2_{\beta,\mu}}\right)=0. \tag{3.59}$$

Moreover, as in the last part of the existence proof in Theorem 3.1, by setting $\hat{K}_\tau^{\phi,\delta}:=K_\tau^{\phi,\delta}+C_{\tau-}^{\phi,\delta}$ for any $\tau\in\mathcal{T}_{0,T}$, we can easily derive from (3.59) that $\{\hat{K}^\phi\}_{0<\phi\le 1}$ forms a Cauchy sequence in $\mathcal{S}^2$. Consequently, there exist two increasing processes $(K,C)\in\mathcal{S}^2\times\mathcal{S}^2$, corresponding to the Mertens processes of the strong optional supermartingale defined in Proposition 3.2, associated with $(\xi,\mathfrak{f},h,\mathfrak{g},A)$.

Step 5: The limiting process (Y,Z,U,K,C) is a the unique solution of the RGBDSDEJ (3.1) with data $(\xi,\mathfrak{f},h,\mathfrak{g},A)$.
From the definition of Yosida approximation (see (3.35)), we can write

$$\begin{aligned}
F_\phi\left(s,Y_s^\phi\right)+\lambda_sY_s^\phi&=\mathfrak{f}\left(s,Y_s^\phi+\phi F_\phi\left(s,Y_s^\phi\right)\right)-\phi\lambda_sF_\phi\left(s,Y_s^\phi\right),\\
H_\phi\left(s,Y_s^\phi\right)+\varrho_sY_s^\phi&=h\left(s,Y_s^\phi+\phi H_\phi\left(s,Y_s^\phi\right)\right)-\phi\varrho_sH_\phi\left(s,Y_s^\phi\right).
\end{aligned}$$

Due to the Cauchy-Schwarz inequality, the uniform estimate (3.58), assumption **(H-M)**-(vi), the fact that $\phi\le 1$, and standard computations, we have

$$\left\{\left(\phi\lambda_tF_\phi\left(t,Y_t^\epsilon\right),\phi F_\phi\left(t,Y_t^\phi\right),t\in[0,T]\right)\right\}_{\epsilon\in]0,1]}\in\mathcal{H}^1\times\mathcal{H}^2.$$

Also

$$\phi\lambda.F_\phi\left(\cdot,Y_\cdot^\phi\right)\xrightarrow[\phi\to 0^+]{\mathcal{H}^1}0\quad\text{and}\quad\phi F_\phi\left(\cdot,Y_\cdot^\phi\right)\xrightarrow[\phi\to 0^+]{\mathcal{H}^2}0.$$

Then, applying the partial reciprocal of the dominated convergence theorem, we deduce the existence of two sub-sequences $\left(\phi_k\lambda.F_{\phi_k}\left(\cdot,Y_\cdot^{\phi_k}\right)\right)_{k\in\mathbb{N}}$ and $\left(\phi_kF_{\phi_k}\left(\cdot,Y_\cdot^{\phi_k}\right)\right)_{k\in\mathbb{N}}$ such that $\phi_k\to 0$ as $k\to+\infty$ and

$$\lim_{k\to+\infty}\phi_k\alpha_s(\omega)F_{\phi_k}\left(\omega,s,Y_s^{\phi_k}(\omega)\right)=0,\ \textit{and}\ \phi_k\lim_{k\to+\infty}F_{\phi_k}\left(\omega,s,Y_s^{\phi_k}(\omega)\right)=0,\ d\mathbb{P}\otimes dt\text{-}a.e.$$

Making use of the continuity of the driver $\mathfrak{f}$ and the fact that $\lim_{k\to+\infty}Y_t^{\phi_k}(\omega)=Y_t(\omega)$, $\mathbb{P}\otimes dt$-a.e., we infer, by the dominated convergence theorem and (3.59), that

$$\lim_{k\to+\infty}\mathbb{E}\left[\int_t^T\left|\mathfrak{f}\left(s,Y_s^{\phi_k}+\phi_kF_\phi\left(s,Y_s^{\phi_k}\right)\right)-\mathfrak{f}(s,Y_s)\right|ds\right]=0.$$

A similar argument gives

$$\lim_{k\to+\infty} \mathbb{E}\left[\int_t^T \left|h\left(s, Y_s^{\phi_k} + \phi_k H_{\phi_k}\left(s, Y_s^{\phi_k}\right)\right) - g(s, Y_s)\right| dA_s\right] = 0.$$

Consequently, $F_{\phi_k}\left(\cdot, Y_\cdot^{\phi_k}\right) + \lambda . Y_\cdot^{\phi_k} \xrightarrow[k\to+\infty]{\mathcal{H}^1} \mathfrak{f}(\cdot, Y_\cdot)$ and $H_{\phi_k}\left(\cdot, Y_\cdot^{\phi_k}\right) + \varrho . Y_\cdot \xrightarrow[k\to+\infty]{\mathcal{C}^{1,A}} h(\cdot, Y_\cdot)$.
By passing to the limit along a subsequence $\left\{\left(Y^{\phi_k}, Z^{\phi_k}, U^{\phi_k}, K^{\phi_k}, C^{\phi_k}\right)\right\}_{k\in\mathbb{N}}$ in the approximating equation (3.36), we deduce that (Y, Z, N, K, C) is a solution of the RGBDSDEJ (3.1) associated with $(\xi, \mathfrak{f}, h, \mathfrak{g}, A)$. Furthermore, by construction, we have $(Y, Z, N, K, C) \in \mathfrak{B}^2_{\beta,\mu}$ for any $(\beta, \mu) \in (2, +\infty) \times (\mu, +\infty)$. Finally, the uniqueness result is derived in a manner similar to that in Theorem 3.1. □

The following corollary may be derived from Theorem 3.2 by following the proof of Corollary 3.5.

Corollary 3.6 *Let $(\beta, \mu) \in (2, +\infty) \times (0, +\infty)$ and $(y, z, u) \in \mathfrak{L}^2_{\beta,\mu}$. Under **(H-M)**, there exists a process $(Y, Z, U, K, M) \in \mathfrak{B}^2_{\beta,\mu}$ that is the unique solution of the RGBDSDEJ (3.1) associated with data $(\xi, f(\cdot, \cdot, z_\cdot, u_\cdot), h, g(\cdot, y_\cdot, z_\cdot, u_\cdot), A)$.*

2. General stochastic monotone coefficients:
The main result of the current paper is given in the following theorem. We use the result of Corollary 3.6 and follow the same procedure as in Theorem 3.1 to prove the existence result using the Picard iteration method. The uniqueness is derived similarly, using Corollary 3.1, Proposition 3.1, and the uniqueness of Mertens processes.

Theorem 3.3 *Let $(\beta, \mu) \in (3, +\infty) \times (0, +\infty)$. Under **(H-M)**, there exists a unique solution $(Y, Z, U, K, C) \in \mathfrak{B}^2_{\beta,\mu}$ of the RGBDSDEJ (3.1) associated with (ξ, f, h, g, A).*

4. Comparison theorem

In this section, we present a comparison theorem for the RGBDSDEJs (3.1). To achieve this, we consider two sets of data: (ξ^1, f^1, h^1, g, A) and (ξ^2, f^2, h^2, g, A), along with their associated solutions $(\Theta^i, K^i, C^i) \in \mathfrak{B}^2_{\beta,\mu}$ for $(\beta, \mu) \in (2, +\infty) \times (0, +\infty)$, where $i \in \{1, 2\}$ and $\Theta^i = (Y^i, Z^i, U^i)$. These solutions exist under the assumptions outlined in Theorem 3.3 concerning the data (i.e., the **(H-M)** assumptions). The following comparison result then holds.

Theorem 4.1 *Assume that **(H-M)** holds and that*

$$\begin{cases} \xi_t^1 \leq \xi_t^2 & a.s. \quad \forall t \in [0, T], \\ f^1(t, y, z, u) \leq f^2(t, y, z, u), & \forall (t, y, z, u) \in [0, T] \times \mathbb{R}^2 \times \mathbb{L}^2_Q, \\ h^1(t, y) \leq h^2(t, y), & \forall (t, y, z, u) \in [0, T] \times \mathbb{R}^2 \times \mathbb{L}^2_Q. \end{cases}$$

Then, we have

$$Y_t^1 \leq Y_t^2, \quad a.s. \quad \forall t \in [0, T].$$

Proof: Define $\bar{\mathcal{R}} = \mathcal{R}^1 - \mathcal{R}^2$ for $\mathcal{R} \in \{Y, Z, U, K, C, f, h\}$. The process $\bar{Y}$ satisfies the following BSDE:

$$\begin{aligned} \bar{Y}_t =& \bar{\xi}_T + \int_t^T \left(f^1(s, \Theta_s^1) - f^2(s, \Theta_s^2)\right) ds + \int_t^T \left(h^1(s, Y_s^1) - h^2(s, Y_s^2)\right) dA_s + \int_t^T \left(g(s, \Theta_s^1) - g(s, \Theta_s^2)\right) \overleftarrow{dB}_s \\ &+ (\bar{K}_T - \bar{K}_t) + (\bar{C}_{T-} - \bar{C}_{t-}) - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e) \tilde{N}(ds, de). \end{aligned} \tag{4.1}$$

By applying Proposition A.1 to $\bar{Y}^+$ with the dynamic (4.1), then using the Gal'chouk-Lenglart formula on $\Phi_t^{\beta,\mu} \left|\bar{Y}_t^+\right|^2$, and with the equalities

$$f^1(s, \Theta_s^1) - f^2(s, \Theta_s^1) = \left\{f^1(s, \Theta_s^1) - f^1(s, \Theta_s^2)\right\} + \bar{f}(s, \Theta_s^2),$$

and

$$h^1(s,Y_s^1)-h^2(s,Y_s^2)=\left\{h^1(s,Y_s^1)-h^1(s,Y_s^2)\right\}+\bar{h}(s,Y_s^2),$$

taking the expectation, along with Proposition 3.2, and under the assumptions that $\bar{f},\ \bar{h}\le 0$, $\xi^1\le\xi^2\le Y^2$, $\xi_-^1\le\xi_-^2\le Y_-^2$, as well as the Skorokhod condition, and performing similar computations as in Proposition 3.3, we derive that, for any $\beta>2$ and $\mu>0$, and all $\tau\in\mathcal{T}_{0,T}$,

$$\begin{aligned}
&\mathbb{E}\left[\Phi_\tau^{\beta,\mu}\left|\bar{Y}_\tau^+\right|^2\right]+\beta\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\left|\bar{Y}_s\right|^2dV_s+\mu\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\left|\bar{Y}_s\right|^2dA_s\\
&+\mathbb{E}\int_\tau^T\Phi_s^{\beta,\mu}\left|\bar{Z}_s\right|^2ds+\mathbb{E}\int_\tau^T\Phi_s^{\beta,\mu}\left\|\bar{U}_s\right\|_Q^2ds\\
&\le\mathbb{E}\left[\Phi_T^{\beta,\mu}\left|\bar{\xi}_T^+\right|^2\right]+2\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\bar{Y}_s^+\left(f^1(s,\Theta_s^1)-f^1(s,\Theta_s^2)\right)ds\\
&+2\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_{s-}>0\}}\Phi_s^{\beta,\mu}\bar{Y}_s^+\left(h^1(s,Y_s^1)-h^1(s,Y_s^2)\right)dA_s+\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\left|g^1(s,\Theta_s^1)-g(s,\Theta_s^2)\right|^2ds.
\end{aligned}\tag{4.2}$$

On the other hand, form **(H-M)**-(i)-(ii)-(iii)-(iv), we have, for any $\phi>0$,

$$\begin{aligned}
&\mathbb{1}_{\{\bar{Y}_s>0\}}\bar{Y}_s^+\left(f^1(s,\Theta_s^1)-f^1(s,\Theta_s^2)\right)ds\\
&\le\left(2+\frac{2}{\phi}\right)\mathbb{1}_{\{\bar{Y}_s>0\}}\left|\bar{Y}_s^+\right|^2dV_s+\phi\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|_Q^2\right)ds,
\end{aligned}$$

and

$$\mathbb{1}_{\{\bar{Y}_{s-}>0\}}\bar{Y}_{s-}^+\left(h^1(s,Y_s^1)-h^1(s,Y_s^2)\right)\le\mathbb{1}_{\{\bar{Y}_{s-}>0\}}\bar{Y}_s\varrho_s\left|\bar{Y}_s\right|^2\le 0,$$

and

$$\mathbb{1}_{\{\bar{Y}_s>0\}}\left|g^1(s,\Theta_s^1)-g(s,\Theta_s^2)\right|^2ds\le\mathbb{1}_{\{\bar{Y}_s>0\}}\left|\bar{Y}_s\right|^2dV_s+\alpha\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|_Q^2\right)ds.$$

Plugging these inequalities into (4.2), we obtain,

$$\begin{aligned}
&\mathbb{E}\left[\Phi_\tau^{\beta,\mu}\left|\bar{Y}_\tau^+\right|^2\right]+\beta\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\left|\bar{Y}_s\right|^2dV_s+\mathbb{E}\int_\tau^T\Phi_s^{\beta,\mu}\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|_Q^2\right)ds\\
&\le\left(3+\frac{2}{\phi}\right)\mathbb{E}\int_\tau^T\mathbb{1}_{\{\bar{Y}_s>0\}}\Phi_s^{\beta,\mu}\left|\bar{Y}_s\right|^2dV_s+(\alpha+\phi)\,\mathbb{E}\int_\tau^T\left(\left|\bar{Z}_s\right|^2+\left\|\bar{U}_s\right\|_Q^2\right)ds.
\end{aligned}$$

Choosing $\phi=(1-\alpha)/2$ and $\beta>3+\frac{2}{\phi}$, we derive that for every $\tau\in\mathcal{T}_{0,T}$, $\left|\bar{Y}_\tau^+\right|^2=0$ a.s. Thus, for every $\tau\in\mathcal{T}_{0,T}$, we have $Y_\tau^1\le Y_\tau^2$ a.s. Since Y^1 and Y^2 are optional processes, then using the optional section theorem (see, e.g., [29, Theorem IV.4.10]), we derive that $Y_t^1\le Y_t^2$ a.s. $\forall t\in[0,T]$. □

Remark 4.1 (An interesting extension) *Note that using Theorem 4.1, we can establish the existence of a minimal solution in the case where the drivers f and h are jointly continuous and satisfy a linear stochastic growth condition. This allows us to work under weaker assumptions than the stochastic monotonicity and Lipschitz conditions imposed on the generators in Subsection 3.3. This result serves as an application of the theoretical findings obtained in this paper*[1].

A. Classical results in stochastic theory beyond right-continuity

A.1. Mertens' decomposition

We start this section by the following definition (see, e.g., [9, Appendix 1]):

[1] The result stated in this remark was recently established by the same authors using the results of the present paper.

Definition A.1 *A real valued optional process $(\mathcal{X}_t)_{t\leq T}$ is said to be a strong optional supermartingale, if*

- *For each stopping time $\tau \in \mathcal{T}_{0,T}$, $\mathcal{X}_\tau$ is integrable.*
- *For any pair $(S,\tau) \in \mathcal{T}_{0,T} \times \mathcal{T}_{0,T}$ such that $S \leq \tau$ a.s., we have $\mathcal{X}_S \geq \mathbb{E}\left[\mathcal{X}_\tau \mid \mathcal{G}_S\right]$ a.s.*

Remark A.1 *The term "strong" refers to the utilization of stopping times instead of deterministic times.*

Definition A.2 *A real valued optional process $(\mathcal{X}_t)_{t\leq T}$ is said to be of class (D), if the family $\{\mathcal{X}_\tau : \tau \in \mathcal{T}_{0,T}\}$ is uniformly integrable.*

Remark A.2 *Note that any strong optional supermartingale of class (D) has paths of almost finite right and left limits over $[0,T]$.*

We recall the decomposition of strong optional supermartingales, known as Mertens' decomposition (see, e.g. [26, Theorem A.1]). The proof is based on using [9, Theorem 20, p.429] combined with [9, Remark 3(b), p.205] and [9, Appendix 1, Theorem 20, equalities (20.2)]).

Theorem A.1 (Mertens decomposition) *Let $(\mathcal{X}_t)_{t\leq T}$ be a strong optional supermartingale of class (D). There exists a unique right-continuous left-limited uniformly integrable martingale $(M_t)_{t\leq T}$, a unique predictable right-continuous non-decreasing process $(A_t)_{t\leq T}$ with $A_0 = 0$ and $\mathbb{E}\left[A_T\right] < \infty$, and a unique right-continuous adapted non-decreasing process $(C_t)_{t\leq T}$, which is purely discontinuous, i.e., $C_t = \sum_{0\leq s\leq t} \Delta C_s$, with $C_{0-} = 0$ and $\mathbb{E}\left[C_T\right] < \infty$, such that*

$$\mathcal{X}_t = M_t - A_t - C_{t-}, \quad t \in [0,T], \quad a.s.$$

A.2. Gal'chouk-Lenglart formula

Now, we revisit the notion of optional semimartingale in our particular case, where every local martingale have RCLL paths, due to an extension of the martingale representation theorem (see, e.g., [10, Section 2.4, p.25] and [38, p.171]), which can be viewed as a particular case of the general definition presented in [25, Definition 8.1, p.462].

Definition A.3 *The process $(\mathcal{X}_t)_{t\leq T}$ is called an optional semimartingale if $\mathcal{X}_t = \mathcal{X}_0 + A_t + M_t$, where M is an RCLL local martingale, K is an optional process of finite variation with $A_0 = M_0 = 0$, and $\mathcal{X}_0$ is $\mathcal{G}_0$-measurable finite variable. Moreover, the process A can be decomposed as $A_t = K_t + C_t$, where $C_t = \sum_{s<t} \Delta_+ A_s$ converges absolutely, and K is an adapted right continuous process of finite variation.*

Remark A.3 *Note that any optional semimartingale has paths of almost finite right and left limits over $[0,T]$.*

In the case of optional semimartingale that are not necessary RCLL, a significant new change of variable formula is introduced. This result generalizes the classical Ito's formula for RCLL semimartingale (see, e.g., [45, Theorem II.32]) and can be found [25, Theorem 8.2] or in [35, Section 3, p. 538].

Theorem A.2 (Gal'chouk-Lenglart) *Let $n \geq 1$. Let $\mathcal{X}$ be an n-dimensional optional semimartingale, i.e. $\mathcal{X} = \left(\mathcal{X}^1, \cdots, \mathcal{X}^n\right)$ is an n-dimensional optional process with decomposition $\mathcal{X}^k_t = \mathcal{X}^k_0 + \mathcal{M}^k_t + \mathcal{A}^k_t + \mathcal{B}^k_t$, for all $k \in \{1, \cdots, n\}$ where $\mathcal{M}^k$ is an RCLL local martingale, $\mathcal{A}^k$ is a right-continuous process of finite variation such that $\mathcal{A}^k_0 = 0$ and $\mathcal{B}^k$ is a left-continuous process of finite variation which is purely discontinuous and such that $\mathcal{B}^k_{0-} = 0$. Let F be a twice continuously differentiable function on $\mathbb{R}^n$. Then,* *almost surely, for all $t \leq T$,*

$$F(\mathcal{X}_t) = F(\mathcal{X}_0) + \sum_{k=1}^{n} \int_{]0,t]} D^k F(\mathcal{X}_{s-}) d\left(\mathcal{A}_s^k + \mathcal{M}_s^k\right) + \frac{1}{2} \sum_{k,l=1}^{n} \int_{]0,t]} D^k D^l F(\mathcal{X}_{s-}) d\left\langle \mathcal{M}^{k,c}, \mathcal{M}^{l,c} \right\rangle_s + \sum_{k=1}^{n} \int_{[0,t[} D^k F(\mathcal{X}_s) d\mathcal{B}_{s+}^k + \sum_{0<s\leq t} \left\{ F(\mathcal{X}_s) - F(\mathcal{X}_{s-}) - \sum_{k=1}^{n} D^k F(\mathcal{X}_{s-}) \Delta \mathcal{X}_s^k \right\} + \sum_{0\leq s<t} \left\{ F(\mathcal{X}_{s+}) - F(\mathcal{X}_s) - \sum_{k=1}^{n} D^k F(\mathcal{X}_s) \Delta_+ \mathcal{X}_s^k \right\},$$

where D^k denotes the differentiation operator with respect to the k^{th} coordinate, and $\mathcal{M}^{k,c}$ denotes the continuous part of the local martingale $\mathcal{M}^k$.

A.3. Tanaka's type formula

Using Theorem A.2, we could also introduce the calculation rules for convex functions, particularly the local time formula, which will be needed in deriving the comparison principal and generalizing the classical theorem for the RCLL case (see, e.g., [45, Theorem IV.66]). For a given process $\mathcal{Y}$, we note $\mathcal{Y}^+ = \max(\mathcal{Y}, 0)$ and $\mathcal{Y}^- = -\min(\mathcal{Y}, 0)$. Then, we have the following (see, e.g., [35, Section 3, p.538–539]):

Proposition A.1 (Tanaka's type formula) *Let $\mathcal{X}$ be an optional semimartingale. Then $\mathcal{X}^+$ is also an optional semimartingale with the decomposition*

$$\mathcal{X}_t^+ = \mathcal{X}_0^+ + \int_0^t \mathbb{1}_{\{\mathcal{X}_{s-}>0\}} d\mathcal{X}_s + \frac{1}{2}\mathcal{L}_t^0(\mathcal{X}) + \sum_{0\leq s<t} \left(\mathcal{X}_{s+}^+ \mathbb{1}_{\{\mathcal{X}_{s-}\leq 0\}} + \mathcal{X}_s^- \mathbb{1}_{\{\mathcal{X}_{s-}>0\}}\right) + \mathcal{X}_t^+ \mathbb{1}_{\{\mathcal{X}_{t-}\leq 0\}} + \mathcal{X}_t^- \mathbb{1}_{\{\mathcal{X}_{t-}>0\}},$$

where $\left(\mathcal{L}_t^0(\mathcal{X})\right)_{t\leq T}$ is a non-decreasing continuous process corresponding to the local time of $(\mathcal{X}_{t+})_{t\leq T}$ at $t = 0$.

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