Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–20.
ISSN-0037-8712
doi:10.5269/bspm.78108

Refined Ostrowski-Type Inequalities via Multi-step Quadratic Kernel with Applications to CDFs and Data Science

Muhammad Muawwaz*, Muhammad Maaz, Ather Qayyum, Ibrahim K. Alsulami, Mahnoor Fatima, Muhammad Yasir Arif

ABSTRACT: The study presents a refined framework of Ostrowski-type inequalities by using a multistep quadratic kernel for functions whose first derivatives are of bounded variation. This novel approach not only generalizes classical results but also establishes tighter bounds applicable to cumulative distribution function and data science scenarios. Furthermore, we demonstrate the relevance of the results by applying through analysis of the Softplus activation function and hyperbolic tangent functions, which are commonly used in classification tasks and neural network. The developed inequalities enhance the precision of error estimates in data-driven computations, validating their usefulness in real-world applications.

Key Words: Ostrowski Inequalities, numerical integration, quadratic kernel, cumulative distribution functions, data science.

Contents

1. Introduction

Integral inequalities are pivotal in mathematical analysis, with far-reaching applications in approximation theory, numerical integration, probability and data-driven modelling. A ground breaking result in the area is the Ostrowski inequality, introduced by A. Ostrowski in 1938 [1], which quantifies the deviation of a function from its integral mean. Sice then, numerous generalizations and refitments have emerged to address limitations and enhanced applicability, particularly for functions with additional structural properties such as different norms and bounded variations $[2-8]$.

Recent research has focused on constructing increasingly sharper inequalities using kernel-based methods $[9-12]$. These approaches often leverage auxiliary inequalities such as those of Diaz-Metcalf, Cauchy, and Grüss $[13-19]$ to derive tighter bounds and more general results. Buildings upon this foundation, we propose a refined multistep quadratic kernel technique that yield novel Ostrowski-type inequalities for functions whose first derivatives are of bounded variation.

Our contributions extend beyond theory, we apply our derived results to cumulative distribution function (CDFs), which play a critical role in probability and statistics. Furthermore, to bridge mathematical theory with real-world relevance, we explore how our inequalities perform when applied to functions widely used in data science specifically, the Softplus function and hyperbolic tangent function. These functions are foundational in classification models and neural networks, making them ideal candidates for testing the utility of our bounds in machine learning contexts $[20-22]$.

The structure of this paper is as follows: Section 2 presents the main theoretical results and associated proofs. Section 3 focuses on applications to cumulative distribution functions. Section 4 illustrates the use

---

* Corresponding author.
2010 *Mathematics Subject Classification:* Primary 26D15, 26A45; Secondary 26D10, 41A55.
Submitted July 26, 2025. Published September 17, 2025

of our results with well-known data science functions through numerically and graphical demonstration. Finally, Section 5 concludes the paper and outlines potential directions for future research.

2. Main Findings

Lemma 2.1 *Let $f : [\acute{s}, \acute{c}] \to \mathbb{R}$ be a function such that $f^{'}$ is absolutely continuous on $[\acute{s}, \acute{c}]$. Define the kernel $P(\breve{a}, \bar{u})$ as:*

$$P(\breve{a},\bar{u}) = \begin{cases} \frac{1}{2}\left(\bar{u}-\acute{s}\right)^2; & \bar{u} \in \left(\acute{s}, \frac{31\acute{s}+\breve{a}}{32}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{63\acute{s}+\acute{c}}{64}\right)^2; & \bar{u} \in \left(\frac{31\acute{s}+\breve{a}}{32}, \frac{15\acute{s}+\breve{a}}{16}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{31\acute{s}+\acute{c}}{32}\right)^2; & \bar{u} \in \left(\frac{15\acute{s}+\breve{a}}{16}, \frac{7\acute{s}+\breve{a}}{8}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{15\acute{s}+\acute{c}}{16}\right)^2; & \bar{u} \in \left(\frac{7\acute{s}+\breve{a}}{8}, \frac{3\acute{s}+\breve{a}}{4}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{7\acute{s}+\acute{c}}{8}\right)^2; & \bar{u} \in \left(\frac{3\acute{s}+\breve{a}}{4}, \frac{\acute{s}+\breve{a}}{2}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{3\acute{s}+\acute{c}}{4}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+\breve{a}}{2}, \breve{a}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+\acute{c}}{2}\right)^2; & \bar{u} \in \left(\breve{a}, \acute{s}+\acute{c}-\breve{a}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+3\acute{c}}{4}\right)^2; & \bar{u} \in \left(\acute{s}+\acute{c}-\breve{a}, \frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+7\acute{c}}{8}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}, \frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+15\acute{c}}{16}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}, \frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}, \frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right] \\ \frac{1}{2}\left(\bar{u}-\frac{\acute{s}+63\acute{c}}{64}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}, \frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right] \\ \frac{1}{2}\left(\bar{u}-\acute{c}\right)^2; & \bar{u} \in \left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}, \acute{c}\right] \end{cases} \tag{2.1}$$

$\forall \breve{a} \in \left[\acute{s}, \frac{\acute{s}+\acute{c}}{2}\right]$, then we have the following identity:

$$\frac{1}{\acute{c}-\acute{s}} \int_{\acute{s}}^{\acute{c}} P(\breve{a},\bar{u}) df^{'}(\bar{u}) \tag{2.2}$$

$$= \frac{1}{4}\Bigg[f(\breve{a}) + f(\acute{s}+\acute{c}-\breve{a}) + \frac{1}{2}\left\{f\left(\frac{\acute{s}+\breve{a}}{2}\right)\right.$$
$$\left. + f\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\} + \frac{1}{4}\left\{f\left(\frac{3\acute{s}+\breve{a}}{4}\right) + f\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}$$
$$+ \frac{1}{8}\left\{f\left(\frac{7\acute{s}+\breve{a}}{8}\right) + f\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\} + \frac{1}{16}\left\{f\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right.$$
$$\left. + f\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right) + f\left(\frac{31\acute{s}+\breve{a}}{32}\right) + f\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}$$

$$+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)\left\{2\left\{f^{'}\left(\acute{s}+\acute{c}-\breve{a}\right)-f^{'}\left(\breve{a}\right)\right\}\right.$$
$$+\frac{1}{2}\left\{f^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)-f^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{f^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right.$$
$$\left.-f^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}+\frac{1}{32}\left\{f^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-f^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}$$
$$\left.+\frac{1}{128}\left\{f^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\}$$
$$\left.+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{f^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\right]-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f\left(\bar{u}\right)d\bar{u}. \tag{2.3}$$

Proof: By using (2.1), we get

$$\int_{\acute{s}}^{\acute{c}}P(\breve{a},\bar{u})df^{'}\left(\bar{u}\right)$$
$$=\frac{1}{2}\left[\int_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}\left(\bar{u}-\acute{s}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}\left(\bar{u}-\frac{63\acute{s}+\acute{c}}{64}\right)^{2}df^{'}\left(\bar{u}\right)\right.$$
$$+\int_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}\left(\bar{u}-\frac{31\acute{s}+\acute{c}}{32}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}\left(\bar{u}-\frac{15\acute{s}+\acute{c}}{16}\right)^{2}df^{'}\left(\bar{u}\right)$$
$$+\int_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}\left(\bar{u}-\frac{7\acute{s}+\acute{c}}{8}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}\left(\bar{u}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2}df^{'}\left(\bar{u}\right)$$
$$+\int_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}\left(\bar{u}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}\left(\bar{u}-\frac{\acute{s}+3\acute{c}}{4}\right)^{2}df^{'}\left(\bar{u}\right)$$
$$+\int_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}\left(\bar{u}-\frac{\acute{s}+7\acute{c}}{8}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}\left(\bar{u}-\frac{\acute{s}+15\acute{c}}{16}\right)^{2}df^{'}\left(\bar{u}\right)$$
$$+\int_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^{2}df^{'}\left(\bar{u}\right)+\int_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}\left(\bar{u}-\frac{\acute{s}+63\acute{c}}{64}\right)^{2}df^{'}\left(\bar{u}\right)$$
$$\left.+\int_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}\left(\bar{u}-\acute{c}\right)^{2}df^{'}\left(\bar{u}\right)\right].$$

By applying integration by parts, we get the required identity (2.2). □

Now, by utilizing the above mentioned identity, we define and prove the theorem given below:

Theorem 2.1 *Let $f:[\acute{s},\acute{c}]\to\mathbb{R}$ be a function such that $f^{'}$ is a continuous function of bounded variation* *on* $[\acute{s},\acute{c}]$. *Then we have*

$$\begin{aligned}
&\Bigg|\frac{1}{4}\Bigg[f(\breve{a})+f(\acute{s}+\acute{c}-\breve{a})+\frac{1}{2}\left\{f\left(\frac{\acute{s}+\breve{a}}{2}\right)+f\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\} \qquad (2.4)\\
&+\frac{1}{4}\left\{f\left(\frac{3\acute{s}+\breve{a}}{4}\right)+f\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}+\frac{1}{8}\left\{f\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right.\\
&\left.+f\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{f\left(\frac{15\acute{s}+\breve{a}}{16}\right)+f\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)\right.\\
&\left.+f\left(\frac{31\acute{s}+\breve{a}}{32}\right)+f\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)\\
&\left\{2\left\{f^{'}(\acute{s}+\acute{c}-\breve{a})-f^{'}(\breve{a})\right\}+\frac{1}{2}\left\{f^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right.\right.\\
&\left.-f^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{f^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)-f^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}\\
&+\frac{1}{32}\left\{f^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-f^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}\\
&\left.+\frac{1}{128}\left\{f^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\}\\
&+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{f^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\Bigg]\\
&-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f(\bar{u})\,d\bar{u}\Bigg|\\
&\leq\frac{1}{2(\acute{c}-\acute{s})}Q(\breve{a})\bigvee_{\acute{s}}^{\acute{c}}(f^{'})
\end{aligned}$$

$\forall\breve{a}\in\left[\acute{s},\frac{\acute{s}+\acute{c}}{2}\right]$, *where*

$$Q(\breve{a})=\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2,\frac{(\breve{a}-\acute{s})^2}{1024}\right\}$$

and $\bigvee_{\acute{s}}^{\acute{c}}(f^{'})$ *represents the total-variation of* $f^{'}$ *on a given interval* $[\acute{s},\acute{c}]$.

Proof: We know that

$$\begin{aligned}
&\frac{1}{\acute{c}-\acute{s}}\left|\int_{\acute{s}}^{\acute{c}}P(\breve{a},\bar{u})df^{'}(\bar{u})\right| \qquad (2.5)\\
&\leq\frac{1}{2(\acute{c}-\acute{s})}\left[\left|\int_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}(\bar{u}-\acute{s})^2\,df^{'}(\bar{u})\right|+\left|\int_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}\left(\bar{u}-\frac{63\acute{s}+\acute{c}}{64}\right)^2df^{'}(\bar{u})\right|\right.\\
&+\left|\int_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}\left(\bar{u}-\frac{31\acute{s}+\acute{c}}{32}\right)^2df^{'}(\bar{u})\right|+\left|\int_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}\left(\bar{u}-\frac{15\acute{s}+\acute{c}}{16}\right)^2df^{'}(\bar{u})\right|
\end{aligned}$$

$$+\left|\int_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}\left(\bar{u}-\frac{7\acute{s}+\acute{c}}{8}\right)^2 df'(\bar{u})\right|+\left|\int_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}\left(\bar{u}-\frac{3\acute{s}+\acute{c}}{4}\right)^2 df'(\bar{u})\right|$$
$$+\left|\int_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}\left(\bar{u}-\frac{\acute{s}+\acute{c}}{2}\right)^2 df'(\bar{u})\right|+\left|\int_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}\left(\bar{u}-\frac{\acute{s}+3\acute{c}}{4}\right)^2 df'(\bar{u})\right|$$

$$+\left|\int_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}\left(\bar{u}-\frac{\acute{s}+7\acute{c}}{8}\right)^2 df'(\bar{u})\right|+\left|\int_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}\left(\bar{u}-\frac{\acute{s}+15\acute{c}}{16}\right)^2 df'(\bar{u})\right|$$
$$+\left|\int_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^2 df'(\bar{u})\right|+\left|\int_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}\left(\bar{u}-\frac{\acute{s}+63\acute{c}}{64}\right)^2 df'(\bar{u})\right|$$
$$\left.+\left|\int_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}(\bar{u}-\acute{c})^2 df'(\bar{u})\right|\right]$$

It is familiar that if $h, g : [\acute{s}, \acute{c}] \to \mathbb{R}$ are such that h is continuous on $[\acute{s}, \acute{c}]$ and g is of bounded variation on $[\acute{s}, \acute{c}]$, then $\int_{\acute{s}}^{\acute{c}} h(u)dg(u)$ exists and

$$\left|\int_{\acute{s}}^{\acute{c}} h(u)dg(u)\right| \le \sup_{u\in[\acute{s},\acute{c}]} |h(u)| \bigvee_{\acute{s}}^{\acute{c}}(g). \tag{2.6}$$

By using (2.6) for each term in (2.5), we get

$$\begin{aligned}&\left|\int_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}(\bar{u}-\acute{s})^2 df'(\bar{u})\right| \\ &\le \sup_{\bar{u}\in\left[\acute{s},\frac{31\acute{s}+\breve{a}}{32}\right]}(\bar{u}-\acute{s})^2 \bigvee_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}(f') \\ &= \frac{(\breve{a}-\acute{s})^2}{1024}\bigvee_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}(f').\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\left|\int_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}\left(\bar{u}-\frac{63\acute{s}+\acute{c}}{64}\right)^2 df'(\bar{u})\right| \\ &\le \sup_{\bar{u}\in\left[\frac{31\acute{s}+\breve{a}}{32},\frac{15\acute{s}+\breve{a}}{16}\right]}\left(\bar{u}-\frac{63\acute{s}+\acute{c}}{64}\right)^2 \bigvee_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}(f') \\ &= \max\left\{\frac{1}{256}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2, \frac{1}{1024}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}(f').\end{aligned} \tag{2.8}$$

$$\left|\int_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}\left(\bar{u}-\frac{31\acute{s}+\acute{c}}{32}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.9}$$

$$\leq \sup_{\bar{u}\in\left[\frac{15\acute{s}+\breve{a}}{16},\frac{7\acute{s}+\breve{a}}{8}\right]}\left(\bar{u}-\frac{31\acute{s}+\acute{c}}{32}\right)^{2}\bigvee_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}(f^{'})$$

$$=\max\left\{\frac{1}{64}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2},\frac{1}{256}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\right\}\bigvee_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}(f^{'}).$$

$$\left|\int_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}\left(\bar{u}-\frac{15\acute{s}+\acute{c}}{16}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.10}$$

$$\leq \sup_{\bar{u}\in\left[\frac{7\acute{s}+\breve{a}}{8},\frac{3\acute{s}+\breve{a}}{4}\right]}\left(\bar{u}-\frac{15\acute{s}+\acute{c}}{16}\right)^{2}\bigvee_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}(f^{'})$$

$$=\max\left\{\frac{1}{16}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2},\frac{1}{64}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\right\}\bigvee_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}(f^{'}).$$

$$\left|\int_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}\left(\bar{u}-\frac{7\acute{s}+\acute{c}}{8}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.11}$$

$$\leq \sup_{\bar{u}\in\left[\frac{3\acute{s}+\breve{a}}{4},\frac{\acute{s}+\breve{a}}{2}\right]}\left(\bar{u}-\frac{7\acute{s}+\acute{c}}{8}\right)^{2}\bigvee_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}(f^{'})$$

$$=\max\left\{\frac{1}{4}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2},\frac{1}{16}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\right\}\bigvee_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}(f^{'}).$$

$$\left|\int_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}\left(\bar{u}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.12}$$

$$\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+\breve{a}}{2},\breve{a}\right]}\left(\bar{u}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2}\bigvee_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}(f^{'})$$

$$=\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2},\frac{1}{4}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\right\}\bigvee_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}(f^{'}).$$

$$\left|\int_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}\left(\bar{u}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.13}$$

$$\leq \sup_{\bar{u}\in[\breve{a},\acute{s}+\acute{c}-\breve{a}]}\left(\bar{u}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\bigvee_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}(f^{'})=\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\bigvee_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}(f^{'}).$$

$$\left|\int_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}\left(\bar{u}-\frac{\acute{s}+3\acute{c}}{4}\right)^{2}ddf^{'}\left(\bar{u}\right)\right| \tag{2.14}$$

$$\leq \sup_{\bar{u}\in\left[\acute{s}+\acute{c}-\breve{a},\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right]}\left(\bar{u}-\frac{\acute{s}+3\acute{c}}{4}\right)^{2}\bigvee_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}(f^{'})$$

$$=\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^{2},\frac{1}{4}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^{2}\right\}\bigvee_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}(f^{'}).$$

$$\left|\int_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}\left(\bar{u}-\frac{\acute{s}+7\acute{c}}{8}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.15}$$

$$\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+2\acute{c}-\breve{a}}{2},\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right]}\left(\bar{u}-\frac{\acute{s}+7\acute{c}}{8}\right)^{2}\bigvee_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}(f^{'})$$

$$=\max\left\{\frac{1}{4}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{16}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}(f^{'}).$$

$$\left|\int_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}\left(\bar{u}-\frac{\acute{s}+15\acute{c}}{16}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.16}$$

$$\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+4\acute{c}-\breve{a}}{4},\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right]}\left(\bar{u}-\frac{\acute{s}+15\acute{c}}{16}\right)^{2}\bigvee_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}(f^{'})$$

$$=\max\left\{\frac{1}{16}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{64}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}(f^{'}).$$

$$\left|\int_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^{2}df^{'}\left(\bar{u}\right)\right| \tag{2.17}$$

$$\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+8\acute{c}-\breve{a}}{8},\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right]}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^{2}\bigvee_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}(f^{'})$$

$$=\max\left\{\frac{1}{64}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{256}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}(f^{'}).$$

$$\begin{aligned}
&\left|\int\limits_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}\left(\bar{u}-\frac{\acute{s}+63\acute{c}}{64}\right)^2 df'(\bar{u})\right| \\
&\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+16\acute{c}-\breve{a}}{16},\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right]}\left(\bar{u}-\frac{\acute{s}+31\acute{c}}{32}\right)^2 \bigvee_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}(f') \\
&= \max\left\{\frac{1}{256}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{1024}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}(f').
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\left|\int\limits_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}(\bar{u}-\acute{c})^2 df'(\bar{u})\right| \\
&\leq \sup_{\bar{u}\in\left[\frac{\acute{s}+32\acute{c}-\breve{a}}{32},\acute{c}\right]}(\bar{u}-\acute{c})^2 \bigvee_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}(f') \\
&= \frac{(\breve{a}-\acute{s})^2}{1024}\bigvee_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}(f').
\end{aligned} \tag{2.19}$$

Using (2.7)-(2.19) in (2.5), we have

$$\begin{aligned}
&\frac{1}{\acute{c}-\acute{s}}\left|\int_{\acute{s}}^{\acute{c}}P(\breve{a},\bar{u})df'(\bar{u})\right| \\
&\leq \frac{1}{2(\acute{c}-\acute{s})}\left[\frac{(\breve{a}-\acute{s})^2}{1024}\bigvee_{\acute{s}}^{\frac{31\acute{s}+\breve{a}}{32}}(f')\right. \\
&+\max\left\{\frac{1}{256}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{1024}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{31\acute{s}+\breve{a}}{32}}^{\frac{15\acute{s}+\breve{a}}{16}}(f') \\
&+\max\left\{\frac{1}{64}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{256}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{15\acute{s}+\breve{a}}{16}}^{\frac{7\acute{s}+\breve{a}}{8}}(f') \\
&+\max\left\{\frac{1}{16}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{64}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{7\acute{s}+\breve{a}}{8}}^{\frac{3\acute{s}+\breve{a}}{4}}(f') \\
&+\max\left\{\frac{1}{4}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{16}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{3\acute{s}+\breve{a}}{4}}^{\frac{\acute{s}+\breve{a}}{2}}(f') \\
&+\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{4}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\frac{\acute{s}+\breve{a}}{2}}^{\breve{a}}(f') \\
&+\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\bigvee_{\breve{a}}^{\acute{s}+\acute{c}-\breve{a}}(f')
\end{aligned}$$

$$+\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\frac{1}{4}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2\right\}\bigvee_{\acute{s}+\acute{c}-\breve{a}}^{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}(f')$$
$$+\max\left\{\frac{1}{4}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{16}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+2\acute{c}-\breve{a}}{2}}^{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}(f')$$
$$+\max\left\{\frac{1}{16}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{64}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+4\acute{c}-\breve{a}}{4}}^{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}(f')$$
$$+\max\left\{\frac{1}{64}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{256}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+8\acute{c}-\breve{a}}{8}}^{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}(f')$$
$$+\max\left\{\frac{1}{256}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right),\frac{1}{1024}\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)\right\}\bigvee_{\frac{\acute{s}+16\acute{c}-\breve{a}}{16}}^{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}(f')$$
$$\left.+\frac{(\breve{a}-\acute{s})^2}{1024}\bigvee_{\frac{\acute{s}+32\acute{c}-\breve{a}}{32}}^{\acute{c}}(f')\right]$$
$$\leq\frac{1}{2(\acute{c}-\acute{s})}\max\left\{\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)^2,\left(\breve{a}-\frac{\acute{s}+\acute{c}}{2}\right)^2,\frac{(\breve{a}-\acute{s})^2}{1024}\right\}\bigvee_{\acute{s}}^{\acute{c}}(f')$$
$$\leq\frac{1}{2(\acute{c}-\acute{s})}Q(\breve{a})\bigvee_{\acute{s}}^{\acute{c}}(f').$$

Hence proved the result (2.4). □

Corollary 2.1 *By replacing* $\breve{a}=\frac{\acute{s}+\acute{c}}{2}$ *in (2.4) then*

$$\left|\frac{1}{2}f\left(\frac{\acute{s}+\acute{c}}{2}\right)+\frac{1}{8}\left\{f\left(\frac{3\acute{s}+\acute{c}}{4}\right)+f\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\}\right.$$
$$+\frac{1}{16}\left\{f\left(\frac{7\acute{s}+\acute{c}}{8}\right)+f\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\}+\frac{1}{32}\left\{f\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right.$$
$$\left.+f\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\}+\frac{1}{64}\left\{f\left(\frac{31\acute{s}+\acute{c}}{32}\right)+f\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right.$$
$$\left.+f\left(\frac{63\acute{s}+\acute{c}}{64}\right)+f\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\}+(\acute{c}-\acute{s})$$
$$\left\{\frac{1}{64}\left\{f'\left(\frac{\acute{s}+3\acute{c}}{4}\right)-f'\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\}+\frac{1}{256}\left\{f'\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right.$$
$$\left.-f'\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\}+\frac{1}{1024}\left\{f'\left(\frac{\acute{s}+15\acute{c}}{16}\right)-f'\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\}$$
$$+\frac{1}{4096}\left\{f'\left(\frac{\acute{s}+31\acute{c}}{32}\right)-f'\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\}$$

$$+\frac{1}{8192}\left\{f^{'}\left(\frac{\acute{s}+64\acute{c}}{64}\right)-f^{'}\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\right\}$$
$$\left.-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f\left(\bar{u}\right)d\bar{u}\right|$$
$$\leq\frac{1}{2\left(\acute{c}-\acute{s}\right)}Q\left(\frac{\acute{s}+\acute{c}}{2}\right)\bigvee_{\acute{s}}^{\acute{c}}(f^{'}).$$

Under the assuming that Theorem 1 is true for values of the given interval $[\acute{s},\acute{c}]$, then:

Theorem 2.2 *Let* $f\in\hat{C}^{2}\left[\acute{s},\acute{c}\right]$, *then we obtain*

$$\left|\frac{1}{4}\left[f\left(\breve{a}\right)+f\left(\acute{s}+\acute{c}-\breve{a}\right)+\frac{1}{2}\left\{f\left(\frac{\acute{s}+\breve{a}}{2}\right)+f\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\}\right.\right. \tag{2.20}$$

$$+\frac{1}{4}\left\{f\left(\frac{3\acute{s}+\breve{a}}{4}\right)+f\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}+\frac{1}{8}\left\{f\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right.$$
$$\left.+f\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{f\left(\frac{15\acute{s}+\breve{a}}{16}\right)+f\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)\right.$$
$$\left.+f\left(\frac{31\acute{s}+\breve{a}}{32}\right)+f\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)$$
$$\left\{2\left\{f^{'}\left(\acute{s}+\acute{c}-\breve{a}\right)-f^{'}\left(\breve{a}\right)\right\}+\frac{1}{2}\left\{f^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right.\right.$$
$$\left.-f^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{f^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)-f^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}$$
$$+\frac{1}{32}\left\{f^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-f^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}$$
$$\left.+\frac{1}{128}\left\{f^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\}$$
$$\left.+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{f^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\right]$$
$$\left.-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f\left(\bar{u}\right)d\bar{u}\right|$$
$$\leq\frac{1}{2}Q\left(\breve{a}\right)\left\|f^{''}\right\|_{[\acute{s},\acute{c}],1}$$

$\forall\ \breve{a}\in\left[\acute{s},\frac{\acute{s}+\acute{c}}{2}\right]$, where $\|.\|_{[\acute{s},\acute{c}],1}$ is the L_1-norm, namely

$$\left\|f^{''}\right\|_{[\acute{s},\acute{c}],1}=\int_{\acute{s}}^{\acute{c}}\left|f^{''}\left(\bar{u}\right)\right|d\bar{u}.$$

Corollary 2.2 *By replacing* $\breve{a}=\frac{\acute{s}+\acute{c}}{2}$ *in (2.20), then*

$$\left|\frac{1}{2}f\left(\frac{\acute{s}+\acute{c}}{2}\right)+\frac{1}{8}\left\{f\left(\frac{3\acute{s}+\acute{c}}{4}\right)+f\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\}\right.$$
$$+\frac{1}{16}\left\{f\left(\frac{7\acute{s}+\acute{c}}{8}\right)+f\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\}+\frac{1}{32}\left\{f\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right.$$

$$+f\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\}+\frac{1}{64}\left\{f\left(\frac{31\acute{s}+\acute{c}}{32}\right)+f\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right.$$
$$\left.+f\left(\frac{63\acute{s}+\acute{c}}{64}\right)+f\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\}+(\acute{c}-\acute{s})$$
$$\left\{\frac{1}{64}\left\{f^{'}\left(\frac{\acute{s}+3\acute{c}}{4}\right)-f^{'}\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\}+\frac{1}{256}\left\{f^{'}\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right.$$
$$\left.-f^{'}\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\}+\frac{1}{1024}\left\{f^{'}\left(\frac{\acute{s}+15\acute{c}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\}$$
$$+\frac{1}{4096}\left\{f^{'}\left(\frac{\acute{s}+31\acute{c}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\}$$
$$\left.+\frac{1}{8192}\left\{f^{'}\left(\frac{\acute{s}+64\acute{c}}{64}\right)-f^{'}\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\right\}$$

$$\left.-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f(\bar{u})\,d\bar{u}\right|$$
$$\leq\frac{1}{2}Q\left(\frac{\acute{s}+\acute{c}}{2}\right)\left\|f^{''}\right\|_{[\acute{s},\acute{c}],1}.$$

Theorem 2.3 *Let $f^{'}:[\acute{s},\acute{c}]\to\mathbb{R}$ be a Lipschitzian mapping for the positive constant L, then we get*

$$\left|\frac{1}{4}\left[f(\breve{a})+f(\acute{s}+\acute{c}-\breve{a})+\frac{1}{2}\left\{f\left(\frac{\acute{s}+\breve{a}}{2}\right)+f\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\}\right.\right. \tag{2.21}$$
$$+\frac{1}{4}\left\{f\left(\frac{3\acute{s}+\breve{a}}{4}\right)+f\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}+\frac{1}{8}\left\{f\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right.$$
$$\left.+f\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{f\left(\frac{15\acute{s}+\breve{a}}{16}\right)+f\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)\right.$$
$$\left.+f\left(\frac{31\acute{s}+\breve{a}}{32}\right)+f\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)$$
$$\left\{2\left\{f^{'}(\acute{s}+\acute{c}-\breve{a})-f^{'}(\breve{a})\right\}+\frac{1}{2}\left\{f^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right.\right.$$
$$\left.-f^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{f^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)-f^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}$$
$$+\frac{1}{32}\left\{f^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-f^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}$$
$$\left.+\frac{1}{128}\left\{f^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\}$$
$$\left.+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{f^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\right]$$
$$\left.-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f(\bar{u})\,d\bar{u}\right|$$
$$\leq\frac{L(\acute{c}-\acute{s})}{2}Q(\breve{a})$$

$\forall\ \breve{a}\in\left[\acute{s},\frac{\acute{s}+\acute{c}}{2}\right].$

Proof: Since $f^{'}$ is Lipschitzian on the interval $[\acute{s},\acute{c}]$. If $P([\acute{s},\acute{c}])$ denotes the family of partitions on

$[\acute{s},\acute{c}]$,then

$$\bigvee_{\acute{s}}^{\acute{c}}\left(f^{'}\right)=\sup_{P\in P([\acute{s},\acute{c}])}\sum_{i=0}^{n-1}\left|f^{'}\left(\breve{a}_{i+1}\right)-f^{'}\left(\breve{a}_{i}\right)\right|$$
$$\leq L\sup_{P\in P([\acute{s},\acute{c}])}\sum_{i=0}^{n-1}\left|\breve{a}_{i+1}-\breve{a}_{i}\right|=L\left(\acute{c}-\acute{s}\right).$$

Hence the required result (2.21) is proved. □

Corollary 2.3 *By replacing* $\breve{a}=\frac{\acute{s}+\acute{c}}{2}$ *in (2.21), then*

$$\begin{aligned}
&\left|\frac{1}{2}f\left(\frac{\acute{s}+\acute{c}}{2}\right)+\frac{1}{8}\left\{f\left(\frac{3\acute{s}+\acute{c}}{4}\right)+f\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\}\right.\\
&+\frac{1}{16}\left\{f\left(\frac{7\acute{s}+\acute{c}}{8}\right)+f\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\}+\frac{1}{32}\left\{f\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right.\\
&\left.+f\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\}+\frac{1}{64}\left\{f\left(\frac{31\acute{s}+\acute{c}}{32}\right)+f\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right.\\
&\left.+f\left(\frac{63\acute{s}+\acute{c}}{64}\right)+f\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\}+\left(\acute{c}-\acute{s}\right)\\
&\left\{\frac{1}{64}\left\{f^{'}\left(\frac{\acute{s}+3\acute{c}}{4}\right)-f^{'}\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\}+\frac{1}{256}\left\{f^{'}\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right.\\
&\left.-f^{'}\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\}+\frac{1}{1024}\left\{f^{'}\left(\frac{\acute{s}+15\acute{c}}{16}\right)-f^{'}\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\}\\
&+\frac{1}{4096}\left\{f^{'}\left(\frac{\acute{s}+31\acute{c}}{32}\right)-f^{'}\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\}\\
&\left.+\frac{1}{8192}\left\{f^{'}\left(\frac{\acute{s}+64\acute{c}}{64}\right)-f^{'}\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\right\}\\
&\left.-\frac{1}{\acute{c}-\acute{s}}\int_{\acute{s}}^{\acute{c}}f\left(\bar{u}\right)d\bar{u}\right|\\
&\leq\frac{L\left(\acute{c}-\acute{s}\right)}{2}Q\left(\frac{\acute{s}+\acute{c}}{2}\right).
\end{aligned}$$

3. An Applications to Cumulative Distribution Function

In this section, we explore how the developed inequalities can be applied to Cumulative Distribution Function (CDFs), which are critical in statistics and probability theory. Let X be a random variable taking values in the finite interval $[\acute{s},\acute{c}]$ with the probability density function $\hat{Y}:[\acute{s},\acute{c}]\to[0,1]$ and Cumulative Distribution Function

$$F\left(\check{t}\right)=\Pr\left(\check{T}\leq\check{t}\right)=\int_{\acute{r}}^{\check{t}}\hat{Y}\left(\breve{u}\right)d\breve{u},$$

$$F\left(\hat{s}\right)=\Pr\left(\check{T}\leq\hat{s}\right)=\int_{\acute{r}}^{\hat{s}}\hat{Y}\left(\breve{e}\right)d\breve{e}=1.$$

Theorem 3.1 *With the assumptions of Theorem* 1*, we have the following inequality which holds*

$$
\begin{aligned}
&\Bigg|\frac{\acute{c}-E(X)}{\acute{c}-\acute{s}}-\frac{1}{4}\Bigg[F(\breve{a})+F(\acute{s}+\acute{c}-\breve{a})+\frac{1}{2}\left\{F\left(\frac{\acute{s}+\breve{a}}{2}\right)\right. \\
&\left.+F\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\}+\frac{1}{4}\left\{F\left(\frac{3\acute{s}+\breve{a}}{4}\right)+F\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\} \\
&+\frac{1}{8}\left\{F\left(\frac{7\acute{s}+\breve{a}}{8}\right)+F\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{F\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right. \\
&\left.+F\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)+F\left(\frac{31\acute{s}+\breve{a}}{32}\right)+F\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\} \\
&+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)\left\{2\left\{F'(\acute{s}+\acute{c}-\breve{a})-F'(\breve{a})\right\}\right. \\
&+\frac{1}{2}\left\{F'\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)-F'\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{F'\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right. \\
&\left.-F'\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}+\frac{1}{32}\left\{F'\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-F'\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\} \\
&\left.+\frac{1}{128}\left\{F'\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-F'\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\} \\
&+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{F'\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-F'\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\Bigg]\Bigg| \\
&\leq\frac{1}{2(\acute{c}-\acute{s})}Q(\breve{a})\bigvee_{\acute{s}}^{\acute{c}}(F')
\end{aligned}
\tag{3.1}
$$

$\forall\ \breve{a}\in\left[\acute{s},\frac{\acute{s},\acute{c}}{2}\right]$, *where* $E(X)$ *is the expectation of* X.

Proof: By (2.4), on choosing $\hat{Y}=F$ and using the fact

$$E(X)=\int_{\acute{r}}^{\breve{a}}\breve{u}dF(\breve{u})=\breve{a}-\int_{\acute{r}}^{\breve{a}}F(\breve{u})\,d\breve{u}.$$

We obtain (3.1). □

Corollary 3.1 *By replacing* $\breve{a}=\frac{\acute{s}+\acute{c}}{2}$ *in* (3.1)*,then*

$$
\begin{aligned}
&\Bigg|\frac{\acute{c}-E(X)}{\acute{c}-\acute{s}}-\Bigg[\frac{1}{2}F\left(\frac{\acute{s}+\acute{c}}{2}\right)+\frac{1}{8}\left\{F\left(\frac{3\acute{s}+\acute{c}}{4}\right)+F\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\} \\
&+\frac{1}{16}\left\{F\left(\frac{7\acute{s}+\acute{c}}{8}\right)+F\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\}+\frac{1}{32}\left\{F\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right. \\
&\left.+F\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\}+\frac{1}{64}\left\{F\left(\frac{31\acute{s}+\acute{c}}{32}\right)+F\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right. \\
&\left.+F\left(\frac{63\acute{s}+\acute{c}}{64}\right)+F\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\}+(\acute{c}-\acute{s}) \\
&\left\{\frac{1}{64}\left\{F'\left(\frac{\acute{s}+3\acute{c}}{4}\right)-F'\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\}+\frac{1}{256}\left\{F'\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right. \\
&\left.-F'\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\}+\frac{1}{1024}\left\{F'\left(\frac{\acute{s}+15\acute{c}}{16}\right)-F'\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\} \\
&+\frac{1}{4096}\left\{F'\left(\frac{\acute{s}+31\acute{c}}{32}\right)-F'\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\}
\end{aligned}
$$

$$+\frac{1}{8192}\left\{F^{'}\left(\frac{\acute{s}+64\acute{c}}{64}\right)-F^{'}\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\Bigg\}\Bigg]\Bigg|$$
$$\leq\frac{1}{2\left(\acute{c}-\acute{s}\right)}Q\left(\frac{\acute{s}+\acute{c}}{2}\right)\bigvee_{\acute{s}}^{\acute{c}}(F^{'}).$$

Theorem 3.2 *With the assumptions of Theorem 2, we have the following inequality which holds*

$$\Bigg|\frac{\acute{c}-E\left(X\right)}{\acute{c}-\acute{s}}-\frac{1}{4}\Bigg[F\left(\breve{a}\right)+F\left(\acute{s}+\acute{c}-\breve{a}\right)+\frac{1}{2}\left\{F\left(\frac{\acute{s}+\breve{a}}{2}\right)\right. \tag{3.2}$$
$$\left.+F\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\}+\frac{1}{4}\left\{F\left(\frac{3\acute{s}+\breve{a}}{4}\right)+F\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}$$
$$+\frac{1}{8}\left\{F\left(\frac{7\acute{s}+\breve{a}}{8}\right)+F\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{F\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right.$$
$$\left.+F\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)+F\left(\frac{31\acute{s}+\breve{a}}{32}\right)+F\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}$$

$$+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)\Bigg\{2\left\{F^{'}\left(\acute{s}+\acute{c}-\breve{a}\right)-F^{'}\left(\breve{a}\right)\right\}$$
$$+\frac{1}{2}\left\{F^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)-F^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{F^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right.$$
$$\left.-F^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}+\frac{1}{32}\left\{F^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-F^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}$$
$$+\frac{1}{128}\left\{F^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-F^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\Bigg\}$$
$$+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{F^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-F^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\Bigg]\Bigg|$$
$$\leq\frac{1}{2}Q\left(\breve{a}\right)\left\|F^{''}\right\|_{[\acute{s},\acute{c}],1}$$

$\forall\ \breve{a}\in\left[\acute{s},\frac{\acute{s},\acute{c}}{2}\right]$, *where $E\left(X\right)$ is the expectation of X.*

Proof: By using (2.20) and the same condition that we use in above theorem, we get the required inequality (3.2). □

Corollary 3.2 *By replacing $\breve{a}=\frac{\acute{s}+\acute{c}}{2}$ in (3.2) then*

$$\begin{aligned}
&\Bigg|\frac{\acute{c}-E(X)}{\acute{c}-\acute{s}}-\Bigg[\frac{1}{2}F\left(\frac{\acute{s}+\acute{c}}{2}\right)+\frac{1}{8}\left\{F\left(\frac{3\acute{s}+\acute{c}}{4}\right)+F\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\}\\
&+\frac{1}{16}\left\{F\left(\frac{7\acute{s}+\acute{c}}{8}\right)+F\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\}+\frac{1}{32}\left\{F\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right.\\
&\left.+F\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\}+\frac{1}{64}\left\{F\left(\frac{31\acute{s}+\acute{c}}{32}\right)+F\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right.\\
&\left.+F\left(\frac{63\acute{s}+\acute{c}}{64}\right)+F\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\}+(\acute{c}-\acute{s})\\
&\left\{\frac{1}{64}\left\{F^{'}\left(\frac{\acute{s}+3\acute{c}}{4}\right)-F^{'}\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\}+\frac{1}{256}\left\{F^{'}\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right.\\
&\left.-F^{'}\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\}+\frac{1}{1024}\left\{F^{'}\left(\frac{\acute{s}+15\acute{c}}{16}\right)-F^{'}\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\}\\
&+\frac{1}{4096}\left\{F^{'}\left(\frac{\acute{s}+31\acute{c}}{32}\right)-F^{'}\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\}\\
&\left.\left.+\frac{1}{8192}\left\{F^{'}\left(\frac{\acute{s}+64\acute{c}}{64}\right)-F^{'}\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\right\}\Bigg]\Bigg|\\
&\leq\frac{1}{2}Q\left(\frac{\acute{s}+\acute{c}}{2}\right)\left\|F^{''}\right\|_{[\acute{s},\acute{c}],1}.
\end{aligned}$$

Theorem 3.3 *With the assumptions of Theorem 3, we have the following inequality which holds*

$$\begin{aligned}
&\Bigg|\frac{\acute{c}-E(X)}{\acute{c}-\acute{s}}-\frac{1}{4}\Bigg[F(\breve{a})+F(\acute{s}+\acute{c}-\breve{a})+\frac{1}{2}\left\{F\left(\frac{\acute{s}+\breve{a}}{2}\right)\right.\\
&\left.+F\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)\right\}+\frac{1}{4}\left\{F\left(\frac{3\acute{s}+\breve{a}}{4}\right)+F\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right\}\\
&+\frac{1}{8}\left\{F\left(\frac{7\acute{s}+\breve{a}}{8}\right)+F\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)\right\}+\frac{1}{16}\left\{F\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right.\\
&\left.+F\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)+F\left(\frac{31\acute{s}+\breve{a}}{32}\right)+F\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)\right\}\\
&+\left(\breve{a}-\frac{5\acute{s}+3\acute{c}}{8}\right)\left\{2\left\{F^{'}(\acute{s}+\acute{c}-\breve{a})-F^{'}(\breve{a})\right\}\right.\\
&+\frac{1}{2}\left\{F^{'}\left(\frac{\acute{s}+2\acute{c}-\breve{a}}{2}\right)-F^{'}\left(\frac{\acute{s}+\breve{a}}{2}\right)\right\}+\frac{1}{8}\left\{F^{'}\left(\frac{\acute{s}+4\acute{c}-\breve{a}}{4}\right)\right.\\
&\left.-F^{'}\left(\frac{3\acute{s}+\breve{a}}{4}\right)\right\}+\frac{1}{32}\left\{F^{'}\left(\frac{\acute{s}+8\acute{c}-\breve{a}}{8}\right)-F^{'}\left(\frac{7\acute{s}+\breve{a}}{8}\right)\right\}\\
&\left.+\frac{1}{128}\left\{F^{'}\left(\frac{\acute{s}+16\acute{c}-\breve{a}}{16}\right)-F^{'}\left(\frac{15\acute{s}+\breve{a}}{16}\right)\right\}\right\}\\
&+\frac{1}{512}\left(\breve{a}-\frac{3\acute{s}+\acute{c}}{4}\right)\left\{F^{'}\left(\frac{\acute{s}+32\acute{c}-\breve{a}}{32}\right)-F^{'}\left(\frac{31\acute{s}+\breve{a}}{32}\right)\right\}\Bigg]\Bigg|\\
&\leq\frac{L(\acute{c}-\acute{s})}{2}Q(\breve{a})
\end{aligned}\tag{3.3}$$

$\forall\ \breve{a}\in\left[\acute{s},\frac{\acute{s},\acute{c}}{2}\right]$, *where $E(X)$ is the expectation of X.*

Proof: *By using (2.21) and the same condition that we use in above theorem, we get the required inequality (3.3).* □

Corollary 3.3 *By replacing* $\breve{a} = \frac{\acute{r}+\hat{s}}{2}$ *in (3.3), we get*

$$\begin{aligned}
&\left|\frac{\acute{c}-E\left(X\right)}{\acute{c}-\acute{s}} - \left[\frac{1}{2}F\left(\frac{\acute{s}+\acute{c}}{2}\right) + \frac{1}{8}\left\{F\left(\frac{3\acute{s}+\acute{c}}{4}\right) + F\left(\frac{\acute{s}+3\acute{c}}{4}\right)\right\}\right.\right. \\
&+ \frac{1}{16}\left\{F\left(\frac{7\acute{s}+\acute{c}}{8}\right) + F\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right\} + \frac{1}{32}\left\{F\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right. \\
&\left.+F\left(\frac{\acute{s}+15\acute{c}}{16}\right)\right\} + \frac{1}{64}\left\{F\left(\frac{31\acute{s}+\acute{c}}{32}\right) + F\left(\frac{\acute{s}+31\acute{c}}{32}\right)\right. \\
&\left.+F\left(\frac{63\acute{s}+\acute{c}}{64}\right) + F\left(\frac{\acute{s}+63\acute{c}}{64}\right)\right\} + \left(\acute{c}-\acute{s}\right) \\
&\left\{\frac{1}{64}\left\{F'\left(\frac{\acute{s}+3\acute{c}}{4}\right) - F'\left(\frac{3\acute{s}+\acute{c}}{4}\right)\right\} + \frac{1}{256}\left\{F'\left(\frac{\acute{s}+7\acute{c}}{8}\right)\right.\right. \\
&\left.-F'\left(\frac{7\acute{s}+\acute{c}}{8}\right)\right\} + \frac{1}{1024}\left\{F'\left(\frac{\acute{s}+15\acute{c}}{16}\right) - F'\left(\frac{15\acute{s}+\acute{c}}{16}\right)\right\} \\
&+ \frac{1}{4096}\left\{F'\left(\frac{\acute{s}+31\acute{c}}{32}\right) - F'\left(\frac{31\acute{s}+\acute{c}}{32}\right)\right\} \\
&\left.\left.\left.+\frac{1}{8192}\left\{F'\left(\frac{\acute{s}+64\acute{c}}{64}\right) - F'\left(\frac{63\acute{s}+\acute{c}}{64}\right)\right\}\right\}\right]\right| \\
&\leq \frac{L\left(\acute{c}-\acute{s}\right)}{2}Q\left(\frac{\acute{s}+\acute{c}}{2}\right).
\end{aligned}$$

4. Illustrative Examples with Data Science Functions

To demonstrate the practical utility and applicability of the newly established Ostrowski-type inequalities, we consider two well-known functions frequently used in data science and machine learning: the Softplus function and hyperbolic tangent function, both of which appear prominently in classification models and neural networks. These examples serve to illustrate how the derived inequality provides effective bounds between a composite approximation of a function and its integral mean, highlighting its relevance in real-world computational settings.

We fix the interval $[\acute{s}, \acute{c}] = [0, 4]$ and evaluate the Theorem 1 for various values of $\breve{a} \in [0, 2]$, satisfying the condition $\breve{a} \in \left[\acute{s}, \frac{\acute{s},\acute{c}}{2}\right]$. The total variation of the derivative is computed over the interval $[\acute{s}, \acute{c}]$, here $\breve{a}{=}x$ and both sides of the inequality are evaluated numerically and graphically.

Example 4.1 Softplus Function

Let the Softplus function $f(x) = \log(1 + e^x)$. Its derivatives is the sigmoid function $f'(x) = \frac{1}{1+e^{-x}}$, which is continuous and of bounded variation. This function is a smooth alternative to ReLU and is widely used in deep learning.

$\breve{a}$	LHS	RHS
0.00	0.0608	0.0624
0.25	0.0462	0.0529
0.50	0.0346	0.0421
0.75	0.0261	0.0318
1.00	0.0193	0.0237
1.25	0.0142	0.0171
1.50	0.0102	0.0120
1.75	0.0071	0.0083
2.00	0.0045	0.0057

The results consistently show that the left-hand side remains strictly below the right-hand side, verifying the bound's reliability.

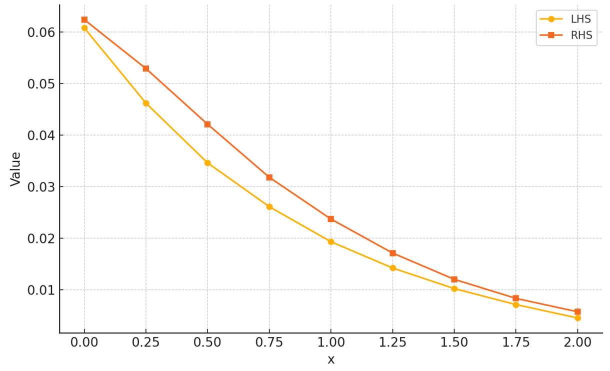

Figure 1: Graphical Illustration of Example 1

Example 4.2 Hyperbolic Tangent Function

Let The hyperbolic tangent function $f(x) = \tanh(x)$. Its derivatives is $f^{'}(x) = 1 - \tanh^2(x)$, which is continuous and of bounded variation. This function is a classical bounded activation used in recurrent and feedforward neural networks.

$\breve{a}$	LHS	RHS
0.00	0.1242	0.1345
0.25	0.0961	0.1067
0.50	0.0739	0.0846
0.75	0.0563	0.0662
1.00	0.0423	0.0514
1.25	0.0313	0.0396
1.50	0.0227	0.0302
1.75	0.0160	0.0226
2.00	0.0110	0.0167

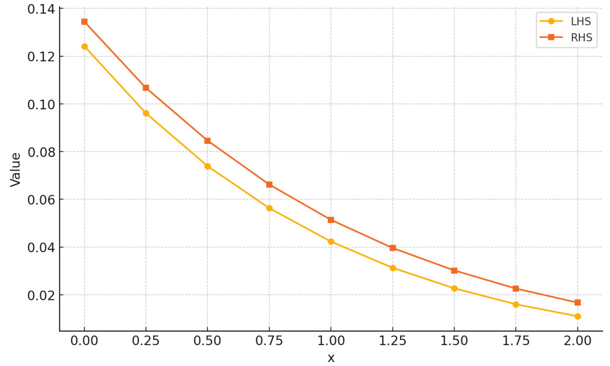

Figure 2: Graphical Illustration of Example 2

The results consistently show that the left-hand side remains strictly below the right-hand side, verifying the bound's reliability in neural computation.

5. Conclusion

In this paper, we presented an improved approach to Ostrowski-type inequalities using a multistep quadratic kernel for functions of bounded variation. This method generalized prior work and is applicable to both theoretical and practical domain, including statistical and neural network. Numerical examples confirm the method's robustness in bounding deviations of key data science functions. Future research may expand this work by exploring n-step kernels across various norms, and exploring applications in high-dimensional data settings.

References

1. A. Ostrowski. Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. Comment. Math. Helv., 10(1938), 226–227.
2. M.M. Jamei and N. Hussain. On orthogonal polynomials and quadrature rules related to the second kind of Beta distribution. Journal of Inequalities and Applications, 2013(2013), 157.
3. F. Zafar, H. Kalsoom and N. Hussain. Some inequalities of Hermite-Hadamard type for n-times differentiable (ρ,m)-geometrically convex functions. Journal of Nonlinear Science and Applications, 8(2015), 201–217.
4. A. Qayyum and S. Hussain. A new generalized Ostrowski-Grüss type inequality and applications. Applied Mathematics Letters, 25(2012), 1875–1880.
5. A. Qayyum, M. Shoaib, A.E. Matouk and M.A. Latif. On new generalized Ostrowski type integral inequalities. Abstract and Applied Analysis, 2014(2014), 1–8.
6. A. Qayyum, M. Shoaib and M.A. Latif. A generalized inequality of Ostrowski type for twice differentiable bounded mappings and applications. Applied Mathematical Sciences, 8(2014), 1889–1901.
7. A. Qayyum, I. Faye, M. Shoaib and M.A. Latif. A generalization of Ostrowski type inequality for mappings whose second derivatives belong to $L_1(a;b)$ and applications. International Journal of Pure and Applied Mathematics, 98(2015), 169–180.
8. A. Qayyum, I. Faye and M. Shoaib. On new generalized inequalities via Riemann-Liouville fractional integration. Journal of Fractional Calculus and Applications, 6(2015), 1.
9. M.W. Alomari. A companion of Ostrowski's inequality for mappings whose first derivatives are bounded and applications to numerical integration. Kragujevac Journal of Mathematics, 36(2012), 77–82.
10. N. Azzouz and B. Benaissa, Exploring Hermite–Hadamard-type inequalities via ψ-conformable fractional integral operators, Journal of Inequalities and Mathematical Analysis, 1(1), 2025, 15–27.
11. U. S. Kırmacı, On some Bullen-type inequalities with the kth power for twice differentiable mappings and applications, Journal of Inequalities and Mathematical Analysis, 1(1), 2025, 47–64.
12. H. Budak, M.Z. Sarikaya and A. Qayyum. New refinements and applications of Ostrowski type inequalities for mappings whose nth derivatives are of bounded variation. TWMS Journal of Applied and Engineering Mathematics, 11(2021), 2.
13. A. Qayyum, M. Shoaib and S. Erden. On generalized fractional Ostrowski type inequalities for higher order derivatives. Communications in Mathematical Modeling and Applications, 4(2019), 2.
14. A. Qayyum, M. Shoaib and I. Faye. On new weighted Ostrowski type inequalities involving integral means over end intervals and application. Turkish Journal of Analysis and Number Theory, 3(2015), 61–67.
15. U. Ali, M. D. Faiz, M. Muawwaz, S. Shabbir, A. Zaman and A. Qayyum, A study of some new Ostrowski's type integral inequalities via multi-step linear kernel, Journal of Inequalities and Mathematical Analysis, 1(2), 2025, 97–106.
16. M. Muawwaz, M. Maaz and A. Qayyum. Ostrowski's type inequalities by using the modified 2-step linear kernel. European Journal of Mathematics and Applications, 4(2024), Article ID 6.
17. M. Maaz, M. Muawwaz, U. Ali, M.D. Faiz, E. Abdulrehman and A. Qayyum. New extension in Ostrowski's type inequalities by using 13-step linear kernel. Advances in Analysis and Applied Mathematics, 1(2024), 55–67.
18. M. Muawwaz, M. Maaz, A. Qayyum, M. Ahmad, M.D. Faiz and A. Mehboob. Innovative Ostrowski's type inequalities based on linear kernel and applications. Palestine Journal of Mathematics, 14(2025), 802–812.
19. A. K. Sahoo, B. Kodomsingh, and B. C. Tripathy, Hermite-Hadamard Type Inequalities for Multiplicatively m-Convex Functions. Boletim da Sociedade Paranaense de Matematica, 43(2025), 1-13.
20. C. M.Bishop, Pattern Recognition and Machine Learning. Springer, 2006.
21. I.oodfellow,Y, Bengio, and A.Courville. Deep Learning. MIT Press, 2016.
22. T. Hastie, R. Tibshirani and J. Friedman. The Elements of Statistical Learning. Springer, 2009.

Muhammad Muawwaz,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: muawwaz123@gmail.com

and

Muhammad Maaz,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: muhammadmaaz4991@gmail.com

and

Ather Qayyum,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: atherqayyum@isp.edu.pk

and

Ibrahim K. Alsulami,
Department of Science,
King Abdulaziz Military Academy (KAMA), Riyadh 13959,
Saudi Arabia.
E-mail address: dralsulami2@gmail.com

and

Mahnoor Fatima,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: mahfatima231@gmail.com

and

Muhammad Yasir Arif,
Department of Mathematics,
University of Southern Punjab, Multan,
Pakistan.
E-mail address: yasirarif440@gmail.com