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Complex Intuitionistic Fuzzy Neutrosophic Structures in Lie Algebras: Subalgebras, Ideals and Homomorphisms

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ABSTRACT: This work develops new concepts of *complex intuitionistic fuzzy neutrosophic Lie subalgebras and ideals* within a framework governed by neutrosophic norm functions, namely the *t-norm* ($\mathcal{T}^*$) and *s-norm* ($\acute{\mathcal{S}}$). The connections between these newly defined objects and traditional Lie subalgebras and ideals are carefully analyzed. Moreover, algebraic constructions including *intersection, sum, and homomorphic images* are introduced for these structures, and a collection of essential structural properties and theoretical results is established.

Keywords: Complex neutrosophic intuitionistic fuzzy sets, neutrosophic sets, neutrosophic norm, neutrosophic Lie subalgebra, t-norm and s-norm, neutrosophic ideal, fuzzy Lie structures, homomorphisms.

Contents

1. Introduction

During the 1870s, Marius Sophus Lie proposed the concept of Lie algebras, which eventually became a fundamental framework in mathematics and theoretical physics. In 1965, Lotfi A. Zadeh developed the fuzzy set theory [27] that provides a tool to depict uncertainty by permitting truth values to exist anywhere within the range $[0, 1]$, instead of following traditional binary logic. The fuzzy set [27] led to the idea of "fuzziness," in which statements are not confined to being completely true or false. The fuzzy set theory has been utilized in various fields, such as artificial intelligence, computer science, control systems, decision-making, formal logic, and management. Expanding on these advancements, numerous studies [8], [25], [26] explored the structures of fuzzy Lie subalgebras that combine the analytical precision of Lie theory with the adaptability of fuzzy logic.

As a refinement of Zadeh's model, Atanassov presented the intuitionistic fuzzy set [3] as a broadened version of Zadeh's foundational fuzzy set theory [27]. In contrast to conventional fuzzy sets, the intuitionistic is set [3] includes both the level of membership and the level of non-membership offering a more comprehensive structure for depicting uncertainty. Its improved descriptive capability renders it especially useful in contemporary mathematical uses.

Building on these concepts, Ramot et al. [11] introduced the idea of complex fuzzy sets by broadening membership functions into the realm of complex numbers. Subsequently, Alkouri and Salleh [2] introduced a more advanced version termed complex intuitionistic fuzzy sets, which incorporate complex-valued functions for membership and non-membership, providing an even more adaptable framework for modeling uncertainty.

The distinctiveness of the complex intuitionistic fuzzy set lies in its greater expressive capacity: the functions for membership and non-membership can now take values on the unit circle in the complex

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plane, in contrast to the conventional real interval $[0,1]$. This broadened value range enhances the set's flexibility in expressing nuanced degrees of uncertainty and hesitation in complex systems.

In a related direction, the neutrosophic extension of the real number system introduces an indeterminate element I, satisfying $I^2 = I$, $I^n = I$ for all $n \in \mathbb{N}$, while lacking a multiplicative inverse [28,29]. The neutrosophic framework has attracted considerable attention and has been successfully applied in diverse areas, including decision-making [30,31], artificial intelligence and machine learning [32,33], medical diagnosis [34,35,40], communication systems [36], pattern recognition [37], educational technologies [38], theoretical physics [39], and data science [41]. Later, representative applications include medical diagnosis, where machine learning models based on neutrosophic sets improve cancer classification [43], and decision analysis, where hybrid neutrosophic frameworks address uncertainty in seismic risk evaluation [45].

As discussed in [29], F. Smarandache introduced the notion of single-valued neutrosophic probability, formulated as a mapping

$$P_N : X \to [0,1]^3$$

where X represents a neutrosophic sample space that accommodates indeterminate elements. The neutrosophic probability of an event A is expressed as

$$P_N(\tilde{\mathcal{Q}}) = \Big(\mathcal{T}^*(\tilde{\mathcal{Q}}), I(\text{neut}\check{\mathfrak{d}}_1), F(\text{anti}\check{\mathfrak{d}}_1)\Big) = (\mathcal{T}^*, I, F)$$

with each component $\mathcal{T}^*, I, F \in [0,1]$ and their sum satisfying $0 \leq \mathcal{T}^* + I + F \leq 3$.

In recent studies, Abobala and Hatip [42] collaborators introduced the framework of two-dimensional AH-isometry to establish a relationship between the neutrosophic plane $\mathbb{R}(I) \times \mathbb{R}(I)$ and the conventional module $\mathbb{R}^2 \times \mathbb{R}^2$. They also discussed a one-dimensional AH-isometry connecting $\mathbb{R}(I)$ with $\mathbb{R} \times \mathbb{R}$. This approach has proven instrumental in formulating inner products, organizing norms, and modeling geometric structures in a neutrosophic setting [42].

Several researchers have explored the characteristics of fuzzy algebraic structures through the application of norms [12], [13]–[24]. In this work, we propose the notions of complex intuitionistic fuzzy neutrosophic Lie subalgebras and Lie ideals, formulated using t-norm($\mathcal{T}^*$) and s-norm($\mathcal{S}^*$) within the framework of Lie subalgebras. We investigate their foundational properties, and further define operations such as intersection and addition for these constructs. The algebraic features of these operations are thoroughly analyzed. Additionally, we study how these structures behave under Lie subalgebra homomorphisms.

2. Preliminaries

This section introduces essential definitions related to neutrosophic logic and neutrosophic probability. We begin by recalling fundamental concepts that are crucial for the subsequent analysis. It includes basic Defns and preliminary results that will be used throughout the paper. For more detailed discussions, the reader is referred to references [1]–[10].

Definition 2.1 *[44] A Lie algebra is a vector space L over a field F (typically $\mathbb{R}$ or $\mathbb{C}$), equipped with a binary operation $[\cdot,\cdot] : L \times L \to L$, written as $(\breve{\varrho}, \tilde{\varsigma}) \mapsto [\breve{\varrho}, \tilde{\varsigma}]$, that satisfies the following conditions:*

1. *The operation $[\breve{\varrho}, \tilde{\varsigma}]$ is bilinear in both arguments,*
2. *$[\breve{\varrho}, \breve{\varrho}] = 0$ for every $\breve{\varrho} \in L$,*
3. *The identity*

$$[[\breve{\varrho}, \tilde{\varsigma}], z] + [[\tilde{\varsigma}, z], \breve{\varrho}] + [[z, \breve{\varrho}], \tilde{\varsigma}] = 0,$$

holds for all $\breve{\varrho}, \tilde{\varsigma}, z \in L$. This is known as the Jacobi identity.

In this work, the symbol L will represent a Lie algebra. Note that the Lie bracket is generally not associative; that is, $[[\breve{\varrho}, \tilde{\varsigma}], z] \neq [\breve{\varrho}, [\tilde{\varsigma}, z]]$ in general. However, the bracket is anti-commutative: $[\breve{\varrho}, \tilde{\varsigma}] = -[\tilde{\varsigma}, \breve{\varrho}]$. A vector subspace $H \subseteq L$ that remains closed under the bracket operation is referred to as a Lie subalgebra. Similarly, a subspace $I \subseteq L$ for which $[I, L] \subseteq I$ holds is called a Lie ideal. Clearly, every Lie ideal is also a Lie subalgebra.

Definition 2.2 *[44] Let L_1 and L_2 be Lie algebras over a field F. A linear transformation $\mho : L_1 \to L_2$ is called a Lie homomorphism if $\mho([\breve{\varrho}, \tilde{\varsigma}]) = [\mho(\breve{\varrho}), \mho(\tilde{\varsigma})]$ for all $\breve{\varrho}, \tilde{\varsigma} \in L_1$.*

Definition 2.3 *[44] Let $\breve{\varrho}, \tilde{\varsigma}$, and Z be sets. A function $\mho = (\mho_1, \mho_2) : X \to Y \times Z$ is referred to as a complex mapping if it consists of two component functions: $\mho_1 : X \to Y$ and $\mho_2 : X \to Z$.*

Definition 2.4 *[44] Let X be a nonempty set. A mapping*

$$A = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}) : X \to [0,1] \times [0,1]$$

is called an intuitionistic fuzzy set *(IFS) in X if*

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq 1 \qquad \text{for all } x \in X,$$

where the mappings $\mathcal{E}_{\tilde{\mathcal{Q}}} : X \to [0,1]$ and $\Upsilon_{\tilde{\mathcal{Q}}} : X \to [0,1]$ denote the membership and non-membership functions of A, respectively. In particular, $0^{\sim}$ and $1^{\sim}$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X, defined by

$$0^{\sim}(\breve{\varrho}) = (0,1), \qquad 1^{\sim}(\breve{\varrho}) = (1,0),$$

for all $x \in X$. We denote the family of all intuitionistic fuzzy sets in X by $\mathrm{IFS}(\breve{\varrho})$.

Definition 2.5 *[11] Let X be a nonempty set, and suppose $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}})$ and $\tilde{\mathcal{D}} = (\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}})$ are two intuitionistic fuzzy sets (IFSs) defined on X. Then the following hold:*

1. *$\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ if and only if, for every $\breve{\varrho} \in X$,*

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq \mathcal{E}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \quad \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq \Upsilon_{\tilde{\mathcal{D}}}(\breve{\varrho}).$$

2. *$\tilde{\mathcal{Q}} = \tilde{\mathcal{D}}$ if and only if both $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{Q}}$ hold.*

Definition 2.6 *[11] Let X be a nonempty set. A complex fuzzy set $\tilde{\mathcal{Q}}$ defined on X is a collection of ordered pairs of the form*

$$\tilde{\mathcal{Q}} = \{(\breve{\varrho}, \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho})) \mid \breve{\varrho} \in X\},$$

where $\mathcal{E}_{\tilde{\mathcal{Q}}}$ is a function assigning to each $\breve{\varrho} \in X$ a complex-valued membership degree $\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ that lies within the unit disk in the complex plane.

Each membership value is assumed to be expressed in polar form:

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})},$$

where $i = \sqrt{-1}$, $\hat{\rho}_{\tilde{\mathcal{Q}}} : X \to [0,1]$ represents the modulus, and $\hat{\zeta}_{\tilde{\mathcal{Q}}} : X \to [0, 2\pi]$ denotes the phase angle.

Setting $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = 0$ for all $\breve{\varrho} \in X$ retrieves the standard (real-valued) fuzzy set as a special case.

Further, let $\mathcal{E}_1 = r_1 e^{iw_1}$, $\mathcal{E}_2 = r_2 e^{iw_2}$, and $\mathcal{E}_3 = r_3 e^{iw_3}$ be complex numbers within the unit circle. Then, the expression

$$\mathcal{E}_1 \leq \mathcal{E}_2 \leq \mathcal{E}_3$$

denotes the condition

$$r_1 \leq r_2 \leq r_3 \quad \text{and} \quad w_1 \leq w_2 \leq w_3.$$

Definition 2.7 *[11] A complex intuitionistic fuzzy set $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \gamma_{\tilde{\mathcal{Q}}})$, defined over a universe U, consists of two complex-valued functions: $\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ representing the degree of membership and $\gamma_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ representing the degree of non-membership of each element $\breve{\varrho} \in U$.*

Both functions take values inside the unit disk in the complex plane and are represented in polar coordinates as:

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) e^{i\omega_{\mathcal{E}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})}, \qquad \gamma_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = k_{\tilde{\mathcal{Q}}}(\breve{\varrho}) e^{i\omega_{\gamma_{\tilde{\mathcal{Q}}}}(\breve{\varrho})},$$

where $i = \sqrt{-1}$*, the magnitude functions* $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *and* $k_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *satisfy* $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), k_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \in [0,1]$*, and their sum adheres to the condition:*

$$0 \leq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + k_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq 1.$$

The phase functions $\omega_{\mathcal{E}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})$ *and* $\omega_{\gamma_{\tilde{\mathcal{Q}}}}(\breve{\varrho})$ *are real-valued functions mapping into the interval* $[0, 2\pi]$*.*

This framework allows for complex-valued assessments of both membership and non-membership, while ensuring their combined intensity remains within the allowed bounds.

Definition 2.8 *Let* X *be a nonempty set. A mapping*

$$\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}},\, \Upsilon_{\tilde{\mathcal{Q}}},\, \mathcal{G}_{\tilde{\mathcal{Q}}}\right) : X \to [0,1] \times [0,1] \times [0,1]$$

is called a neutrosophic intuitionistic fuzzy set (NIFS) *in* X *if for every* $\breve{\varrho} \in X$*,*

$$0 \leq \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\ \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\ \mathcal{G}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq 1, \qquad 0 \leq \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + \mathcal{G}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq 3.$$

Here:

1. $\Omega_{\tilde{\mathcal{Q}}}(\breve{\varrho}) : X \to [0,1]$ *denotes the truth (membership) degree,*
2. $\Upsilon_{\tilde{\mathcal{Q}}} : X \to [0,1]$ *denotes the indeterminacy degree,*
3. $\mathcal{G}_{\tilde{\mathcal{Q}}} : X \to [0,1]$ *denotes the falsity (non-membership) degree.*

In particular, the neutrosophic intuitionistic fuzzy empty and whole sets, denoted by $0^{\sim}$ *and* $1^{\sim}$*, are defined by*

$$0^{\sim}(\breve{\varrho}) = (0,1,1), \qquad 1^{\sim}(\breve{\varrho}) = (1,0,0) \quad \text{for all } \breve{\varrho} \in X.$$

The family of all neutrosophic intuitionistic fuzzy sets on X *is denoted by* NIFS(X)*.*

Definition 2.9 *Let* X *be a nonempty set, and suppose* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \mathcal{G}_{\tilde{\mathcal{Q}}})$ *and* $B = (\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \mathcal{G}_{\tilde{\mathcal{D}}})$ *are two NIFSs defined on* X*. Then the following hold:*

1. $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ *if and only if, for every* $\breve{\varrho} \in X$*,*

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq \mathcal{E}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \quad \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq \Upsilon_{\tilde{\mathcal{D}}}(\breve{\varrho}), \quad \mathcal{G}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq \mathcal{G}_{\tilde{\mathcal{D}}}(\breve{\varrho}).$$

2. $\tilde{\mathcal{Q}} = \tilde{\mathcal{D}}$ *if and only if both* $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ *and* $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{Q}}$ *hold.*

Definition 2.10 *Let* X *be a nonempty set, and suppose* $\tilde{\mathcal{Q}} = (\Omega_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \Theta_{\tilde{\mathcal{Q}}})$ *and* $\tilde{\mathcal{D}} = (\Omega_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \Theta_{\tilde{\mathcal{D}}})$ *are NIFSs defined on* X*. Then the following hold:*

1. $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ *if and only if, for every* $\breve{\varrho} \in X$*,*

$$\Omega_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq \Omega_{\tilde{\mathcal{D}}}(\breve{\varrho}), \qquad \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq \Upsilon_{\tilde{\mathcal{D}}}(\breve{\varrho}), \qquad \Theta_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq \Theta_{\tilde{\mathcal{D}}}(\breve{\varrho}).$$

2. $\tilde{\mathcal{Q}} = \tilde{\mathcal{D}}$ *if and only if both* $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{D}}$ *and* $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{Q}}$ *hold.*

Definition 2.11 *A complex neutrosophic intuitionistic fuzzy set* (CNIFS) $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \mathcal{G}_{\tilde{\mathcal{Q}}})$ *over a universe* U *is determined by three complex-valued functions:* $\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *(membership) and* $\Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *(non-membership), and* $\mathcal{G}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *(indeterminacy) defined for each* $\breve{\varrho} \in U$*. Both functions take values inside the unit disc of the complex plane.*

The membership, non-membership and indeterminacy degrees are represented in polar form:

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\, e^{\,i\omega_{\mathcal{E}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})}, \qquad \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = k_{\tilde{\mathcal{Q}}}(\breve{\varrho})\, e^{\,i\omega_{\Upsilon_{\tilde{\mathcal{Q}}}}(\breve{\varrho})}, \qquad \mathcal{G}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{q}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\, e^{\,i\omega_{\mathcal{G}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})}$$

where $i = \sqrt{-1}$*, the magnitudes* $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$*,* $k_{\tilde{\mathcal{Q}}}(\breve{\varrho})$ *and* $\dot{q}_{\tilde{\mathcal{Q}}}$ *belong to* $[0,1]$*, and the phase angles* $\omega_{\mathcal{E}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})$*,* $\omega_{\Upsilon_{\tilde{\mathcal{Q}}}}(\breve{\varrho})$*,* $\omega_{\mathcal{G}_{\tilde{\mathcal{Q}}}}(\breve{\varrho})$ *lie in* $[0, 2\pi]$*, and*

$$0 \leq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + k_{\tilde{\mathcal{Q}}}(\breve{\varrho}) + \omega_{\mathcal{G}_{\tilde{\mathcal{Q}}}}(\breve{\varrho}) \leq 3$$

subject to the constraint

$$\omega_{\mathcal{G}_{\tilde{\mathcal{Q}}}}(\breve{\varrho}) = 3 - \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) - k_{\tilde{\mathcal{Q}}}(\breve{\varrho}).$$

This formulation extends the standard neutrosophic intuitionistic fuzzy set NIFS *structure into the complex domain, allowing complex-valued degrees of truth, indeterminacy, and falsity.*

Definition 2.12 *A tri-nary t-norm $\mathcal{T}^*$ is a function*

$$\mathcal{T}^* : [0,1]^3 \to [0,1]$$

satisfying the following properties for all $\breve{\varrho}, \tilde{\varsigma}, z, w \in [0,1]$:

(T1) $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, 1) = \mathcal{T}^*(\breve{\varrho}, 1, \tilde{\varsigma}) = \mathcal{T}^*(1, \breve{\varrho}, \tilde{\varsigma}) = \mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma})$ *(neutral element),*

(T2) $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z) \leq \mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, w)$ *if* $z \leq w$ *(monotonicity in each argument),*

(T3) $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z)$ *is symmetric in its arguments (commutativity),*

(T4) $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, \mathcal{T}^*(z, w, u)) = \mathcal{T}^*(\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z), w, u)$ *(associativity in a generalized form),*

That is, a tri-nary t-norm extends the binary t-norm to operate on three inputs while preserving the essential properties of neutrality, monotonicity, commutativity, and associativity.

Moreover, it follows that if $\breve{\varrho}_1 \geq \breve{\varrho}_2$, $\tilde{\varsigma}_1 \geq \tilde{\varsigma}_2$, and $z_1 \geq z_2$, then:

$$\mathcal{T}^*(\breve{\varrho}_1, \tilde{\varsigma}_1, z_1) \geq \mathcal{T}^*(\breve{\varrho}_2, \tilde{\varsigma}_2, z_2).$$

Example 2.1

1. *Tri-nary minimum $\mathcal{T}^*$-norm:*

$$\mathcal{T}^*_{\min}(\breve{\varrho}, \tilde{\varsigma}, z) = \min\{\breve{\varrho}, \tilde{\varsigma}, z\}.$$

This is a natural extension of the standard intersection t-norm to three variables.

2. *Tri-nary algebraic product $\mathcal{T}^*$-norm:*

$$\mathcal{T}^*_p(\breve{\varrho}, \tilde{\varsigma}, z) = \breve{\varrho} \cdot y \cdot z.$$

This generalizes the binary algebraic product t-norm and preserves associativity and commutativity.

3. *Tri-nary bounded sum $\mathcal{T}^*$-norm:*

$$\mathcal{T}^*_{\tilde{\mathcal{D}}}(\breve{\varrho}, \tilde{\varsigma}, z) = \max\{0, \breve{\varrho} + \tilde{\varsigma} + z - 2\}.$$

This is the 3-variable extension of the bounded sum, ensuring the output remains in $[0,1]$.

4. *Tri-nary drastic $\mathcal{T}^*$-norm:*

$$\mathcal{T}^*_D(\breve{\varrho}, \tilde{\varsigma}, z) = \begin{cases} \min\{\breve{\varrho}, \tilde{\varsigma}, z\}, & \textit{if at least two of } \breve{\varrho}, \tilde{\varsigma}, z \textit{ are equal to } 1, \\ 0, & \textit{otherwise.} \end{cases}$$

This extends the drastic t-norm to three variables with a more selective condition for nonzero output.

5. *Tri-nary Hamacher product:*

$$\mathcal{T}^*_h(\breve{\varrho}, \tilde{\varsigma}, z) = \begin{cases} 0, & \textit{if } \breve{\varrho} = \tilde{\varsigma} = z = 0, \\ \dfrac{\breve{\varrho}\tilde{\varsigma}z}{\breve{\varrho}\tilde{\varsigma} + \tilde{\varsigma}z + z\breve{\varrho} - 2\breve{\varrho}\tilde{\varsigma}z}, & \textit{otherwise.} \end{cases}$$

This is an extension of the Hamacher product to tri-nary input while preserving the form of the denominator to maintain proper normalization.

Example 2.2 *The tri-nary drastic t-norm represents the minimal tri-nary t-norm in a pointwise sense, whereas the tri-nary minimum t-norm serves as the maximal one. That is, for all $\breve{\varrho}, \tilde{\varsigma}, z \in [0,1]$, the following holds:*

$$\mathcal{T}^*_D(\breve{\varrho}, \tilde{\varsigma}, z) \le \mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z) \le \mathcal{T}^*_{\min}(\breve{\varrho}, \tilde{\varsigma}, z),$$

where

$$\mathcal{T}^*_D(\breve{\varrho}, \tilde{\varsigma}, z) = \begin{cases} \min\{\breve{\varrho}, \tilde{\varsigma}, z\}, & \text{if at least two of } \breve{\varrho}, \tilde{\varsigma}, z \text{ equal } 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{T}^*_{\min}(\breve{\varrho}, \tilde{\varsigma}, z) = \min\{\breve{\varrho}, \tilde{\varsigma}, z\}.$$

Note that a tri-nary t-norm $\mathcal{T}^*$ is said to be *idempotent* if it satisfies the condition $\mathcal{T}^*\breve{\varrho}, \breve{\varrho}, \breve{\varrho}) = \breve{\varrho}$ for every $\breve{\varrho} \in [0,1]$.

Lemma 2.1 *Let $\mathcal{T}^* : [0,1]^3 \to [0,1]$ be a tri-nary t-norm. Then, for all $\breve{\varrho}, \tilde{\varsigma}, w, z \in [0,1]$, the following equality holds:*

$$\mathcal{T}^*\big(\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, 1), \mathcal{T}^*(w, z, 1), 1\big) = \mathcal{T}^*\big(\mathcal{T}^*(\breve{\varrho}, w, 1), \mathcal{T}^*(\tilde{\varsigma}, z, 1), 1\big).$$

Lemma 2.2 *Let $\mathcal{T}^* : [0,1]^3 \to [0,1]$ be a tri-nary t-norm. Thus, for all $\breve{\varrho}, \tilde{\varsigma}, z \in [0,1]$, the following properties hold:*

1. $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z) \le \min\{\breve{\varrho}, \tilde{\varsigma}, z\}$.
2. $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, z) = 0$ *if and only if at least one of $\breve{\varrho}, \tilde{\varsigma}, z$ is zero.*
3. $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma}, 1) = \mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma})$ *where $\mathcal{T}^*(\breve{\varrho}, \tilde{\varsigma})$ is the induced binary t-norm by fixing the third argument as 1.*
4. $\mathcal{T}^*\breve{\varrho}, 1, 1) = \breve{\varrho}$ *(neutral element property).*

Definition 2.13 *A tri-nary $\acute{\mathcal{S}}$-norm is a function*

$$\acute{\mathcal{S}} : [0,1]^3 \to [0,1]$$

that meets the following criteria for any $\breve{\varrho}, \tilde{\varsigma}, z, w \in [0,1]$:

(S1) $\acute{\mathcal{S}}(\breve{\varrho}, 0, 0) = \breve{\varrho}$ *(identity element),*

(S2) If $\tilde{\varsigma} \le w$ and $z \le w$, then

$$\acute{\mathcal{S}}(\breve{\varrho}, \tilde{\varsigma}, z) \le \acute{\mathcal{S}}(\breve{\varrho}, w, w)$$

which expresses monotonicity,

(S3) $\acute{\mathcal{S}}$ is symmetric with respect to its arguments; that is,

$$\acute{\mathcal{S}}(\breve{\varrho}, \tilde{\varsigma}, z) = \acute{\mathcal{S}}(\pi(\breve{\varrho}, \tilde{\varsigma}, z))$$

for any permutation π of the set $\{\breve{\varrho}, \tilde{\varsigma}, z\}$,

(S4) $\acute{\mathcal{S}}$ satisfies associativity in the following form

$$\acute{\mathcal{S}}\big(\breve{\varrho}, \acute{\mathcal{S}}(\tilde{\varsigma}, z, w), 0\big) = \acute{\mathcal{S}}\big(\acute{\mathcal{S}}(\breve{\varrho}, \tilde{\varsigma}, 0), z, w\big).$$

Furthermore, the function $\acute{\mathcal{S}}$ is called idempotent *if for every $\breve{\varrho} \in [0,1]$,*

$$\acute{\mathcal{S}}(\breve{\varrho}, \breve{\varrho}, \breve{\varrho}) = \breve{\varrho}.$$

Example 2.3 *Some fundamental tri-nary $\acute{\mathcal{S}}$-norms are defined as follows for any $\breve{\varrho}, \tilde{\varsigma}, z \in [0,1]$:*

$$\acute{\mathcal{S}}_m(\breve{\varrho}, \tilde{\varsigma}, z) = \max\{\breve{\varrho}, \tilde{\varsigma}, z\},$$

$$\acute{\mathcal{S}}_{\tilde{\mathcal{D}}}(\breve{\varrho}, \tilde{\varsigma}, z) = \min\{1, \breve{\varrho} + \tilde{\varsigma} + z\},$$

$$\acute{\mathcal{S}}_p(\breve{\varrho}, \tilde{\varsigma}, z) = \breve{\varrho} + \tilde{\varsigma} + z - \breve{\varrho}\tilde{\varsigma} - \tilde{\varsigma}z - z\breve{\varrho} + \breve{\varrho}\tilde{\varsigma}z.$$

Here, $\acute{\mathcal{S}}_m$ is referred to as the standard union, *$\acute{\mathcal{S}}_{\tilde{\mathcal{D}}}$ as the* bounded sum, *and $\acute{\mathcal{S}}_p$ as the* algebraic sum *in the context of tri-nary operations.*

3. Main Results

Characterization of Complex Intuitionistic Fuzzy Lie Subalgebras via Neutrosophic Norms

Definition 3.1 *Let L be a Lie subalgebra. Suppose that*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \text{and} \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$$

are three complex-valued functions on L, representing the membership, non-membership, and indeterminacy degrees, respectively, of a complex fuzzy set on L.

Then the triplet $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \mathrm{NIFS}(L)$ is referred to as a complex intuitionistic fuzzy neutrosophic Lie subalgebra of L, and is denoted by CIFNLS(L), *with respect to a chosen t-norm $\mathcal{T}^*$ and $\acute{\mathcal{S}}$-norm $\acute{\mathcal{S}}$, if the following conditions are satisfied:*

1. $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \geq \mathcal{T}^*(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
2. $\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
3. $\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \geq \mathcal{T}^*(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
4. $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \geq \min\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
5. $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
6. $\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \geq \min\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
7. $\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \leq \acute{\mathcal{S}}(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
8. $\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
9. $\dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \acute{\mathcal{S}}(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
10. $\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \leq \max\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
11. $\dot{\chi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
12. $\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \max\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \theta_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
13. $\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \leq \acute{\mathcal{S}}(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
14. $\tilde{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
15. $\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \acute{\mathcal{S}}(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
16. $\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}) \leq \max\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
17. $\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
18. $\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \max\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$.

for all $\breve{\varrho}, \tilde{\varsigma} \in L$ and $\check{\tau} \in \mathcal{F}$.

Definition 3.2 *Let L be a Lie subalgebra. Suppose that*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \text{and} \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$$

are three complex-valued functions on L, representing membership, non-membership, and indeterminacy degrees, respectively, of complex fuzzy sets on L.

Then the triplet $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \mathrm{NIFS}(L)$is known as a complex neutrosophic intuitionistic fuzzy Lie ideal *of L under the operations defined by the t-norm $\mathcal{T}^*$ and $\acute{\mathcal{S}}$-norm $\acute{\mathcal{S}}$, if the following conditions hold:*

1. $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq \mathcal{T}^*(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
2. $\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
3. $\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \geq \mathcal{T}^*(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
4. $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq \min\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
5. $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
6. $\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \geq \min\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
7. $\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
8. $\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
9. $\dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \acute{\mathcal{S}}(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
10. $\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \max\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
11. $\dot{\chi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
12. $\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \max\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \theta_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
13. $\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
14. $\tilde{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
15. $\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \acute{\mathcal{S}}(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}))$,
16. $\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \max\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$,
17. $\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$,
18. $\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) \leq \max\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\}$.

for every $\breve{\varrho}, \tilde{\varsigma} \in L$ *and* $\check{\tau} \in \mathcal{F}$. *We denote by* CNIFIS($L$) *the collection of all complex intuitionistic fuzzy neutrosophic Lie ideals of* L*, considered under the norms given by a t-norm* T *and an* $\acute{\mathcal{S}}$*-norm* $\acute{\mathcal{S}}$.

Example 3.1 *Let* $L = \mathbb{R}^3$ *be a Lie algebra over* $\mathbb{R}$*, with Lie bracket defined by the cross product:*

$$[\breve{\varrho}, \tilde{\varsigma}] = \breve{\varrho} \times \tilde{\varsigma}, \quad \textit{for all } \breve{\varrho}, \tilde{\varsigma} \in L.$$

Define a complex Neutrosophic set $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \mathrm{NIFS}(L)$ *on* L*, where:*

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1, \breve{\varrho}_2, \breve{\varrho}_3) = \begin{cases} 1, & \textit{if } \breve{\varrho}_1 = \breve{\varrho}_2 = \breve{\varrho}_3 = 0, \\ 0.7, & \textit{if } \breve{\varrho}_1 = \breve{\varrho}_2 = 0,\ \breve{\varrho}_3 \neq 0, \\ 0.3, & \textit{otherwise}, \end{cases}$$

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1,\breve{\varrho}_2,\breve{\varrho}_3)=\begin{cases}0, & \text{if } \breve{\varrho}_1=\breve{\varrho}_2=\breve{\varrho}_3=0,\\ \frac{\pi}{6}, & \text{if } \breve{\varrho}_3\neq 0,\\ \frac{\pi}{4}, & \text{otherwise,}\end{cases}$$

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1,\breve{\varrho}_2,\breve{\varrho}_3)=\begin{cases}0.2, & \text{if } \breve{\varrho}_1=\breve{\varrho}_2=\breve{\varrho}_3=0,\\ 0.4, & \text{if } \breve{\varrho}_1^2+\breve{\varrho}_2^2+\breve{\varrho}_3^2\leq 4,\\ 0.6, & \text{otherwise,}\end{cases}$$

$$\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1,\breve{\varrho}_2,\breve{\varrho}_3)=\begin{cases}0, & \text{if } \breve{\varrho}_1=\breve{\varrho}_2=\breve{\varrho}_3=0,\\ \frac{\pi}{2}, & \text{if } \breve{\varrho}_2\neq 0,\\ \pi, & \text{otherwise,}\end{cases}$$

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1,\breve{\varrho}_2,\breve{\varrho}_3)=\begin{cases}0, & \text{if } \breve{\varrho}_1=\breve{\varrho}_2=\breve{\varrho}_3=0,\\ 0.1, & \text{if } \breve{\varrho}_1=0,\ \breve{\varrho}_2^2+\breve{\varrho}_3^2>0,\\ 0.25, & \text{otherwise,}\end{cases}$$

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}_1,\breve{\varrho}_2,\breve{\varrho}_3)=\begin{cases}0, & \text{if } \breve{\varrho}_1=\breve{\varrho}_2=\breve{\varrho}_3=0,\\ \frac{3\pi}{4}, & \text{otherwise.}\end{cases}$$

Let the t-norm and s-norm be defined by

$$\mathcal{T}^*(\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2)=\breve{\mathfrak{d}}_1\breve{\mathfrak{d}}_2,\quad \acute{\mathcal{S}}(\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2)=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2-\breve{\mathfrak{d}}_1\breve{\mathfrak{d}}_2.$$

Then $\tilde{\mathcal{Q}}=(\mathcal{E}_{\tilde{\mathcal{Q}}},\Upsilon_{\tilde{\mathcal{Q}}},\check{\mathcal{G}}_{\tilde{\mathcal{Q}}})\in IFS(L)$ satisfies all the required conditions for a complex neutrosophic intuitionistic fuzzy Lie subalgebra, i.e., $A\in \mathrm{CIFNLS}(L)$.

Lemma 3.1 *Let $\tilde{\mathcal{Q}}=\left(\mathcal{E}_{\tilde{\mathcal{Q}}},\Upsilon_{\tilde{\mathcal{Q}}},\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right)\in \mathrm{CIFNLS}(L)$, and suppose $\mathcal{T}^*$ and $\acute{\mathcal{S}}$ are idempotent t- and $\acute{\mathcal{S}}$-norms, respectively. Then the following properties hold:*

1. *$\tilde{\mathcal{Q}}(0)\supseteq\tilde{\mathcal{Q}}(\breve{\varrho})$ for every $\breve{\varrho}\in L$.*
2. *$\tilde{\mathcal{Q}}(\breve{\varrho})=\tilde{\mathcal{Q}}(-\breve{\varrho})$ for all $\breve{\varrho}\in L$.*
3. *$\tilde{\mathcal{Q}}([\breve{\varrho},\tilde{\varsigma}])=\tilde{\mathcal{Q}}([\tilde{\varsigma},\breve{\varrho}])$ for all $\breve{\varrho},\tilde{\varsigma}\in L$.*
4. *$\tilde{\mathcal{Q}}(\breve{\varrho}-\tilde{\varsigma})=\tilde{\mathcal{Q}}(0)$ implies that $\tilde{\mathcal{Q}}(\breve{\varrho})=\tilde{\mathcal{Q}}(\tilde{\varsigma})$ for all $\breve{\varrho},\tilde{\varsigma}\in L$.*

Proof: (1) Let $\breve{\varrho}\in L$. Then

$$\begin{aligned}
\hat{\rho}_{\tilde{\mathcal{Q}}}(0)=\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\geq\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\hat{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right)\geq\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right)=\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
\hat{\zeta}_{\tilde{\mathcal{Q}}}(0)=\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\geq\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\hat{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right\}\geq\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\}=\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
\tilde{\rho}_{\tilde{\mathcal{Q}}}(0)=\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\leq\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\tilde{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right)\leq\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right)=\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
\acute{\zeta}_{\tilde{\mathcal{Q}}}(0)=\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\leq\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\acute{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right\}\leq\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\}=\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
\dot{\xi}_{\tilde{\mathcal{Q}}}(0)=\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\leq\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\dot{\xi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right)\leq\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right)=\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
\dot{\chi}_{\tilde{\mathcal{Q}}}(0)=\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-\breve{\varrho}))\leq\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\dot{\chi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\right\}\leq\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}),\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\}=\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}).
\end{aligned}$$

As a result

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(0)=\hat{\rho}_{\tilde{\mathcal{Q}}}(0)e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(0)}\geq\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})}=\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\Upsilon_{\tilde{\mathcal{Q}}}(0)=\tilde{\rho}_{\tilde{\mathcal{Q}}}(0)e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}(0)}\leq\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})}=\Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(0) = \dot{\xi}_{\tilde{\mathcal{Q}}}(0)e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}(0)} \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{idot\theta_{\tilde{\mathcal{Q}}}(\breve{\varrho})} = \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and so

$$\tilde{\mathcal{Q}}(0) = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}(0), \Upsilon_{\tilde{\mathcal{Q}}}(0), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(0)\big) \supseteq \big(\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(0)\big) = \tilde{\mathcal{Q}}(\breve{\varrho}).$$

(2) Let $\breve{\varrho} \in L$. As

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$
$$\hat{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \hat{\zeta}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\zeta}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \tilde{\rho}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \tilde{\rho}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$
$$\acute{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \acute{\zeta}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \acute{\zeta}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \dot{\xi}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\xi}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$
$$\dot{\chi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}) = \dot{\chi}_{\tilde{\mathcal{Q}}}((-1)\breve{\varrho}) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\chi}_{\tilde{\mathcal{Q}}}(-(-\breve{\varrho})) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}).$$

Accordingly

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})} = \hat{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})} = \mathcal{E}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$

and

$$\Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})} = \tilde{\rho}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})} = \Upsilon_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$

and

$$\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})} = \dot{\xi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})} = \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})$$

therefore

$$\tilde{\mathcal{Q}}(\breve{\varrho}) = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\big) = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}(-\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}}(-\breve{\varrho}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(-\breve{\varrho})\big) = \tilde{\mathcal{Q}}(-\breve{\varrho}).$$

(3) Let $\breve{\varrho}, \tilde{\varsigma} \in L$. Then by using part(2) we get that

$$\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \hat{\rho}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \hat{\rho}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$
$$\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \hat{\zeta}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \hat{\zeta}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$
$$\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \tilde{\rho}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \tilde{\rho}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$
$$\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \acute{\zeta}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \acute{\zeta}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$
$$\dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \dot{\xi}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \dot{\xi}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$
$$\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \dot{\chi}_{\tilde{\mathcal{Q}}}(-[\tilde{\varsigma}, \breve{\varrho}]) = \dot{\chi}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$

then

$$\mathcal{E}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])} = \hat{\rho}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])} = \mathcal{E}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$

and

$$\Upsilon_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])} = \tilde{\rho}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])} = \Upsilon_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$

and

$$\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]) = \dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])} = \dot{\xi}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])} = \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])$$

so

$$\tilde{\mathcal{Q}}([\breve{\varrho}, \tilde{\varsigma}]) = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]), \Upsilon_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}])\big) = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}]), \Upsilon_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}]), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}([\tilde{\varsigma}, \breve{\varrho}])\big) = \tilde{\mathcal{Q}}([\tilde{\varsigma}, \breve{\varrho}]).$$

(4) Let $\breve{\varrho}, \tilde{\varsigma} \in L$. Therefore

$$\begin{aligned}
\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - (\breve{\varrho} - \tilde{\varsigma})) \\
&= \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + (-(\breve{\varrho} - \tilde{\varsigma}))) \\
&\geq \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho} - \tilde{\varsigma}))\right) \\
&= \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma})\right) \\
&= \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(0)\right) \\
&\geq \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right) \\
&= \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \\
&= \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma} + \tilde{\varsigma}) \\
&\geq \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&= \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(0), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&\geq \mathcal{T}^* \left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&= \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})
\end{aligned}$$

thus $\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. As

$$\begin{aligned}
\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - (\breve{\varrho} - \tilde{\varsigma})) \\
&= \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + (-(\breve{\varrho} - \tilde{\varsigma}))) \\
&\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho} - \tilde{\varsigma}))\right\} \\
&= \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma})\right\} \\
&= \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(0)\right\} \\
&\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\} \\
&= \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \\
&= \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma} + \tilde{\varsigma}) \\
&\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \\
&= \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(0), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right. \\
&\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \\
&= \left.\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}.
\end{aligned}$$

As a result, we obtain $\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. Consequently, it follows that

$$\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})} = \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})} = \mathcal{E}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).$$

In addition,

$$\begin{aligned}
\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - (\breve{\varrho} - \tilde{\varsigma})) \\
&= \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + (-(\breve{\varrho} - \tilde{\varsigma}))) \\
&\leq \acute{\mathcal{S}} \left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho} - \tilde{\varsigma}))\right) \\
&= \acute{\mathcal{S}} \left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma})\right) \\
&= \acute{\mathcal{S}} \left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(0)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right)\\
&= \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
&= \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma} + \tilde{\varsigma})\\
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)\\
&= \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(0), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)\\
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)\\
&= \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).
\end{aligned}
$$

Thus, $\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. Also

$$
\begin{aligned}
\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - (\breve{\varrho} - \tilde{\varsigma}))\\
&= \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + (-(\breve{\varrho} - \tilde{\varsigma})))\\
&\leq \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho} - \tilde{\varsigma}))\right\}\\
&= \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma})\right\}\\
&= \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(0)\right\}\\
&\leq \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\}\\
&= \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
&= \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma} + \tilde{\varsigma})\\
&\leq \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}\\
&= \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(0), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}\\
&\leq \mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}\\
&= \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\Big\}
\end{aligned}
$$

this implies that $\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. As a consequence, we have

$$
\Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \tilde{\rho}(\breve{\varrho})e^{i\acute{\zeta}(\breve{\varrho})} = \tilde{\rho}(\tilde{\varsigma})e^{i\acute{\zeta}(\tilde{\varsigma})} = \Upsilon_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).
$$

Also

$$
\begin{aligned}
\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - (\breve{\varrho} - \tilde{\varsigma}))\\
&= \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + (-(\breve{\varrho} - \tilde{\varsigma})))\\
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho} - \tilde{\varsigma}))\right)\\
&= \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma})\right)\\
&= \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(0)\right)\\
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right)\\
&= \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\\
&= \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho} - \tilde{\varsigma} + \tilde{\varsigma})
\end{aligned}
$$

$$\begin{aligned}
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}-\tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&= \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(0), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&\leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \\
&= \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})
\end{aligned}$$

Hence, we conclude that $\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. Moreover,

$$\begin{aligned}
\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) &= \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}-(\breve{\varrho}-\tilde{\varsigma})) \\
&= \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+(-(\breve{\varrho}-\tilde{\varsigma}))) \\
&\leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(-(\breve{\varrho}-\tilde{\varsigma}))\right\} \\
&= \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}-\tilde{\varsigma})\right\} \\
&= \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(0)\right\} \\
&\leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right\} \\
&= \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \\
&= \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}-\tilde{\varsigma}+\tilde{\varsigma}) \\
&\leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}-\tilde{\varsigma}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \\
&= \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(0), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right. \\
&\leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \\
&= \left.\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}
\end{aligned}$$

It follows that $\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})$. Consequently, we have

$$\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = \dot{\xi}(\breve{\varrho})e^{i\dot{\chi}(\breve{\varrho})} = \dot{\xi}(\tilde{\varsigma})e^{i\dot{\chi}(\tilde{\varsigma})} = \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).$$

Thus,

$$\tilde{\mathcal{Q}}(\breve{\varrho}) = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right) = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \Upsilon_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) = \tilde{\mathcal{Q}}(\tilde{\varsigma}).$$

□

Proposition 3.1 *Let L be a Lie subalgebra and consider $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{NIFS}(L)$, where*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}}e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}}e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}}e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}},$$

are three complex fuzzy functions defined on L. Suppose $\mathcal{T}^*$ and $\acute{\mathcal{S}}$ are idempotent norms. Then the following conditions are equivalent:

1. *$\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right)$ belongs to* CIFNLS(L).
2. *For any $t, s, l \in [0,1]$, the set*

$$\begin{aligned}
\tilde{\mathcal{Q}}^t_{s,l} &= \{\breve{\varrho} \in L \mid \tilde{\mathcal{Q}}(\breve{\varrho}) \succeq (t,s,l)\} = \left\{\breve{\varrho} \in L \mid \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t, \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq l\right\} \\
&= \left\{\breve{\varrho} \in L \mid \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t, \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t, \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t, \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s, \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s, \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s\right\}
\end{aligned}$$

   forms a subalgebra of L.

Proof: Let $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CIFNLS}(L)$, and let $\breve{\varrho}, \tilde{\varsigma} \in \tilde{\mathcal{Q}}^t_{s,l}$ with $\check{\tau} \in F$. We will demonstrate that the elements $\breve{\varrho}+\tilde{\varsigma}$, $\check{\tau}\breve{\varrho}$, and $[\breve{\varrho}, \tilde{\varsigma}]$ also belong to $\tilde{\mathcal{Q}}^t_{s,l}$. Namely, as

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \geq \mathcal{T}^*(t,t) = t,$$

and

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq \operatorname{Min}\{w(\breve{\varrho}), w(\tilde{\varsigma})\} \geq \operatorname{Min}\{t,t\} = t$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \leq \acute{\mathcal{S}}(s,s) = s$$

and

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \leq \operatorname{Max}\{s,s\} = s$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \leq \acute{\mathcal{S}}(s,s) = s$$

and

$$\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \leq \operatorname{Max}\{s,s\} = s.$$

Hence, it follows that $\breve{\varrho}+\tilde{\varsigma} \in \tilde{\mathcal{Q}}^t_{s,l}$. Furthermore, considering the inequalities

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t, \quad \hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t$$

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s, \quad \acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s, \quad \dot{\chi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s.$$

We conclude that $\check{\tau}\breve{\varrho}$ also belongs to $\tilde{\mathcal{Q}}^t_{s,l}$. Additionally,

$$\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \geq \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \geq \mathcal{T}^*(t,t) = t$$

and

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \geq \operatorname{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}\{t,t\} = t$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \leq \acute{\mathcal{S}}(s,s) = s$$

and

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq \operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}\{s,s\} = s$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \leq \acute{\mathcal{S}}(s,s) = s$$

and

$$\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\} \leq \dot{\chi}_{\tilde{\mathcal{Q}}}\{s,s\} = s$$

implying that $[\breve{\varrho},\tilde{\varsigma}] \in \tilde{\mathcal{Q}}^t_{s,l}$. Consequently, $\tilde{\mathcal{Q}}^t_{s,l}$ forms a subalgebra of L for every $t,s,l \in [0,1]$.

Conversely, suppose that for all $s,t,l \in [0,1]$, the set $\tilde{\mathcal{Q}}^t_{s,l}$ is a Lie subalgebra of L. For arbitrary elements $\breve{\varrho},\tilde{\varsigma} \in L$ and scalar $\check{\tau} \in F$, assume

$$\tilde{\mathcal{Q}}(\tilde{\varsigma}) = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \Upsilon_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right) \supseteq \tilde{\mathcal{Q}}(\breve{\varrho}) = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho})\right) = (t,s,l).$$

This means that

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \geq \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = t, \quad \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \geq \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = t,$$

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \leq \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = s, \quad \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \leq \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = s,$$

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \leq \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = l, \quad \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}) \leq \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) = l.$$

Hence, both $\breve{\varrho}$ and $\tilde{\varsigma}$ belong to $\tilde{\mathcal{Q}}^t_{s,l}$, which implies that their sum $\breve{\varrho}+\tilde{\varsigma}$, scalar multiple $\check{\tau}\breve{\varrho}$, and bracket $[\breve{\varrho},\tilde{\varsigma}]$ also lie in $\tilde{\mathcal{Q}}^t_{s,l}$. Since

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq t = \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \geq t = \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}$$

and

$$\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \geq t = \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \geq t = \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \geq t = \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \geq t = \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right.$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq s = \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq s = \text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \leq s = \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \leq s = \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq s = \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\acute{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq s = \text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).\right.$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq s = \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}) \leq s = \text{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \leq s = \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\tau}\breve{\varrho}) \leq s = \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho})$$

and

$$\dot{\xi}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq s = \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right)$$

and

$$\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]) \leq s = \text{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}).\right.$$

Therefore $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in CNIFLS(L)$.

□

Corollary 3.1 *Let L be a Lie subalgebra and consider $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \text{NIFS}(L)$, where*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$$

are complex fuzzy mappings defined on L. Assume the norms $\mathcal{T}^*$ and $\acute{\mathcal{S}}$ are idempotent. Then the following conditions are equivalent:

1. $\tilde{\mathcal{Q}} = \big(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\big)$ *belongs to the class* CIFNLS(L).
2. *For each triple* $(t, s, l) \in [0,1]^3$*, the set*

$$\tilde{\mathcal{Q}}^t_{s,l} = \{\breve{\varrho} \in L \mid \tilde{\mathcal{Q}}(\breve{\varrho}) \succeq (t,s,l)\} = \{\breve{\varrho} \in L \mid \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < l\}$$

$$= \Big\{\breve{\varrho} \in L \mid \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s\Big\}$$

forms a Lie subalgebra of L.

The argument closely follows the proof of Proposition 3.1.

Proposition 3.2 *Let* L *be a Lie subalgebra, and consider* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in$ NIFS(L) *where*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$$

are complex fuzzy sets defined over L. *Suppose* $\mathcal{T}^*$ *and* $\acute{\mathcal{S}}$ *are idempotent norms. Then, the following are equivalent:*

1. *The triple* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}})$ *is an element of* CIFNLS(L).
2. *For any* $t, s, l \in [0,1]$*, the subset*

$$\tilde{\mathcal{Q}}^t_{s,l} = \{\breve{\varrho} \in L \mid \tilde{\mathcal{Q}}(\breve{\varrho}) \succeq (t,s,l)\} = \{\breve{\varrho} \in L \mid \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t,\ \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq l\}$$

$$= \Big\{\breve{\varrho} \in L \mid \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t,\ \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \geq t,\ \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s,\ \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s,\ \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) \leq s\Big\}$$

forms an ideal of the Lie algebra L.

Corollary 3.2 *Let* L *be a Lie subalgebra, and suppose*

$$\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \text{NIFS}(L),$$

where the complex fuzzy sets on L *are given by*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}.$$

Assuming $\mathcal{T}^*$ *and* $\acute{\mathcal{S}}$ *are idempotent norms, the following conditions are equivalent:*

1. *The element* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}})$ *belongs to the set* CIFNLS(L).
2. *For all* $t, s, l \in [0,1]$*, the subset*

$$\tilde{\mathcal{Q}}^t_{s,l} = \{\breve{\varrho} \in L \mid \tilde{\mathcal{Q}}(\breve{\varrho}) \succ (t,s,l)\} = \{\breve{\varrho} \in L \mid \mathcal{E}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \Upsilon_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < l\}$$

$$= \Big\{\breve{\varrho} \in L \mid \hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) > t,\ \tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s,\ \dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}) < s\Big\}$$

is an ideal of the Lie algebra L.

Definition 3.3 *Let* L *be a Lie subalgebra, and consider*

$$A = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}), \quad B = (\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}) \in \text{NIFS}(L),$$

where

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}},$$

and

$$\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}} e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}, \quad \Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}} e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}} e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$$

are complex fuzzy sets defined on L. The intersection $A \cap B$ is then defined by assigning, for every $\breve{\varrho} \in L$,

$$(A \cap B)(\breve{\varrho}) = \left(\mathcal{E}_{A\cap B}(\breve{\varrho}),\ \Upsilon_{A\cap B}(\breve{\varrho}),\ \check{\mathcal{G}}_{A\cap B}(\breve{\varrho})\right),$$

where the components are specified accordingly.

$$\begin{aligned}(A \cap B)(\breve{\varrho}) &= \left(\left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \cap \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right)\right)(\breve{\varrho}) = \left(\mathcal{E}_{A\cap B}(\breve{\varrho}), \Upsilon_{A\cap B}(\breve{\varrho}), I_{A\cap B}(\breve{\varrho})\right)\\ &= \left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}, \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}, \left(\dot{\xi}_{\tilde{\mathcal{Q}}} \cap \dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}\right)\end{aligned}$$

such that $\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}} : L \to [0,1]$ and $\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}} : L \to [0,2\pi]$ and $\dot{\xi}_{\tilde{\mathcal{Q}}} \cap \dot{\xi}_{\tilde{\mathcal{D}}} : L \to [0,1]$ and $\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}} : L \to [0,1]$ and $\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}} : L \to [0,2\pi]$ and $\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}} : L \to [0,2\pi]$. Define

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right)$$

and

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \operatorname{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\}$$

and

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right)$$

and

$$\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\}$$

and

$$\left(\dot{\xi}_{\tilde{\mathcal{Q}}} \cap \dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\xi}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right)$$

and

$$\left(\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \operatorname{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\}$$

for all $\breve{\varrho} \in G$.

Proposition 3.3 *Let $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CIFNLS}(L)$ and $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right) \in \mathrm{CIFNLS}(L)$ such that $\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}$, $\Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}$ and $\check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$ and $\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}} e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}$, $\Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}} e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}$ and $\check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}} e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$ be complex fuzzy sets on L. Then $\tilde{\mathcal{Q}} \cap \tilde{\mathcal{D}} \in \mathrm{CIFNLS}(L)$.*

Proof: Let $\breve{\varrho}, \tilde{\varsigma} \in L$ and $\check{\tau} \in F$. Then

(1)

$$\begin{aligned}\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) &= \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho} + \tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho} + \tilde{\varsigma})\right)\\ &\geq \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right)\\ &= \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right)\\ &= \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

Then

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) \geq \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).$$

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) = \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\tau}\breve{\varrho})\right) \geq \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right) = \left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) \quad (2)$$

$$\begin{aligned}\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho}, \tilde{\varsigma}]) &= \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho}, \tilde{\varsigma}]), \hat{\rho}_{\tilde{\mathcal{D}}}([\breve{\varrho}, \tilde{\varsigma}])\right) \quad (3)\\ &\geq \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right)\\ &= \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \hat{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right)\\ &= \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

So,

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho}, \tilde{\varsigma}] \geq \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} \cap \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right)$$

(4)

$$\begin{aligned}\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho}+\tilde{\varsigma})\right\} \\ &\geq \text{Min}\left\{\text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}, \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}\right\} \\ &= \text{Min}\left\{\text{Min}\left\{r_1(\breve{\varrho}), r_2(\breve{\varrho})\right\}, \text{Min}\left\{r_1(\tilde{\varsigma}), r_2(\tilde{\varsigma})\right\}\right\} \\ &= \text{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

Thus,

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \geq \text{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.$$

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) = \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\tau}\breve{\varrho})\right\} \geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\} = \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}). \tag{5}$$

$$\begin{aligned}\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]), \hat{\zeta}_{\tilde{\mathcal{D}}}([\breve{\varrho},\tilde{\varsigma}])\right\} \qquad (6) \\ &\geq \text{Min}\left\{\text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}, \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}\right\} \\ &= \text{Min}\left\{\text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\}, \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}\right\} \\ &= \text{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

Accordingly,

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) \geq \text{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} \cap \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}$$

(5)

$$\begin{aligned}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho}+\tilde{\varsigma})\right) \\ &\leq \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right), \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right) \\ &= \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right), \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right) \\ &= \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\Big).\end{aligned}$$

Consequently,

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\Big).$$

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) = \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\check{\tau}\breve{\varrho})\right) \leq \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right) = \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}). \tag{6}$$

(7)

$$\begin{aligned}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\varrho},\tilde{\varsigma}]), \tilde{\rho}_{\tilde{\mathcal{D}}}([\breve{\varrho},\tilde{\varsigma}])\right) \\ &\leq \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right), \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right) \\ &= \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right), \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right)\right) \\ &= \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right)\end{aligned}$$

As a result,

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\varsigma]] \leq \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} \cap \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right). \tag{8}$$

$$\begin{aligned}\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}+\tilde{\varsigma}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho}+\tilde{\varsigma})\right\} \\ &\leq \text{Max}\left\{\text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma})\right\}, \text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}\right\} \\ &= \text{Max}\left\{\text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\varrho}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\varrho})\right\}, \text{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\varsigma}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\tilde{\varsigma})\right\}\right\} \\ &= \text{Max}\left\{\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\acute{\zeta}_{\tilde{\mathcal{Q}}} \cap \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}\end{aligned}$$

Thus,

$$\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \text{Max}\left\{\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\breve{\varrho}), \left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}. \tag{9}$$

$$\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\check{\tau}\breve{\varrho}) = \text{Max}\left\{\acute{\zeta}_{\tilde{Q}}(\check{\tau}\breve{\varrho}), \acute{\zeta}_{\tilde{D}}(\check{\tau}\breve{\varrho})\right\} \leq \text{Max}\left\{\acute{\zeta}_{\tilde{Q}}(\breve{\varrho}), \acute{\zeta}_{\tilde{D}}(\breve{\varrho})\right\} = \left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\breve{\varrho}). \tag{10}$$

$$\begin{aligned}\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{Q}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \text{Max}\left\{\acute{\zeta}_{\tilde{Q}}([\breve{\varrho},\tilde{\varsigma}]), \acute{\zeta}_{\tilde{D}}([\breve{\varrho},\tilde{\varsigma}])\right\}\\ &\leq \text{Max}\left\{\text{Max}\left\{\acute{\zeta}_{\tilde{Q}}(\breve{\varrho}), \acute{\zeta}_{\tilde{Q}}(\tilde{\varsigma})\right\}, \text{Max}\left\{\acute{\zeta}_{\tilde{D}}(\breve{\varrho}), \acute{\zeta}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\text{Max}\left\{\acute{\zeta}_{\tilde{Q}}(\breve{\varrho}), \acute{\zeta}_{\tilde{D}}(\breve{\varrho})\right\}, \text{Max}\left\{\acute{\zeta}_{\tilde{Q}}(\tilde{\varsigma})\acute{\zeta}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\breve{\varrho}), \left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}\end{aligned}$$

So,

$$\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)([\breve{\varrho},\tilde{\varsigma}]) \leq \text{Max}\left\{\left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\breve{\varrho}), \left(\acute{\zeta}_{\tilde{Q}} \cap \acute{\zeta}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}.$$

(11)

$$\begin{aligned}\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}+\tilde{\varsigma}), \dot{\xi}_{\tilde{D}}(\breve{\varrho}+\tilde{\varsigma})\right)\\ &\leq \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}), \dot{\xi}_{\tilde{Q}}(\tilde{\varsigma})\right), \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{D}}(\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\tilde{\varsigma})\right)\right)\\ &= \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\breve{\varrho})\right), \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\tilde{\varsigma}), \dot{\xi}_{\tilde{D}}(\tilde{\varsigma})\right)\right)\\ &= \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

Thus,

$$\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\tilde{\varsigma})\right).$$

$$\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\check{\tau}\breve{\varrho}) = \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\check{\tau}\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\check{\tau}\breve{\varrho})\right) \leq \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\breve{\varrho})\right) = \left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}). \tag{12}$$

(13)

$$\begin{aligned}\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}([\breve{\varrho},\tilde{\varsigma}]), \dot{\xi}_{\tilde{D}}([\breve{\varrho},\tilde{\varsigma}])\right)\\ &\leq \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}), \dot{\xi}_{\tilde{Q}}(\tilde{\varsigma})\right), \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{D}}(\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\tilde{\varsigma})\right)\right)\\ &= \acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\breve{\varrho}), \dot{\xi}_{\tilde{D}}(\breve{\varrho})\right), \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{Q}}(\tilde{\varsigma}), \dot{\xi}_{\tilde{D}}(\tilde{\varsigma})\right)\right)\\ &= \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

So,

$$\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)([\breve{\varrho},\tilde{\varsigma})] \leq \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\xi}_{\tilde{Q}} \cap \dot{\xi}_{\tilde{D}}\right)(\tilde{\varsigma})\right). \tag{14}$$

$$\begin{aligned}\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}+\tilde{\varsigma}), \dot{\chi}_{\tilde{D}}(\breve{\varrho}+\tilde{\varsigma})\right\}\\ &\leq \text{Max}\left\{\text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}), \dot{\chi}_{\tilde{Q}}(\tilde{\varsigma})\right\}, \text{Max}\left\{\dot{\chi}_{\tilde{D}}(\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\breve{\varrho})\right\}, \text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\tilde{\varsigma}), \dot{\chi}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

Therefore,

$$\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \text{Max}\left\{\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}. \tag{15}$$

$$\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\check{\tau}\breve{\varrho}) = \text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\check{\tau}\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\check{\tau}\breve{\varrho})\right\} \leq \text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\breve{\varrho})\right\} = \left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}). \tag{16}$$

$$\begin{aligned}\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{Q}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \text{Max}\left\{\dot{\chi}_{\tilde{Q}}([\breve{\varrho},\tilde{\varsigma}]), \dot{\chi}_{\tilde{D}}([\breve{\varrho},\tilde{\varsigma}])\right\}\\ &\leq \text{Max}\left\{\text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}), \dot{\chi}_{\tilde{Q}}(\tilde{\varsigma})\right\}, \text{Max}\left\{\dot{\chi}_{\tilde{D}}(\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\breve{\varrho}), \dot{\chi}_{\tilde{D}}(\breve{\varrho})\right\}, \text{Max}\left\{\dot{\chi}_{\tilde{Q}}(\tilde{\varsigma})\dot{\chi}_{\tilde{D}}(\tilde{\varsigma})\right\}\right\}\\ &= \text{Max}\left\{\left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\breve{\varrho}), \left(\dot{\chi}_{\tilde{Q}} \cap \dot{\chi}_{\tilde{D}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

As a results,

$$\left(\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho}, \tilde{\varsigma}]) \leq \operatorname{Max}\left\{\left(\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\dot{\chi}_{\tilde{\mathcal{Q}}} \cap \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.$$

(1)–(16) show that $A \cap B \in \mathrm{CIFNLS}(L)$. Using a method akin to the proof of Proposition 3.3, we obtain the following result. □

Proposition 3.4 *Let* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CIFNI}(L)$ *and* $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right) \in \mathrm{CIFNI}(L)$ *such that* $\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}$, $\Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$ *and* $\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}} e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}$, $\Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}} e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}} e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$ *be complex fuzzy sets on* L. *Then* $\tilde{\mathcal{Q}} \cap \tilde{\mathcal{D}} \in \mathrm{CIFNI}(L)$.

Definition 3.4 *Let* L *be a Lie subalgebra, and consider two elements* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right)$ *and* $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right)$ *from* $\mathrm{NIFS}(L)$, *where*

$$\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}},$$

and

$$\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}} e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}, \quad \Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}} e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}, \quad \check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}} e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$$

are complex fuzzy sets on L. *We define the sum* $\tilde{\mathcal{Q}} + \tilde{\mathcal{D}}$ *for each* $\breve{\varrho} \in L$ *as follows:*

$$\begin{aligned}(\tilde{\mathcal{Q}} + \tilde{\mathcal{D}})(\breve{\varrho}) &= \left(\left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) + \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right)\right)(\breve{\varrho}) = \left(\mathcal{E}_{\tilde{\mathcal{Q}}+\tilde{\mathcal{D}}}(\breve{\varrho}), \Upsilon_{\tilde{\mathcal{Q}}+\tilde{\mathcal{D}}}(\breve{\varrho}), I_{\tilde{\mathcal{Q}}+\tilde{\mathcal{D}}}(\breve{\varrho})\right) \\ &= \left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}, \left(\tilde{\rho}_{\tilde{\mathcal{Q}}} + \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} + \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}, \left(\dot{\xi}_{\tilde{\mathcal{Q}}} + \dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) e^{i\left(\dot{\chi}_{\tilde{\mathcal{Q}}} + \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})}\right)\end{aligned}$$

where the functions

$$\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}} : L \to [0,1], \quad \hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}} : L \to [0, 2\pi], \quad \check{\mathcal{G}}_{\tilde{\mathcal{Q}}} + \check{\mathcal{G}}_{\tilde{\mathcal{D}}} : L \to [0,1],$$

and

$$\tilde{\rho}_{\tilde{\mathcal{Q}}} + \tilde{\rho}_{\tilde{\mathcal{D}}} : L \to [0,1], \quad \acute{\zeta}_{\tilde{\mathcal{Q}}} + \acute{\zeta}_{\tilde{\mathcal{D}}} : L \to [0, 2\pi]$$

are defined accordingly. $\dot{\chi}_{\tilde{\mathcal{Q}}} + \dot{\chi}_{\tilde{\mathcal{D}}} : L \to [0, 2\pi]$. *Define*

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \sup_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right)$$

and

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \mathrm{Min}_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right\}$$

and

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}} + \tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \inf_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2} \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \tilde{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right)$$

and

$$\left(\acute{\zeta}_{\tilde{\mathcal{Q}}} + \acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \mathrm{Max}_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right\}$$

and

$$\left(\dot{\xi}_{\tilde{\mathcal{Q}}} + \dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \inf_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2} \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right)$$

and

$$\left(\dot{\chi}_{\tilde{\mathcal{Q}}} + \dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}) = \mathrm{Max}_{\breve{\varrho} = \check{\mathfrak{d}}_1 + \check{\mathfrak{d}}_2}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \dot{\chi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right\}$$

for all $\breve{\varrho} \in L$.

Proposition 3.5 *Let* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CIFNLS}(L)$ *and* $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}}, \Upsilon_{\tilde{\mathcal{D}}}, \check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right) \in \mathrm{CIFNLS}(L)$ *such that* $\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}} e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}$, $\Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}} e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}} e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$ *and* $\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}} e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}$, $\Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}} e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}} e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$ *be complex fuzzy sets on* L. *Then* $\tilde{\mathcal{Q}} + \tilde{\mathcal{D}} \in \mathrm{CIFNLS}(L)$.

Proof: Let $\breve{\varrho}, \tilde{\varsigma}, \check{\mathfrak{d}}_1, \check{\mathfrak{d}}_2, \check{\mathfrak{d}}_3, \check{\mathfrak{d}}_4 \in L$ and $\check{\tau} \in F$.

$$\begin{aligned}\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) &= \sup_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4)\right) \\ &\geq \sup_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4} \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \sup_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4} \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \mathcal{T}^*\left(\sup_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right), \sup_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned} \tag{1}$$

Then,

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) \geq \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).$$

(2)

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) = \sup_{\check{\tau}\breve{\varrho}=\check{\tau}\check{\mathfrak{d}}_1+\check{\tau}\check{\mathfrak{d}}_2} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\check{\mathfrak{d}}_1), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\tau}\check{\mathfrak{d}}_2)\right) \geq \sup_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right) = \left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}).$$

(3)

$$\begin{aligned}\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho}, \tilde{\varsigma}]) &= \sup_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}([\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]), \hat{\rho}_{\tilde{\mathcal{D}}}([\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4])\right) \\ &\geq \sup_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]} \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \sup_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4]} \mathcal{T}^*\left(\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right), \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \mathcal{T}^*\left(\sup_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right), \sup_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4} \mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right) \\ &= \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

So,

$$\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho}, \tilde{\varsigma}]) \geq \mathcal{T}^*\left(\left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\rho}_{\tilde{\mathcal{Q}}} + \hat{\rho}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).$$

(4)

$$\begin{aligned}\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) &= \mathrm{Min}_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4)\right\} \\ &\geq \mathrm{Min}_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4}\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right\}, \mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\} \\ &= \mathrm{Min}_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right\}, \mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\} \\ &= \mathrm{Min}\left\{\mathrm{Min}_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3}\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right\}, \mathrm{Min}_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2), \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\} \\ &= \mathrm{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

Then,

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho} + \tilde{\varsigma}) \geq \mathrm{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}), \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}. \tag{5}$$

$$\begin{aligned}\left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) &= \mathrm{Min}_{\check{\tau}\breve{\varrho}=\check{\tau}\check{\mathfrak{d}}_1+\check{\tau}\check{\mathfrak{d}}_2}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\check{\mathfrak{d}}_1), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\tau}\check{\mathfrak{d}}_2)\right\} \\ &\geq \mathrm{Min}_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}), \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right\} \\ &= \left(\hat{\zeta}_{\tilde{\mathcal{Q}}} + \hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})\end{aligned}$$

(6)

$$\begin{aligned}\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\breve{\varsigma}]) &= \mathrm{Min}_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}([\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]),\hat{\zeta}_{\tilde{\mathcal{D}}}([\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4])\right\}\\ &\geq \mathrm{Min}_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right\},\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Min}_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4]}\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Min}\left\{\mathrm{Min}_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Min}_{\breve{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\hat{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right\}.\end{aligned}$$

Thus,

$$\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\breve{\varsigma}) \geq \mathrm{Min}\left\{\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}+\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right\}.$$

(7)

$$\begin{aligned}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\breve{\varsigma}) &= \inf_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4)\right)\\ &\leq \inf_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(S\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right),\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right)\Big)\\ &= \inf_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right),\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right)\right)\\ &= \acute{\mathcal{S}}\left(\inf_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right),\inf_{\breve{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}S\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\Big)\right)\\ &= \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right).\end{aligned}$$

Then,

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\breve{\varsigma}) \leq \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right).$$

(8)

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\tau}\breve{\varrho}) = \inf_{\breve{\tau}\breve{\varrho}=\breve{\tau}\breve{\mathfrak{d}}_1+\breve{\tau}\breve{\mathfrak{d}}_2}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\tau}\breve{\mathfrak{d}}_1),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\tau}\breve{\mathfrak{d}}_2)\right) \leq \inf_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_2)\right) = \left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})$$

(9)

$$\begin{aligned}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\breve{\varsigma}]) &= \inf_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}([\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]),\tilde{\rho}_{\tilde{\mathcal{D}}}([\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4])\right)\\ &\leq \inf_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}})\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right),\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right)\right)\\ &= \inf_{[\breve{\varrho},\breve{\varsigma}]=[\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right),\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right)\right)\\ &= \acute{\mathcal{S}}\left(\inf_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right),\inf_{\breve{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}S\tilde{\rho}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\tilde{\rho}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\Big)\right)\\ &= \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right).\end{aligned}$$

So,

$$\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\breve{\varsigma}]) \leq \acute{\mathcal{S}}\left(\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}+\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right).$$

(10)

$$\begin{aligned}\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\breve{\varsigma}) &= \mathrm{Max}_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4)\right\}\\ &\leq \mathrm{Max}_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\left\{\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right\},\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Max}_{\breve{\varrho}+\breve{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Max}\left\{\mathrm{Min}_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}_{\breve{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\acute{\zeta}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\ &= \mathrm{Max}\left\{\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varsigma})\right\}.\end{aligned}$$

Then

$$\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \operatorname{Max}\left\{\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.$$

$$\begin{aligned}\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) &= \operatorname{Max}_{\check{\tau}\breve{\varrho}=\check{\tau}\check{\mathfrak{d}}_1+\check{\tau}\check{\mathfrak{d}}_2}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\check{\mathfrak{d}}_1),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\tau}\check{\mathfrak{d}}_2)\right\}\\ &\leq \operatorname{Max}_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right\}\\ &= \left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho})\end{aligned}$$

(11)

$$\begin{aligned}\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \operatorname{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}([\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]),\acute{\zeta}_{\tilde{\mathcal{D}}}([\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4])\right\}\\ &\leq \operatorname{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]}\left\{\operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right\},\operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\}\\ &= \operatorname{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4]}\left\{\operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right\},\operatorname{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\}\\ &= \operatorname{Max}\left\{\operatorname{Max}_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right\},\operatorname{Max}_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right\}\right\}\\ &= \operatorname{Max}\left\{\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.\end{aligned}$$

Thus,

$$\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \operatorname{Max}\left\{\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}+\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.$$

(12)

$$\begin{aligned}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \inf_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4)\right)\\ &\leq \inf_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(S\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right),\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\Big)\\ &= \inf_{\breve{\varrho}+\tilde{\varsigma}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3+\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right),\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right)\\ &= \acute{\mathcal{S}}\left(\inf_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right),\inf_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}S\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\Big)\right)\\ &= \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

Then,

$$\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).$$

(13)

$$\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) = \inf_{\check{\tau}\breve{\varrho}=\check{\tau}\check{\mathfrak{d}}_1+\check{\tau}\check{\mathfrak{d}}_2}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\tau}\check{\mathfrak{d}}_1),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\tau}\check{\mathfrak{d}}_2)\right) \leq \inf_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_2}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_2)\right) = \left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}).$$

(14)

$$\begin{aligned}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \inf_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}([\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]),\dot{\xi}_{\tilde{\mathcal{D}}}([\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4])\right)\\ &\leq \inf_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1,\check{\mathfrak{d}}_2]+[\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}})\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2)\right),\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right)\\ &= \inf_{[\breve{\varrho},\tilde{\varsigma}]=[\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3,\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4]}\acute{\mathcal{S}}\left(\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right),\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\right)\right)\\ &= \acute{\mathcal{S}}\left(\inf_{\breve{\varrho}=\check{\mathfrak{d}}_1+\check{\mathfrak{d}}_3}\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_3)\right),\inf_{\tilde{\varsigma}=\check{\mathfrak{d}}_2+\check{\mathfrak{d}}_4}S\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\mathfrak{d}}_2),\dot{\xi}_{\tilde{\mathcal{D}}}(\check{\mathfrak{d}}_4)\Big)\right)\\ &= \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).\end{aligned}$$

So,

$$\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) \leq \acute{\mathcal{S}}\left(\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\xi}_{\tilde{\mathcal{Q}}}+\dot{\xi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right).$$

$$\begin{aligned}
\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) &= \mathrm{Max}_{\breve{\varrho}+\tilde{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4)\right\}\\
&\leq \mathrm{Max}_{\breve{\varrho}+\tilde{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_4}\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right\},\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}_{\breve{\varrho}+\tilde{\varsigma}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3+\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}\left\{\mathrm{Min}_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}_{\tilde{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}\left\{\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.
\end{aligned}$$

Then,

$$\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \mathrm{Max}\left\{\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}. \tag{15}$$

$$\begin{aligned}
\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\breve{\varrho}) &= \mathrm{Max}_{\check{\tau}\breve{\varrho}=\check{\tau}\breve{\mathfrak{d}}_1+\check{\tau}\breve{\mathfrak{d}}_2}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\check{\tau}\breve{\mathfrak{d}}_1),\dot{\chi}_{\tilde{\mathcal{D}}}(\check{\tau}\breve{\mathfrak{d}}_2)\right\}\\
&\leq \mathrm{Max}_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_2}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_2)\right\}\\
&= \left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}).
\end{aligned}$$

(16)

$$\begin{aligned}
\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)([\breve{\varrho},\tilde{\varsigma}]) &= \mathrm{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}([\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]),\dot{\chi}_{\tilde{\mathcal{D}}}([\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4])\right\}\\
&\leq \mathrm{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\breve{\mathfrak{d}}_1,\breve{\mathfrak{d}}_2]+[\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_4]}\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2)\right\},\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}_{[\breve{\varrho},\tilde{\varsigma}]=[\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3,\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4]}\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}\left\{\mathrm{Max}_{\breve{\varrho}=\breve{\mathfrak{d}}_1+\breve{\mathfrak{d}}_3}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{Q}}),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_3)\right\},\mathrm{Max}_{\tilde{\varsigma}=\breve{\mathfrak{d}}_2+\breve{\mathfrak{d}}_4}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\breve{\mathfrak{d}}_2),\dot{\chi}_{\tilde{\mathcal{D}}}(\breve{\mathfrak{d}}_4)\right\}\right\}\\
&= \mathrm{Max}\left\{\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.
\end{aligned}$$

Thus,

$$\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}+\tilde{\varsigma}) \leq \mathrm{Max}\left\{\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\breve{\varrho}),\left(\dot{\chi}_{\tilde{\mathcal{Q}}}+\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\varsigma})\right\}.$$

Therefore, $\tilde{\mathcal{Q}}+\tilde{\mathcal{D}} \in \mathrm{CIFNLS}(L)$. □

As with Proposition 3.5, the proof of the next result is straightforward.

Proposition 3.6 *Let* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}},\Upsilon_{\tilde{\mathcal{Q}}},\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CIFNI}(L)$ *and* $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}},\Upsilon_{\tilde{\mathcal{D}}},\check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right) \in \mathrm{CIFNI}(L)$ *such that* $\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}}e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}$, $\Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}}e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}}e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$ *and* $\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}}e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}$, $\Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}}e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}}e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$ *be complex fuzzy sets on L. Then* $\tilde{\mathcal{Q}}+\tilde{\mathcal{D}} \in \mathrm{CIFNI}(L)$.

Definition 3.5 *Let* L_1, L_2 *be Lie subalgebras and* $\aleph : L_1 \to L_2$ *be a homomorphism of Lie subalgebras. Let* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}},\Upsilon_{\tilde{\mathcal{Q}}},\check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in NIFS(L_1)$ *and* $\tilde{\mathcal{D}} = \left(\mathcal{E}_{\tilde{\mathcal{D}}},\Upsilon_{\tilde{\mathcal{D}}},\check{\mathcal{G}}_{\tilde{\mathcal{D}}}\right) \in NIFS(L_2)$ *such that* $\mathcal{E}_{\tilde{\mathcal{Q}}} = \hat{\rho}_{\tilde{\mathcal{Q}}}e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}$, $\Upsilon_{\tilde{\mathcal{Q}}} = \tilde{\rho}_{\tilde{\mathcal{Q}}}e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{Q}}} = \dot{\xi}_{\tilde{\mathcal{Q}}}e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}$ *be complex fuzzy sets on* L_1 *and* $\mathcal{E}_{\tilde{\mathcal{D}}} = \hat{\rho}_{\tilde{\mathcal{D}}}e^{i\hat{\zeta}_{\tilde{\mathcal{D}}}}$, $\Upsilon_{\tilde{\mathcal{D}}} = \tilde{\rho}_{\tilde{\mathcal{D}}}e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}$, $\check{\mathcal{G}}_{\tilde{\mathcal{D}}} = \dot{\xi}_{\tilde{\mathcal{D}}}e^{i\dot{\chi}_{\tilde{\mathcal{D}}}}$ *be complex fuzzy sets on* L_2. *Define*

$$\begin{aligned}
\aleph(\tilde{\mathcal{Q}}) = \left(\aleph(\mathcal{E}_{\tilde{\mathcal{Q}}}),\aleph(\Upsilon_{\tilde{\mathcal{Q}}}),\aleph(\check{\mathcal{G}}_{\tilde{\mathcal{Q}}})\right) &= \left(\aleph\left(\hat{\rho}_{\tilde{\mathcal{Q}}}e^{i\hat{\zeta}_{\tilde{\mathcal{Q}}}}\right),\aleph\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}e^{i\acute{\zeta}_{\tilde{\mathcal{Q}}}}\right),\aleph\left(\dot{\xi}_{\tilde{\mathcal{Q}}}e^{i\dot{\chi}_{\tilde{\mathcal{Q}}}}\right)\right)\\
&= \left(\aleph\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)e^{i\aleph(\hat{\zeta}_{\tilde{\mathcal{Q}}})},\aleph\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)e^{i\aleph(\acute{\zeta}_{\tilde{\mathcal{Q}}})},\aleph\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)e^{i\aleph(\dot{\chi}_{\tilde{\mathcal{Q}}})}\right)
\end{aligned}$$

and

$$\begin{aligned}
\aleph^{-1}(\breve{\mathfrak{d}}_2) &= \left(\aleph^{-1}(\mathcal{E}_{\tilde{\mathcal{D}}}),\aleph^{-1}(\Upsilon_{\tilde{\mathcal{D}}}),\aleph^{-1}(\check{\mathcal{G}}_{\tilde{\mathcal{D}}})\right)\\
&= \left(\aleph^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}}e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}),\aleph^{-1}(\tilde{\rho}_{\tilde{\mathcal{D}}}e^{i\acute{\zeta}_{\tilde{\mathcal{D}}}}),\aleph^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}}e^{i\dot{\chi}_{\tilde{\mathcal{D}}}})\right)\\
&= \left(\aleph^{-1}\left(\hat{\rho}_{\tilde{\mathcal{D}}}\right)e^{i\aleph^{-1}(\hat{\zeta}_{\tilde{\mathcal{Q}}})},\aleph^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)e^{i\aleph^{-1}(\acute{\zeta}_{\tilde{\mathcal{D}}})},\aleph^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})e^{i\aleph^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})}\right).
\end{aligned}$$

Now for all $h \in L_2$ *and* $g \in L_1$ *define*
$\aleph\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,1]$ *as*

$$\aleph\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)(h) = \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,1]$ *as*

$$\aleph\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)(h) = \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,1]$ *as*

$$\aleph\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)(h) = \inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,2\pi]$ *as*

$$\aleph\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)(h) = \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,2\pi]$ *as*

$$\aleph\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)(h) = \inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right) : L_2 \to [0,2\pi]$ *as*

$$\aleph\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)(h) = \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(g) \mid g \in L_1, \aleph(g) = h\right\}$$

$\aleph^{-1}\left(\hat{\rho}_{\tilde{\mathcal{D}}}\right) : L_1 \to [0,1]$ *as*

$$\aleph^{-1}\left(\hat{\rho}_{\tilde{\mathcal{D}}}\right)(g) = \hat{\rho}_{\tilde{\mathcal{D}}}(\aleph(g))$$

$\aleph^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right) : L_1 \to [0,1]$ *as*

$$\aleph^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(g) = \tilde{\rho}_{\tilde{\mathcal{D}}}(\aleph(g))$$

$\aleph^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}}) : L_1 \to [0,1]$ *as*

$$\aleph^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(g) = \dot{\xi}_{\tilde{\mathcal{D}}}(\aleph(g))$$

$\aleph^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right) : L_1 \to [0,2\pi]$ *as*

$$\aleph^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(g) = \hat{\zeta}_{\tilde{\mathcal{D}}}(\aleph(g))$$

$\aleph^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right) : L_1 \to [0,2\pi]$ *as*

$$\aleph^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(g) = \acute{\zeta}_{\tilde{\mathcal{D}}}(\aleph(g))$$

$\aleph^{-1}\left(\dot{\chi}_{\tilde{\mathcal{D}}}\right) : L_1 \to [0,2\pi]$ *as*

$$\aleph^{-1}\left(\dot{\chi}_{\tilde{\mathcal{D}}}\right)(g) = \dot{\chi}_{\tilde{\mathcal{D}}}(\aleph(g)).$$

Proposition 3.7 *Let* $\mho : L_1 \to L_2$ *be an epimorphism of Lie algebras and* $\tilde{\mathcal{Q}} = \left(\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}\right) \in \mathrm{CNIFS}(L_1)$. *Then* $\mho(\tilde{\mathcal{Q}}) \in \mathrm{CNIFS}(L_2)$.

Proof: Let $\tilde{\mathbb{O}}_1, \tilde{\mathbb{O}}_2 \in L_2$ and $\ell^*, \ell^{**} \in L_1$ such that $\tilde{\mathbb{O}}_1 = \mho(\ell^*)$ and $\tilde{\mathbb{O}}_2 = \mho(\ell^{**})$. Then

$$\begin{aligned}
\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1 + \tilde{\mathbb{O}}_2\right) &= \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}\left(\ell^* + \ell^{**}\right) \mid \ell^*, \ell^{**} \in L_1, \mho\left(\ell^* + \ell^{**}\right) = \tilde{\mathbb{O}}_1 + \tilde{\mathbb{O}}_2\right\} \qquad (1)\\
&\geq \sup\left\{\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^*), \hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**})\right) \mid \ell^*, \ell^{**} \in L_1, \mho(\ell^*) = \tilde{\mathbb{O}}_1, \mho(\ell^{**}) = \tilde{\mathbb{O}}_2\right\}\\
&= \mathcal{T}^*\left(\sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^* \in L, \mho(\ell^*) = \tilde{\mathbb{O}}_1\right\}, \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**} \in L_1, \mho(\ell^{**}) = \tilde{\mathbb{O}}_2\right\}\right)\\
&= \mathcal{T}^*\left(\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).
\end{aligned}$$

So,

$$\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1 + \tilde{\mathbb{O}}_2\right) \geq \mathcal{T}^*\left(\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right). \qquad (2)$$

$$\begin{aligned}
\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) &= \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&\geq \sup\left\{\mathcal{T}^*\left(\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^*),\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**})\right) \mid \ell^*,\ell^{**}\in L_1, \left([\mho(\ell^*),\mho(\ell^{**})]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&= \mathcal{T}^*\left(\sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\
&= \mathcal{T}^*\left(\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).
\end{aligned}$$

Then,

$$\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) \geq \mathcal{T}^*\left(\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right) \tag{3}$$

$$\begin{aligned}
\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) &= \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}\left(\ell^*+\ell^{**}\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left(\ell^*+\ell^{**}\right)=\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right\}\\
&\geq \sup\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*),\hat{\zeta}_L(\ell^{**})\right\} \mid \ell^*,\ell^{**}\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\\
&= \mathrm{Min}\left\{\sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^{**})\right\} \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\
&= \mathrm{Min}\left\{\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.
\end{aligned}$$

Thus,

$$\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) \geq \mathrm{Min}\left\{\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}$$

$$\begin{aligned}
\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) &= \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&\geq \sup\left\{\mathrm{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*),\hat{\zeta}_L(\ell^{**})\right\} \mid \ell^*,\ell^{**}\in L_1, \left([\mho(\ell^*),\mho(\ell^{**})]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&= \mathrm{Min}\left\{\sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right\}\\
&= \mathrm{Min}\left\{\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.
\end{aligned} \tag{4}$$

Then,

$$\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) \geq \mathrm{Min}\left\{\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.$$

(5)

$$\begin{aligned}
\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) &= \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}\left(\ell^*+\ell^{**}\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left(\ell^*+\ell^{**}\right)=\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right\}\\
&\leq \inf\left\{\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^*),\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**})\right) \mid \ell^*,\ell^{**}\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\\
&= \acute{\mathcal{S}}\left(\inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\
&= \acute{\mathcal{S}}\left(\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).
\end{aligned}$$

Thus,

$$\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) \leq \acute{\mathcal{S}}\left(\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).$$

(6)

$$\begin{aligned}
\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) &= \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&\leq \inf\left\{\acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^*),\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**})\right) \mid \ell^*,\ell^{**}\in L_1, \left([\mho(\ell^*),\mho(\ell^{**})]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\
&= \acute{\mathcal{S}}\left(\inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\
&= \acute{\mathcal{S}}\left(\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\hat{\rho}_L\right)\left(\tilde{\mathbb{O}}_2\right)\right).
\end{aligned}$$

Then,

$$\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right)\leq\acute{\mathcal{S}}\left(\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right)$$

(7)

$$\begin{aligned}\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right)&=\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}\left(\ell^*+\ell^{**}\right)\mid\ell^*,\ell^{**}\in L_1,\mho\left(\ell^*+\ell^{**}\right)=\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right\}\\&\leq\inf\left\{\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*),\hat{\zeta}_L(\ell^{**})\right\}\mid\ell^*,\ell^{**}\in L_1,\mho(\ell^*)=\tilde{\mathbb{O}}_1,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\\&=\mathrm{Max}\left\{\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*)\mid\ell^*\in L_1,\mho(\ell^*)=\tilde{\mathbb{O}}_1\right\},\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^{**})\right\}\mid\ell^{**}\in L_1,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\Big)\\&=\mathrm{Max}\left\{\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.\end{aligned}$$

Then,

$$\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right)\leq\mathrm{Max}\left\{\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}. \tag{8}$$

$$\begin{aligned}\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right)&=\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right)\mid\ell^*,\ell^{**}\in L_1,\mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\&\leq\inf\left\{\mathrm{Max}\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*),\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^{**})\right\}\mid\ell^*,\ell^{**}\in L_1,([\mho(\ell^*),\mho(\ell^{**})])=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\&=\mathrm{Max}\left\{\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^*)\mid\ell^*\in L,\mho(\ell^*)=\tilde{\mathbb{O}}_1\right\},\inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\ell^{**})\mid\ell^{**}\in L_1,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right\}\\&=\mathrm{Max}\left\{\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.\end{aligned}$$

Then,

$$\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right)\leq\mathrm{Max}\left\{\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.$$

(9)

$$\begin{aligned}\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right)&=\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}\left(\ell^*+\ell^{**}\right)\mid\ell^*,\ell^{**}\in L_1,\mho\left(\ell^*+\ell^{**}\right)=\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right\}\\&\leq\inf\left\{\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^*),\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^{**})\right)\mid\ell^*,\ell^{**}\in L_1,\mho(\ell^*)=\tilde{\mathbb{O}}_1,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\\&=\acute{\mathcal{S}}\left(\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^*)\mid\ell^*\in L_1,\mho(\ell^*)=\tilde{\mathbb{O}}_1\right\},\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^{**})\mid\ell^{**}\in\ell^*,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\&=\acute{\mathcal{S}}\left(\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).\end{aligned}$$

Thus,

$$\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right)\leq\acute{\mathcal{S}}\left(\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right).$$

(10)

$$\begin{aligned}\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right)&=\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right)\mid\ell^*,\ell^{**}\in L_1,\mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\&\leq\inf\left\{\acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^*),\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^{**})\right)\mid\ell^*,\ell^{**}\in L_1,([\mho(\ell^*),\mho(\ell^{**})])=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\}\\&=\acute{\mathcal{S}}\left(\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^*)\mid\ell^*\in L_1,\mho(\ell^*)=\tilde{\mathbb{O}}_1\right\},\inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\ell^{**})\mid\ell^{**}\in L_1,\mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right)\\&=\acute{\mathcal{S}}\left(\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\hat{\rho}_L\right)\left(\tilde{\mathbb{O}}_2\right)\right).\end{aligned}$$

Then,

$$\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right)\leq\acute{\mathcal{S}}\left(\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right),\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right)$$

(11)

$$\begin{aligned}
\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) &= \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}\left(\ell^*+\ell^{**}\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left(\ell^*+\ell^{**}\right)=\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right\} \\
&\leq \inf\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^*), \dot{\xi}_L(\ell^{**})\right\} \mid \ell^*,\ell^{**}\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\} \\
&= \mathrm{Max}\left\{\inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L_1, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^{**})\right\} \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\Big) \\
&= \mathrm{Max}\left\{\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.
\end{aligned}$$

Then,

$$\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1+\tilde{\mathbb{O}}_2\right) \leq \mathrm{Max}\left\{\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.$$

(12)

$$\begin{aligned}
\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) &= \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}\left([\ell^*,\ell^{**}]\right) \mid \ell^*,\ell^{**}\in L_1, \mho\left([\ell^*,\ell^{**}]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\} \\
&\leq \inf\left\{\mathrm{Max}\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^*), \dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^{**})\right\} \mid \ell^*,\ell^{**}\in L_1, \left([\mho(\ell^*),\mho(\ell^{**})]\right)=\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right\} \\
&= \mathrm{Max}\left\{\inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^*) \mid \ell^*\in L, \mho(\ell^*)=\tilde{\mathbb{O}}_1\right\}, \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\ell^{**}) \mid \ell^{**}\in L_1, \mho(\ell^{**})=\tilde{\mathbb{O}}_2\right\}\right\} \\
&= \mathrm{Max}\left\{\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}.
\end{aligned}$$

Then,

$$\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\left[\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\right]\right) \leq \mathrm{Max}\left\{\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_1\right), \mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)\left(\tilde{\mathbb{O}}_2\right)\right\}$$

Let $\tilde{\mathbb{O}}_1,\tilde{\mathbb{O}}_2\in L_2$ and $\ell^*,\ell^{**}\in L_1$ such that $\tilde{\mathbb{O}}_1=\mho(\ell^*)$ and $\tilde{\mathbb{O}}_2=\mho(\ell^{**})$. Then

$$\begin{aligned}
\mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}_{)} \mid \tilde{\ell}\in L_1, \mho(\check{\tau}\tilde{\ell}_{)}=\check{\tau}\tilde{\mathbb{O}}\right\} \\
&\geq \sup\left\{\hat{\rho}_{\tilde{\mathcal{Q}}}\tilde{\ell}_{|}\tilde{\ell}\in L_1, \mho(\tilde{\ell})=\tilde{\mathbb{O}}\right\} \\
&= \mho\left(\hat{\rho}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}})
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}_{)} \mid \tilde{\ell}\in L_1, \mho(\check{\tau}\tilde{\ell}_{)}=\check{\tau}\tilde{\mathbb{O}}\right\} \\
&\geq \sup\left\{\hat{\zeta}_{\tilde{\mathcal{Q}}}\tilde{\ell}_{|}\tilde{\ell}\in L_1, \mho(\tilde{\ell})=\tilde{\mathbb{O}}\right\} \\
&= \mho\left(\hat{\zeta}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}})
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}_{)} \mid \tilde{\ell}\in L_1, \mho(\check{\tau}\tilde{\ell}_{)}=\check{\tau}\tilde{\mathbb{O}}\right\} \\
&\leq \inf\left\{\tilde{\rho}_{\tilde{\mathcal{Q}}}\tilde{\ell}_{|}\tilde{\ell}\in L_1, \mho(\tilde{\ell})=\tilde{\mathbb{O}}\right\} \\
&= \mho\left(\tilde{\rho}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}})
\end{aligned} \tag{15}$$

$$\begin{aligned}
\mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}_{)} \mid \tilde{\ell}\in L_1, \mho(\check{\tau}\tilde{\ell}_{)}=\check{\tau}\tilde{\mathbb{O}}\right\} \\
&\leq \inf\left\{\acute{\zeta}_{\tilde{\mathcal{Q}}}(\tilde{\ell}) \mid \tilde{\ell}\in L_1, \mho(\tilde{\ell})=\tilde{\mathbb{O}}\right\} \\
&= \mho\left(\acute{\zeta}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}})
\end{aligned}$$

$$\begin{aligned}\mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}) \mid \tilde{\ell} \in L_1, \mho(\check{\tau}\tilde{\ell}) = \check{\tau}\tilde{\mathbb{O}}\right\}\\ &\leq \inf\left\{\dot{\xi}_{\tilde{\mathcal{Q}}}(\tilde{\ell}) \mid \tilde{\ell} \in L_1, \mho(\tilde{\ell}) = \tilde{\mathbb{O}}\right\}\\ &= \mho\left(\dot{\xi}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}}) \end{aligned} \tag{16}$$

(17)

$$\begin{aligned}\mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)(\check{\tau}\tilde{\mathbb{O}}) &= \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}(\check{\tau}\tilde{\ell}) \mid \tilde{\ell} \in L_1, \mho(\check{\tau}\tilde{\ell}) = \check{\tau}\tilde{\mathbb{O}}\right\}\\ &\leq \inf\left\{\dot{\chi}_{\tilde{\mathcal{Q}}}\tilde{\ell}|\tilde{\ell} \in L_1, \mho(\tilde{\ell}) = \tilde{\mathbb{O}}\right\}\\ &= \mho\left(\dot{\chi}_{\tilde{\mathcal{Q}}}\right)(\tilde{\mathbb{O}}).\end{aligned}$$

Hence, following the arguments presented in (1)–(17), it follows that $\mho(\tilde{\mathcal{Q}}) \in CIFN(L_2)$. □

Its validity can be seen without difficulty.

Proposition 3.8 *Suppose* $\mho : L_1 \to L_2$ *is a surjective Lie algebra homomorphism and* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \text{CNIFI}(L_1)$. *Then the image* $\mho(\tilde{\mathcal{Q}})$ *belongs to* CIFNI(L_2).

Proposition 3.9 *Suppose* $\mho : L_1 \to L_2$ *is a surjective Lie algebra homomorphism and* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \text{CNIFS}(L_1)$. *Then the image* $\mho(\tilde{\mathcal{Q}})$ *belongs to* CNIFS(L_2).

Proof: Let $\ell^*, \ell^{**} \in L_1$. Then
(1)

$$\begin{aligned}\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^* + \ell^{**}) &= \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^* + \ell^{**}))\\ &= \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^*) + \mho(\ell^{**}))\\ &= \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^*) + \mho(\ell^{**}))\\ &\geq \mathcal{T}^*(r_B(\mho(\ell^*)), \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^{**})))\\ &= \mathcal{T}^*\left(\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^{**})\right).\end{aligned}$$

So,

$$\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^* + \ell^{**}) \geq \mathcal{T}^*\left(\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^{**})\right).$$

(2)

$$\begin{aligned}\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})([\ell^*, \ell^{**}]) &= \hat{\rho}_{\tilde{\mathcal{D}}}(\mho([\ell^*, \ell^{**}]))\\ &= \hat{\rho}_{\tilde{\mathcal{D}}}([\mho(\ell^*), \mho(\ell^{**})])\\ &\geq \mathcal{T}^*(\hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^{**})))\\ &= \mathcal{T}^*\left(\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^{**})\right).\end{aligned}$$

Then,

$$\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})([\ell^*, \ell^{**}]) \geq \mathcal{T}^*\left(\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\ell^{**})\right).$$

(3)

$$\begin{aligned}\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^* + \ell^{**}) &= \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\ell^* + \ell^{**}))\\ &= \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\ell^*) + \mho(\ell^{**}))\\ &= \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\ell^*) + \mho(\ell^{**}))\\ &\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\ell^{**}))\right\}\\ &= \text{Min}\left\{\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^*), \mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.\end{aligned}$$

Thus,

$$\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}+\ell^{**}) \geq \text{Min}\left\{\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.$$

(4)

$$\begin{aligned}\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)([\ell^{*},\ell^{**}]) &= \hat{\zeta}_{\tilde{\mathcal{D}}}\left(\mho\left([\ell^{*},\ell^{**}]\right)\right)\\ &= \hat{\zeta}_{\tilde{\mathcal{D}}}\left([\mho(\ell^{*}),\mho(\ell^{**})]\right)\\ &\geq \text{Min}\left\{\hat{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})\right), \hat{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{**})\right)\right\}\\ &= \text{Min}\left\{\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.\end{aligned}$$

Then,

$$\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)([\ell^{*},\ell^{**}]) \geq \text{Min}\left\{\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.$$

(5)

$$\begin{aligned}\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}+\ell^{**}) &= \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho\left(\ell^{*}+\ell^{**}\right)\right)\\ &= \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})+\mho(\ell^{**})\right)\\ &= \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})+\mho(\ell^{**})\right)\\ &\leq \acute{\mathcal{S}}\left(\tilde{\rho}_{B}\left(\mho(\ell^{*})\right), \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{**})\right)\right)\\ &= \acute{\mathcal{S}}\left(\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right).\end{aligned}$$

So,

$$\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}+\ell^{**}) \leq \acute{\mathcal{S}}\left(\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right).$$

(6)

$$\begin{aligned}\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\ell^{*},\ell^{**}]) &= \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho\left([\ell^{*},\ell^{**}]\right)\right)\\ &= \tilde{\rho}_{\tilde{\mathcal{D}}}\left([\mho(\ell^{*}),\mho(\ell^{**})]\right)\\ &\leq \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})\right), \tilde{\rho}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{**})\right)\right)\\ &= \acute{\mathcal{S}}\left(\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right).\end{aligned}$$

Thus,

$$\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)([\ell^{*},\ell^{**}]) \leq \acute{\mathcal{S}}\left(\mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right).$$

(7)

$$\begin{aligned}\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}+\ell^{**}) &= \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho\left(\ell^{*}+\ell^{**}\right)\right)\\ &= \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})+\mho(\ell^{**})\right)\\ &= \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{*})+\mho(\ell^{**})\right)\\ &\leq \text{Max}\left\{\acute{\zeta}_{B}\left(\mho(\ell^{*})\right), \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{**})\right)\right\}\\ &= \text{Max}\left\{\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.\end{aligned}$$

Then,

$$\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}+\ell^{**}) \leq \text{Max}\left\{\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.$$

(8)

$$\begin{aligned}\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)([\ell^{*},\ell^{**}]) &= \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho\left([\ell^{*},\ell^{**}]\right)\right)\\ &= \acute{\zeta}_{\tilde{\mathcal{D}}}\left([\mho(\ell^{*}),\mho(\ell^{**})]\right)\\ &\leq \text{Max}\left\{\acute{\zeta}_{B}\left(\mho(\ell^{*})\right), \acute{\zeta}_{\tilde{\mathcal{D}}}\left(\mho(\ell^{**})\right)\right\}\\ &= \text{Max}\left\{\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{*}), \mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.\end{aligned}$$

Then,

$$\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)([\ell^*,\ell^{**}]) \leq \operatorname{Max}\left\{\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^*), \mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\ell^{**})\right\}.$$

(9)

$$\begin{aligned}
\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*+\ell^{**}) &= \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*+\ell^{**})) \\
&= \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)+\mho(\ell^{**})) \\
&= \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)+\mho(\ell^{**})) \\
&\leq \acute{\mathcal{S}}\left(\dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^{**}))\right) \\
&= \acute{\mathcal{S}}\left(\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^{**})\right).
\end{aligned}$$

So,

$$\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*+\ell^{**}) \leq \acute{\mathcal{S}}\left(\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^{**})\right).$$

(10)

$$\begin{aligned}
\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})([\ell^*,\ell^{**}]) &= \dot{\xi}_{\tilde{\mathcal{D}}}(\mho([\ell^*,\ell^{**}])) \\
&= \dot{\xi}_{\tilde{\mathcal{D}}}([\mho(\ell^*),\mho(\ell^{**})]) \\
&\leq \acute{\mathcal{S}}\left(\tilde{\rho}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^{**}))\right) \\
&= \acute{\mathcal{S}}\left(\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^{**})\right).
\end{aligned}$$

Thus,

$$\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})([\ell^*,\ell^{**}]) \leq \acute{\mathcal{S}}\left(\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\ell^{**})\right).$$

(11)

$$\begin{aligned}
\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*+\ell^{**}) &= \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\ell^*+\ell^{**})) \\
&= \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)+\mho(\ell^{**})) \\
&= \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)+\mho(\ell^{**})) \\
&\leq \operatorname{Max}\left\{\dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\ell^{**}))\right\} \\
&= \operatorname{Max}\left\{\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^{**})\right\}.
\end{aligned}$$

Then,

$$\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*+\ell^{**}) \leq \operatorname{Max}\left\{\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^{**})\right\}.$$

(12)

$$\begin{aligned}
\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})([\ell^*,\ell^{**}]) &= \dot{\chi}_{\tilde{\mathcal{D}}}(\mho([\ell^*,\ell^{**}])) \\
&= \dot{\chi}_{\tilde{\mathcal{D}}}([\mho(\ell^*),\mho(\ell^{**})]) \\
&\leq \operatorname{Max}\left\{\dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\ell^*)), \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\ell^{**}))\right\} \\
&= \operatorname{Max}\left\{\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^{**})\right\}.
\end{aligned}$$

Then,

$$\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})([\ell^*,\ell^{**}]) \leq \operatorname{Max}\left\{\mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^*), \mho^{-1}(\dot{\chi}_{\tilde{\mathcal{D}}})(\ell^{**})\right\}.$$

Let $\tilde{\ell} \in L$ and $\check{\tau} \in F$. Thus

(13)

$$\mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\check{\tau}\tilde{\ell}) = \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \hat{\rho}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \geq \hat{\rho}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}(\hat{\rho}_{\tilde{\mathcal{D}}})(\tilde{\ell})$$

$$\mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\tilde{\ell}) = \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \hat{\zeta}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \geq \hat{\zeta}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}\left(\hat{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\ell}) \tag{14}$$

$$\mho^{-1}(\tilde{\rho}_{\tilde{\mathcal{D}}})(\check{\tau}\tilde{\ell}) = \tilde{\rho}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \tilde{\rho}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \leq \tilde{\rho}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}(\tilde{\rho}_{\tilde{\mathcal{D}}})(\tilde{\ell}). \tag{15}$$

$$\mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\tilde{\ell}) = \acute{\zeta}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \acute{\zeta}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \leq \acute{\zeta}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}\left(\acute{\zeta}_{\tilde{\mathcal{D}}}\right)(\tilde{\ell}). \tag{16}$$

$$\mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\check{\tau}\tilde{\ell}) = \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \dot{\xi}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \leq \dot{\xi}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}(\dot{\xi}_{\tilde{\mathcal{D}}})(\tilde{\ell}). \tag{17}$$

$$\mho^{-1}\left(\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\check{\tau}\tilde{\ell}) = \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\check{\tau}\tilde{\ell})) = \dot{\chi}_{\tilde{\mathcal{D}}}(\check{\tau}\mho(\tilde{\ell})) \leq \dot{\chi}_{\tilde{\mathcal{D}}}(\mho(\tilde{\ell})) = \mho^{-1}\left(\dot{\chi}_{\tilde{\mathcal{D}}}\right)(\tilde{\ell}). \tag{18}$$

Therefore, $\mho^{-1}(\check{\mathfrak{d}}_2) \in CIFN\,(L_1)$.

□

The result below is straightforward to establish.

Proposition 3.10 *Suppose* $\mho : L_1 \to L_2$ *is a surjective Lie algebra homomorphism and* $\tilde{\mathcal{Q}} = (\mathcal{E}_{\tilde{\mathcal{Q}}}, \Upsilon_{\tilde{\mathcal{Q}}}, \check{\mathcal{G}}_{\tilde{\mathcal{Q}}}) \in \mathrm{CNIFS}(L_1)$. *Then the image* $\mho(\tilde{\mathcal{Q}})$ *belongs to* $\mathrm{CNIFS}(L_2)$.

4. Conclusion

In this paper, we introduced and explored the notions of complex intuitionistic fuzzy neutrosophic Lie subalgebras and Lie ideals, defined with respect to neutrosophic t-norms and s-norms. These structures serve as a natural extension of classical Lie subalgebras by incorporating complex-valued membership, non-membership, and indeterminacy components. We examined their fundamental properties and established how they interact under standard operations such as intersection, sum, and homomorphism. The results provide a deeper algebraic foundation for handling uncertainty, indeterminacy, and partial truth in Lie algebraic contexts. These developing frameworks demonstrate potential for application in mathematical modeling, theoretical physics, and information sciences, especially in environments where uncertainty displays intricate behavior. Additional studies may focus on a more detailed structural examination of these ideas and explore their practical significance in decision-making processes and various applied domains.

5. Future Work Problem

This paper establishes the basic definitions and characteristics of complex intuitionistic fuzzy neutrosophic Lie subalgebras and ideals concerning neutrosophic norms (t-norm T and s-norm S), while future research may seek to investigate applications and extensions of these structures within wider algebraic frameworks. Examine the expansion of complex intuitionistic fuzzy neutrosophic Lie subalgebras and ideals to graded Lie algebras, Lie superalgebras, and topological Lie algebras. Additionally, investigate how the proposed structures function with various neutrosophic aggregation operators apart from conventional t-norms and s-norms, and assess their significance in symmetry analysis and fuzzy control systems.

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