Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.78121

Existence of Entropy Solutions for Degenerate Problems with Singular Term in Weighted Sobolev Spaces

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ABSTRACT: In this paper, we will prove the existence of entropy solutions of some nonlinear elliptic equation defined by a degenerate coercive operator and a singular right-hand side whose model case is:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\omega(x)\,|\nabla u|^{p-2}\,\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = \dfrac{f}{u^{\gamma}} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, $1 < p < N$, $\theta \geq 0$, $\gamma \geq 0$, ω is a vector of weight functions and f is a non-negative function that belongs to $L^1(\Omega)$ and we will provide an example.

Key Words: Nonlinear elliptic equation, degenerate coercivity, entropy solutions, existence results, singular non-linearity, weighted Sobolev spaces.

Contents

1. Introduction

In this work, we are interested in the existence of entropy solutions for Dirichlet problems of the form

$$\begin{cases} -\operatorname{div}\left(a(x,u,\nabla u)\right) = fh(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, $\omega(x) = (\omega_i(x))_{\{0 \leq i \leq N\}}$ is a vector of weight functions (i.e. each $\omega_i(x)$ is a measurable almost everywhere strictly positive function on Ω), the data f is a non-negative and it belongs to $L^1(\Omega)$, $h(s)$ is a singular continuous function which behaves as $s^{-\gamma}$ near zero with $\gamma \geq 0$ (i.e. h satisfied (3.5) below) and the Leray-Lions operator $-\operatorname{div}\left(a(x,u,\nabla u)\right)$ defined on the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ such that $a(x,s,\xi)$ is a nonlinear Carathéodory function satisfying some growth and degenerate coercivity assumptions which can be modeled by $a(x,s,\xi) = \dfrac{\omega(x)\,|\xi|^{p-2}\,\xi}{(1+|s|)^{\theta(p-1)}}$ in the simplest case, where $1 < p < N$ and $\theta \geq 0$.

The main characteristics of this type of problem lie in the lack of coercivity of the operator

$$-\operatorname{div}\left(\frac{\omega(x)\,|\nabla u|^{p-2}\,\nabla u}{(1+|u|)^{\theta(p-1)}}\right)$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 46E35; 35J20; 35J70.
Submitted July 28, 2025. Published September 29, 2025

when $\theta > 0$ and ω is not constant. Additionally, the right-hand side of equation (1.1) can exhibit singularities near zero. These factors prevent the application of standard existence theorems to study problem (1.1), highlighting the need for a deeper analysis.

In the following, we provide an overview of problems that have been studied and are closely related to our problem (1.1).

To begin with, let's focus on the coercive case, where $\theta = 0$ and $\omega_i(x) = C_i > 0$ for $i = 0, 1, \ldots, N$. A significant amount of research has been conducted to establish the existence of solutions to problem (1.1). For $p = 2$ and $h(s) = \frac{1}{s^\gamma}$, the seminal works of [7,18,29] were foundational in proving existence and regularity results for classical non-negative solutions, specifically $\mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$ solutions, under smooth data conditions. Later, in [6], the authors extended these results by demonstrating the existence and regularity of distributional solutions when the data f belongs to $L^m(\Omega)$ for $m \geq 1$, while analyzing cases for different values of γ: $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$. For the nonlinear case (i.e., $1 < p < N$), a series of works, including [4,8,9,14,22,23,25], have shown the existence of weak solutions under Leray-Lions type conditions, particularly when f belongs either to $L^m(\Omega)$ for $m \geq 1$ or to the space of Radon measures.

When $\theta = 0$, with ω being a vector of weight functions and $h \equiv 1$, the existence of weak solutions for problem (1.1) with lower-order terms and $f \in W^{-1,p'}(\Omega, \omega^*)$ was tackled by Akdim et al. in [3]. This work was further generalized in [1,2]. Additionally, in [15] the authors provided existence results for the following problem:

$$\begin{cases} -\mathrm{div}\left(\omega(x) \left|\nabla u\right|^{p-2} \nabla u\right) = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a non-negative function belonging to a suitable Lebesgue space, with $\delta > 0$.

In contrast, for the non-coercive case (i.e., $\theta > 0$) where ω is constant and $h(s) = \frac{1}{s^\gamma}$, several authors have proven the existence of solutions in either the entropy sense (see [12,13,21]) or the weak sense (see [27]), depending on the assumptions. We also refer to [26], where the author addressed problem (1.1) in the case of non-constant ω, $h(s) \equiv 1$ and L^1-data.

Recent works addressing problems with degenerate coercivity and singular terms have focused on studying existence and regularity within the framework of classical Sobolev spaces.

In our paper, we extend these problems to the setting of weighted Sobolev spaces, employing the concept of entropy solutions. To tackle the challenges posed by the simultaneous presence of singular terms and degenerate coercivity, we adopt an approximation method for problem (1.1) and utilize Schauder's fixed-point theorem to establish the existence of solutions.

The structure of this paper is as follows. In Section 2, we present the necessary preliminaries and notations. In Section 3, we specify the assumptions concerning the data of our problem, we introduce the definition of an entropy solution for problem (1.1) and we state our main result. Section 4 devoted to the proof of this result. Finally, in Section 5, we provide an example.

2. Preliminary results and notations

We recall in this section some standard definitions and properties which will be used through the paper. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $N \geq 2$, we suppose that $1 < p < N$ and the weight vector $\omega(x)$ satisfies, for any $i = 1, \ldots, N$,

$$\omega_i \in L^1_{loc}(\Omega) \tag{2.1}$$

and

$$\omega_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega). \tag{2.2}$$

We define now the weighted Lebesgue space $L^p(\Omega, \omega_0)$ with weight ω_0 as

$$L^p(\Omega, \omega_0) = \left\{ u : \Omega \to \mathbb{R}, \text{meas.}, \int_\Omega |u(x)|^p \, \omega_0(x) dx < \infty \right\},$$

where the norm is

$$\|u\|_{p,\omega_0} = \left(\int_\Omega |u(x)|^p \, \omega_0(x) dx \right)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, \omega)$ the space of real-valued functions $u \in L^p(\Omega, \omega_0)$ such that $\partial_i u \in L^p(\Omega, \omega_i)$ for $i = 1, \dots, N$, which is a Banach space under the norm

$$\|u\|_{1,p,\omega} = \left(\int_\Omega |u(x)|^p \, \omega_0(x) dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^p \, \omega_i(x) dx \right)^{\frac{1}{p}}. \tag{2.3}$$

By condition (2.1), we can define the subspace $X = W_0^{1,p}(\Omega, \omega)$ of $W^{1,p}(\Omega, \omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, if (2.2) holds, then X and $W^{1,p}(\Omega, \omega)$ are separable and reflexive Banach spaces. We recall that the dual of X is equivalent to $W^{-1,p'}(\Omega, \omega^*)$ where $\omega^*(x) = \left(\omega_i^*(x) = \omega_i^{1-p'}(x) \right)_{\{1 \le i \le N\}}$ and $p' = \frac{p}{p-1}$ is the conjugate of p. (see [10,16] for more details)

The following notations will be used throughout the paper. For any $u : \Omega \to \mathbb{R}$ and $k \ge 0$, we write $\{|u| \le (<, \ge, >, =)k\}$ for the set $\{x \in \Omega : |u(x)| \le (<, \ge, >, =)k\}$.

For a fixed $k > 0$, let the truncation functions T_k, G_k and V_k of real variable defines as follows:

$$T_k(s) = \max(-k, \min(k, s)),$$

$$G_k(s) = s - T_k(s)$$

and

$$V_k(s) = \begin{cases} 1 & |s| \le k, \\ \frac{2k-|s|}{k} & k < |s| < 2k, \\ 0 & |s| \ge 2k. \end{cases}$$

Moreover, we note by $\tau^+ = \max\{\tau, 0\}$ and $\tau^- = -\min\{\tau, 0\}$ the positive and the negative part of τ.

Finally, We mention that the constant C used in our paper may change from line to line and its only depend of the data of our problem but its never depend on the indexes of the sequences we well often introduce.

3. Statement of the problem and main result

Throughout this paper, we assume that the following assumptions hold true. Ω is a bounded open domain in $\mathbb{R}^N$ $(N \ge 2)$, $1 < p < N$, $\omega(x) = (\omega_i(x))_{\{0 \le i \le N\}}$ is a vector of weight functions such that (2.1) and (2.2) hold. The expression

$$\|u\|_X = \left(\sum_{i=1}^{N} \int_\Omega |\partial_i u|^p \, \omega_i(x) dx \right)^{\frac{1}{p}} \tag{3.1}$$

is a norm defined on X which equivalent to the norm (2.3). Furthermore, we suppose that there exist a weight function σ on Ω and a parameter $1 < q < \infty$ such that

$$\sigma^{1-q'} \in L^1_{loc}(\Omega) \tag{3.2}$$

and the Hardy inequality

$$\left(\int_\Omega |u(x)|^q \, \sigma(x) dx \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{N} \int_\Omega |\partial_i u|^p \, \omega_i(x) dx \right)^{\frac{1}{p}} \tag{3.3}$$

holds for every $u \in X$, with C a constant independent of u. Consequently, the embedding

$$X \hookrightarrow L^q(\Omega, \sigma) \tag{3.4}$$

is compact.
The function $h \in \mathcal{C}^0\left([0;+\infty[\,;[0;+\infty]\right)$ is a singular sourcing that is finite outside the origin such that

$$\exists c>0, \gamma \geq 0 \text{ such that : } h(s) \leq \frac{c}{s^\gamma} \quad \text{for all } s \in (0,\infty). \tag{3.5}$$

The Leray-Lions operator $-\operatorname{div}(a(x,u,\nabla u))$ defined from $W_0^{1,p}(\Omega,\omega)$ into its dual, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the assumptions:

$$a(x,s,\xi)\cdot\xi \geq b(|s|)\sum_{i=1}^N \omega_i(x)\,|\xi_i|^p, \tag{3.6}$$

for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$ and $\forall \xi \in \mathbb{R}^N$, where $b : (0,+\infty) \longrightarrow (0,+\infty)$ is a continuous and bounded function such that

$$b(s) \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}} \quad \text{for } \alpha>0 \text{ and } \theta \geq 0. \tag{3.7}$$

For any $i = 1,\ldots,N$,

$$|a_i(x,s,\xi)| \leq \omega_i(x)^{\frac{1}{p}}\left(k(x)+\sigma(x)^{\frac{1}{p'}}\,|s|^{\frac{q}{p'}}+b(|s|)\sum_{j=1}^N \omega_j(x)^{\frac{1}{p'}}\,|\xi_j|^{p-1}\right), \tag{3.8}$$

for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, where σ and q are given by (3.2) and k is a non-negative function in $L^{p'}(\Omega)$.

$$\left(a(x,s,\xi)-a(x,s,\xi')\right)\cdot\left(\xi-\xi'\right)>0, \tag{3.9}$$

for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$ such that $\xi \neq \xi'$.

Finally,

$$f \text{ is a non-negative function that belongs to } L^1(\Omega). \tag{3.10}$$

Now, we give the definition of an entropy solution to (1.1) and we will give our principal result.

Definition 3.1 *We say that a non-negative measurable function u is an entropy solution to problem (1.1) if*

$$T_k(u) \in W_0^{1,p}(\Omega,\omega), \tag{3.11}$$

$$a(x,T_k(u),\nabla T_k(u)) \in \prod_{i=1}^N L^{p'}(\Omega,\omega_i^{1-p'}), \tag{3.12}$$

$$fh(u)T_k(u-\varphi) \in L^1(\Omega) \tag{3.13}$$

and

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-\varphi)dx \leq \int_\Omega fh(u)T_k(u-\varphi)dx \tag{3.14}$$

for every $k>1$ and for any $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$.

Our main result is the following theorem:

Theorem 3.1 *Assume that (3.1)-(3.10) hold with $0 \leq \gamma \leq 1$ and $\theta < 1+\dfrac{\gamma}{p-1}$. Then there exists an entropy solution u of problem (1.1).*

Remark 3.1 *If we assume that the collection of weight functions ω satisfy (2.1), $\omega_0(x)=1$ and the integrability condition: There exists $\tau \in \left]\frac{N}{p};\infty\right[\bigcap\left[\frac{1}{p-1};\infty\right[$ such that for all $i=1,\ldots,N$,*

$$\omega_i^{-\tau} \in L^1(\Omega),$$

then hypothesis (3.1) holds. Moreover, (3.3)-(3.4) are verified for all $1 \leq q < p_1^*$ if $p\tau < N(\tau+1)$ and for all $q \geq 1$ if $p\tau \geq N(\tau+1)$ where $p_1 = \dfrac{p\tau}{\tau+1}$ and p_1^* is the Sobolev conjugate of p_1.

4. Proof of our result

The proof is divided into four steps.

Step 1: Approximating problems
For $n \in \mathbb{N}$, we consider the following problem

$$\begin{cases} -\mathrm{div}(a(x, T_n(u_n), \nabla u_n)) = f_n h_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$ and

$$h_n(s) = \begin{cases} T_n(h(s)) & s \geq 0, \\ \min(n, h(0)) & s < 0. \end{cases}$$

In this approximating problem, we have truncated both the degenerate coercivity and the singularity of the right-hand side. The weak formulation of this problem is given by

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla \varphi \, dx = \int_\Omega f_n h_n(u_n) \varphi \, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega). \tag{4.2}$$

Lemma 4.1 *For each $n \in \mathbb{N}$, there exists a non-negative weak solution u_n to the problem (4.1), such that $u_n \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.*

Proof:
The proof relies on a standard Schauder's fixed point argument. Let $n \in \mathbb{N}$ and fix $v \in L^p(\Omega, \omega)$. We set

$$A : L^p(\Omega, \omega) \to L^p(\Omega, \omega) \tag{4.3}$$

as the map that, for any v, gives the weak solution ϖ to,

$$\begin{cases} -\mathrm{div}(a(x, T_n(\varpi), \nabla \varpi)) = f_n h_n(v) & \text{in } \Omega, \\ \varpi = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

whose existence is guaranteed from the classical results (see [3,19,20]), in particular $\varpi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$. Indeed, since $f_n h_n(v)$ is bounded, we have that $\varpi \in L^\infty(\Omega)$ and there exists a positive constant C_∞, independents of v and ϖ (but possibly depending in n), such that

$$\|\varpi\|_{L^\infty(\Omega)} \leq C_\infty.$$

Now we show that the map A has an invariant ball, it's relatively compact and continuous in $L^p(\Omega, \omega)$. Let us take ϖ as a test function in (4.4), we get

$$\int_\Omega a(x, T_n(\varpi), \nabla \varpi) \cdot \nabla \varpi dx = \int_\Omega f_n h_n(v) \varpi dx,$$

so by (3.6),(3.7) and the definition of f_n and h_n, we have

$$\frac{\alpha}{(1+n)^{\theta(p-1)}} \int_\Omega \sum_{i=1}^N |\partial_i \varpi|^p \omega_i(x) dx \leq \int_\Omega \frac{\alpha}{(1 + T_n(\varpi))^{\theta(p-1)}} \sum_{i=1}^N |\partial_i \varpi|^p \omega_i(x) dx \leq n^2 \int_\Omega |\varpi| \, dx.$$

Using the Sobolev embedding theorem $W^{1,p}(\Omega, \omega) \hookrightarrow L^1(\Omega)$ and the Poincaré inequality to get

$$\|\varpi\|_X^p \leq C,$$

and in view of the compact embedding $W^{1,p}(\Omega, \omega) \hookrightarrow L^p(\Omega, \omega)$, we get

$$\|\varpi\|_{p,\omega} \leq \|\varpi\|_X \leq C, \tag{4.5}$$

where the positive constant C is independent on v and ϖ. Thus, the ball B in $L^p(\Omega,\omega)$ of radius C is invariant for the map A. Moreover, by (4.5) and the compact embedding $W^{1,p}(\Omega,\omega) \hookrightarrow L^p(\Omega,\omega)$, we deduce that $A\left(L^p(\Omega,\omega)\right)$ is relatively compact in $L^p(\Omega,\omega)$.

For the continuity of A, let consider a sequence $v_k \in L^p(\Omega,\omega)$ that strongly converges to $v \in L^p(\Omega,\omega)$ as $k \to \infty$. If we denotes by $\varpi_k = A(v_k)$ then, by (4.5), ϖ_k is bounded in X with respect to k. Moreover, there exists $\varpi \in X$ to which ϖ_k, up to subsequence, converges weakly in X and a.e. in Ω.

Now we need to show that $\varpi = A\left(v\right)$; i.e. we need to pass to the limit with respect to k in the weak formulation

$$\int_\Omega a(x, T_n(\varpi_k), \nabla\varpi_k)\cdot\nabla\varphi dx = \int_\Omega f_n h_n(|v_k|)\varphi dx, \quad \forall\varphi \in X \cap L^\infty(\Omega). \tag{4.6}$$

For the right-hand side of (4.6), since $|f_n h_n(|v_k|)| \leq n^2$ and ϖ_k converges a.e. in Ω to ϖ, by dominated convergence theorem, it's easy to prove that

$$f_n h_n(|v_k|) \longrightarrow f_n h_n(|v|) \text{ in } L^1(\Omega) \text{ as } k \to \infty.$$

Moreover, observe that if we prove that ϖ_k converges strongly to ϖ in X, we can safely pass to the limit on the left-hand side of (4.6). Indeed, we take $\varphi = \varpi_k - \varpi$ in (4.6), we have for $n > C_\infty$

$$\int_\Omega a(x, \varpi_k, \nabla\varpi_k)\cdot\nabla(\varpi_k - \varpi)dx = \int_\Omega f_n h_n(|v_k|)(\varpi_k - \varpi).$$

Since the right-hand side of the previous equality tends to zero when k tends to infinity, we deduce that

$$\int_\Omega a(x, \varpi_k, \nabla\varpi_k)\cdot\nabla(\varpi_k - \varpi)dx \to 0 \text{ as } k \to \infty. \tag{4.7}$$

Moreover, by using (3.8) and the Hardy inequality, one has, for any $i = 1, \ldots, N$,

$$\int_\Omega \omega_i(x)^{1-p'} \left|a_i(x, \varpi_k, \nabla\varpi)\right|^{p'} dx \leq C \int_\Omega \left(k^{p'}(x) + \omega_i(x)\left|\nabla\varpi_k\right|^p + \sum_{j=1}^N \omega_j(x)\left|\partial_j \nabla\varpi\right|^p \right) dx,$$

which implies, from Vitali's theorem that

$$a(x, \varpi_k, \nabla\varpi) \to a_i(x, \varpi, \nabla\varpi) \text{ strongly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}). \tag{4.8}$$

Now, we write

$$\begin{aligned}
&\int_\Omega (a(x, \varpi_k, \nabla\varpi_k) - a(x, \varpi_k, \nabla\varpi))\cdot\nabla(\varpi_k - \varpi)dx \\
&\qquad = \int_\Omega a(x, \varpi_k, \nabla\varpi_k)\cdot\nabla(\varpi_k - \varpi)dx - \int_\Omega a(x, \varpi_k, \nabla\varpi)\cdot\nabla(\varpi_k - \varpi)dx,
\end{aligned}$$

therefore, by (4.7) and (4.8), we obtain

$$\lim_{k\to\infty} \int_\Omega (a(x, \varpi_k, \nabla\varpi_k) - a(x, \varpi_k, \nabla\varpi))\cdot(\nabla\varpi_k - \nabla\varpi)dx = 0, \tag{4.9}$$

according to (3.9), the integrand function in the left-hand side in (4.9) is non-negative, so

$$a(x, \varpi_k, \nabla\varpi_k) - a(x, \varpi_k, \nabla\varpi)\cdot(\nabla\varpi_k - \nabla\varpi) \to 0 \text{ in } L^1(\Omega).$$

So, up a subsequence still indexed by k, we have

$$a(x, \varpi_k, \nabla\varpi_k) - a(x, \varpi_k, \nabla\varpi)\cdot(\nabla\varpi_k - \nabla\varpi) \to 0 \text{ a.e.} x \text{ in } \Omega.$$

Consequently, there exists a subset $\Gamma$of Ω of zero measure, such that for every $x \in \Omega \setminus \Gamma$, one has

$$D_k(x) = a(x, \varpi_k, \nabla\varpi_k) - a(x, \varpi_k, \nabla\varpi) \cdot (\nabla\varpi_k - \nabla\varpi) \to 0,$$

$|\varpi(x)| < \infty$, $|\nabla\varpi(x)| < \infty$, $|k(x)| < \infty$ and $\varpi_k(x) \to \varpi(x)$.

Hence, from (3.6), (3.8) and since $\|\varpi_k\|_{L^\infty(\Omega)} \leq C_\infty$ we can write

$$D_k(x) \geq \frac{\alpha}{(1+C_\infty)^{\theta(p-1)}} |\nabla\varpi_k|^p - c(x)\left(1 + |\nabla\varpi_k| + |\nabla\varpi_k|^{p-1}\right), \tag{4.10}$$

where $c(x)$ is a constant which depends on x but does not depend on k, which shows, in view of (4.10), that the sequence $|\nabla\varpi_k|$ is uniformly bounded in $\mathbb{R}^N$ with respect to k. Then, arguing as in the same way as in Lemma 5 in [5], to obtain

$$\varpi_k \to \varpi \text{ strongly in } X.$$

This is sufficient to conclude that $\varpi = A(v)$, i.e. A is continuous.

Finally, by applying the Schauder's fixed point theorem, there exist a fixed point u_n of the map A that belongs to $W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ which is solution of problem (4.4).

To demonstrate that u_n is non-negative, we take u_n^- as a test function in the weak formulation. We find

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n^- \, dx = \int_\Omega f_n h_n(u_n) u_n^- \, dx.$$

Since the right-hand side is non-positive, and using the properties of coercivity, we obtain

$$\frac{\alpha}{(1+n)^{\theta(p-1)}} \|u_n^-\|_X^p \leq 0.$$

This implies $\|u_n^-\|_X^p = 0$, leading us to conclude that $u_n \geq 0$ almost everywhere in Ω. □

Step 2: A priori estimates and basic convergence results

In this step, we will establish some a priori estimates for the solutions u_n of the problems defined in (4.1). These estimates will help us to prove Theorem 3.1.

Lemma 4.2 *Suppose that the hypotheses of Theorem 3.1 are satisfied and let u_n be a weak solution of (4.1). Then, for every $k > 1$,*

$$\|T_k(u_n)\|_X \leq C,$$

where C is a constant independent of n. Moreover, there exists a non-negative measurable function u that is finite almost everywhere such that

$$u_n \to u \quad \text{a.e. in } \Omega$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } X \text{ as } n \to +\infty.$$

Proof: For $k > 1$, we use $T_k(u_n)$ as a test function in the weak formulation (4.2)

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n) \, dx = \int_\Omega f_n h_n(u_n) T_k(u_n) \, dx.$$

From this, we derive the inequality

$$\int_{\{u_n \leq k\}} a(x, T_n(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \leq \int_\Omega f_n (T_k(u_n))^{1-\gamma} \, dx.$$

By applying conditions (3.5), (3.6) and (3.7), along with the fact that $T_n(u_n) \leq k$ on the set $\{u_n \leq k\}$, we obtain

$$\frac{\alpha}{(1+k)^{\theta(p-1)}} \int_\Omega \sum_{i=1}^N |\partial_i T_k(u_n)|^p \omega_i(x)\,dx \leq k^{1-\gamma} \int_\Omega f\,dx.$$

Consequently, we can conclude

$$\int_\Omega \sum_{i=1}^N |\partial_i T_k(u_n)|^p \omega_i(x)\,dx \leq C k^{\theta(p-1)+1-\gamma} \|f\|_{L^1(\Omega)}, \tag{4.11}$$

where C is independent of k. Since $f \in L^1(\Omega)$, it follows that $T_k(u_n)$ is bounded in X. Thus, there exists some $v_k \in X$ such that

$$\begin{array}{lll} T_k(u_n) \rightharpoonup v_k & \text{weakly in } X, & \\ T_k(u_n) \to v_k & \text{strongly in } L^q(\Omega,\sigma) & \text{and a.e. in } \Omega. \end{array} \tag{4.12}$$

Next, we will demonstrate that u_n converges to some u locally in measure. Let $k > 1$ and B_R be a ball of radius R. From (4.11),the Hölder inequality and Hardy's inequality , we have

$$\begin{aligned} k \text{ meas}\left(\{|u_n| > k\} \cap B_R\right) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx \\ &\leq \left(\int_\Omega |T_k(u_n)|^q \sigma\,dx\right)^{\frac{1}{q}} \left(\int_{B_R} \sigma^{1-q'}\,dx\right)^{\frac{1}{q'}} \\ &\leq C_R \left(\int_\Omega \sum_{i=1}^N |\partial_i T_k(u_n)|^p \omega_i(x)\,dx\right)^{\frac{1}{p}} \\ &\leq C k^{\frac{\theta(p-1)+1-\gamma}{p}}. \end{aligned}$$

This leads us to conclude that, for every $k > 1$,

$$\text{meas}\left(\{|u_n| > k\} \cap B_R\right) \leq \frac{C}{k^{\frac{(p-1)(1-\theta)+\gamma}{p}}}. \tag{4.13}$$

Furthermore, for any $\delta > 0$, we have

$$\begin{aligned} \text{meas}\left(\{|u_n - u_m| > \delta\} \cap B_R\right) \leq &\text{ meas}\left(\{|u_n| > k\} \cap B_R\right) + \text{meas}\left(\{|u_m| > k\} \cap B_R\right) \\ &+ \text{meas}\left(\{|T_k(u_n) - T_k(u_m)| > \delta\}\right). \end{aligned} \tag{4.14}$$

Let $\varepsilon > 0$, by using (4.13) and (4.14), we can find some $k(\varepsilon) > 0$ such that

$$\text{meas}\left(\{|u_n - u_m| > \delta\} \cap B_R\right) \leq \varepsilon \quad \text{for any } n, m \geq n_0(k(\varepsilon), \delta, R).$$

This shows that $(u_n)_n$ is a Cauchy sequence in measure in B_R, so, its converges almost everywhere, for a subsequence, to some measurable function u. Therefore, from (4.12), we obtain

$$\begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } X, & \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^q(\Omega,\sigma) & \text{and a.e. in } \Omega. \end{array} \tag{4.15}$$

□

On top of that, using the growth condition (3.8), we have, for each $i = 1, \ldots, N$

$$\begin{aligned} &\int_\Omega \omega_i(x)^{1-p'} |a_i(x, T_k(u_n), \nabla T_k(u_n))|^{p'}\,dx \\ &\leq C \int_\Omega \left(k^{p'}(x) + \sigma(x)|T_k(u_n)|^q + b(|T_k(u_n)|)^{p'} \sum_{j=1}^N \omega_j(x)|\partial_j T_k(u_n)|^p\right) dx \\ &\leq C \int_\Omega \left(k^{p'}(x) + \sigma(x)|T_k(u_n)|^q + \sum_{j=1}^N \omega_j(x)|\partial_j T_k(u_n)|^p\right) dx. \end{aligned}$$

Thus, from the Hardy inequality and Lemma 4.2, the sequence $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^{1-p'})$.

Therefore, for every $k>1$, there exists a function $\psi_k \in \prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^{1-p'})$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \psi_k \quad \text{weakly in } \prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^{1-p'}). \tag{4.16}$$

Remark 4.1 *if we replace condition (3.2) with a stronger one $\sigma^{1-q'} \in L^1(\Omega)$, we can follow the same reasoning as in (4) to conclude that u is almost everywhere finite in Ω,*

Step 3: Strong convergence of truncations

Before demonstrating the strong convergence of truncations, we need to establish the following proposition, which will be helpful in subsequent arguments.

Proposition 4.1 *For $k>1$, suppose that the hypotheses of Theorem 3.1 are satisfied and let u_n be a weak solution of (4.1). Then, for all non-negative $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, we have*

$$\int_\Omega f_n h_n(u_n)\varphi\,dx \le C, \tag{4.17}$$

where the constant C does not depend on n.

Proof: Let $k>1$ and φ be a non-negative function in $W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. Taking $V_k(u_n)\varphi$ as a test function in (4.1) and ignoring the positive terms, we obtain

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n\le k\}}\varphi\,dx \le \int_\Omega V_k(u_n)a(x,T_n(u_n),\nabla u_n)\cdot\nabla\varphi\,dx. \tag{4.18}$$

Applying Young's inequality and using (3.8), we derive

$$\begin{aligned}
&\int_\Omega V_k(u_n)a(x,T_n(u_n),\nabla u_n)\cdot\nabla\varphi\,dx\\
&\le \sum_{i=1}^{N}\int_\Omega |a_i(x,T_{2k}(u_n),\nabla T_{2k}(u_n))||\partial_i\varphi|\,dx\\
&\le C\sum_{i=1}^{N}\int_\Omega \omega_i(x)^{\frac{1}{p}}k(x)|\partial_i\varphi|\,dx + C\sum_{i=1}^{N}\int_\Omega \omega_i(x)^{\frac{1}{p}}\sigma(x)^{\frac{1}{p'}}|T_{2k}(u_n)|^{\frac{q}{p'}}|\partial_i\varphi|\,dx\\
&+C\sum_{i=1}^{N}\int_\Omega b(T_{2k}(u_n))\omega_i(x)^{\frac{1}{p}}\sum_{j=1}^{N}\omega_j(x)^{\frac{1}{p'}}|\partial_j T_{2k}(u_n)|^{p-1}|\partial_i\varphi|\,dx\\
&\le C\int_\Omega k^{p'}(x)\,dx + C\int_\Omega \sigma(x)|T_{2k}(u_n)|^q\,dx + C\sum_{j=1}^{N}\int_\Omega \omega_j(x)|\partial_j T_{2k}(u_n)|^p\,dx\\
&+C\sum_{i=1}^{N}\int_\Omega \omega_i(x)|\partial_i\varphi|^p\,dx\\
&\le C\left(1+\sum_{i=1}^{N}\int_\Omega \omega_i(x)|\partial_i\varphi|^p\,dx + \int_\Omega \sigma(x)|T_{2k}(u_n)|^q\,dx + \sum_{j=1}^{N}\int_\Omega \omega_j(x)|\partial_j T_{2k}(u_n)|^p\,dx\right).
\end{aligned}$$

Applying Hardy's inequality, we conclude that

$$\begin{aligned}
&\int_\Omega V_k(u_n)a(x,T_n(u_n),\nabla u_n)\cdot\nabla\varphi\,dx \\
&\leq C\left(1+\sum_{i=1}^N\int_\Omega \omega_i(x)|\partial_i\varphi|^p\,dx+\sum_{i=1}^N\int_\Omega \omega_i(x)|\partial_i T_{2k}(u_n)|^p\,dx\right) \\
&\leq C\left(1+\|\varphi\|_X+\|T_{2k}(u_n)\|_X\right).
\end{aligned} \tag{4.19}$$

Combining (4.18), (4.19), and applying Lemma 4.2, we obtain

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n\leq k\}}\varphi\,dx \leq C. \tag{4.20}$$

Moreover, it is straightforward to show, due to the assumptions on h and f, that

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n> k\}}\varphi\,dx \leq C. \tag{4.21}$$

Thus, from (4.20) and (4.21), we conclude that $f_n h_n(u_n)\varphi \in L^1(\Omega)$. □

Now we have all the ingredients to demonstrate the strong convergence of the truncations.

Lemma 4.3 *For $k>1$, suppose that the hypotheses of Theorem 3.1 are satisfied and let u_n be a weak solution of (4.1). Then $T_k(u_n)\to T_k(u)$ strongly in X as $n\to\infty$, where u is given by Lemma 4.2.*

Proof: For all $k>1$, without loss of generality, we can assume that $n>k$. We choose $V_j(u_n)\left(T_k(u_n)-T_k(u)\right)$ (with $j>k$) as the test function in (4.1), yielding

$$\begin{aligned}
&\int_\Omega V_j(u_n)a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)\,dx \\
&=\frac{1}{j}\int_{\{j<u_n<2j\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\left(T_k(u_n)-T_k(u)\right)\,dx \\
&+\int_\Omega f_n h_n(u_n)V_j(u_n)\left(T_k(u_n)-T_k(u)\right)\,dx.
\end{aligned}$$

From this, we derive the equality

$$\begin{aligned}
&\int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx \\
&=-\int_{\{k<u_n<2j\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)V_j(u_n)dx \\
&+\frac{1}{j}\int_{\{j<u_n<2j\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\left(T_k(u_n)-T_k(u)\right)dx \\
&+\int_\Omega f_n h_n(u_n)V_j(u_n)\left(T_k(u_n)-T_k(u)\right)dx \\
&-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx \\
&=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned} \tag{4.22}$$

We proceed to analyze each term $(\mathcal{I}_i)_{i=1,2,3,4}$ in equation (4.22) individually to determine their limits as $j\to\infty$ and $n\to\infty$.

For the first term $\mathcal{I}_1$, we observe that

$$\begin{aligned}\mathcal{I}_1 &= \int_{\{k<u_n<2j\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)V_j(u_n)\,dx\\ &\leq C\int_\Omega |a(x,T_n(u_n),\nabla u_n)|\,V_j(u_n)\,|\nabla T_k(u)|\,\chi_{\{u_n>k\}}\,dx.\end{aligned}$$

By applying condition (3.8) and Lemma 4.2, we recognize that $|a(x,T_n(u_n),\nabla u_n)|\,V_j(u_n)$ remains bounded in $\prod_{i=1}^N L^{p'}(\Omega,\omega_i^{1-p'})$ with respect to n, while $|\nabla T_k(u)|\,\chi_{\{u_n>k\}}$ converges to zero in $\prod_{i=1}^N L^p(\Omega,\omega_i)$. Therefore, we find

$$\limsup_{n\to\infty}\mathcal{I}_1 \leq 0. \tag{4.23}$$

Turning to $\mathcal{I}_2$, we use $1-V_j(u_n)$ as a test function in the weak formulation (4.2). This yields

$$\begin{aligned}\frac{1}{j}\int_{\{j<u_n<2j\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,dx &= \int_\Omega f_n h_n(u_n)\,(1-V_j(u_n))\,dx\\ &\leq \left(\sup_{s\in[j,\infty)} h(s)\right)\int_\Omega f\,(1-V_j(u_n))\,dx,\end{aligned}$$

which implies

$$\mathcal{I}_2 \leq \frac{2k}{j}\left(\sup_{s\in[j,\infty)} h(s)\right)\int_\Omega f\,(1-V_j(u_n))\,dx. \tag{4.24}$$

By applying Lebesgue's theorem twice, the right-hand side of (4.24) tends to zero as $n\to\infty$ and $j\to\infty$. Consequently,

$$\limsup_{j\to\infty}\limsup_{n\to\infty}\mathcal{I}_2 \leq 0. \tag{4.25}$$

For the third term, $\mathcal{I}_3$, we let δ be a small positive number such that

$$\delta \notin \{\eta>0 : \text{meas}\,(\{u=\eta\})>0\}.$$

Splitting $\mathcal{I}_3$ over $\{u_n\leq\delta\}$ and $\{u_n>\delta\}$ and utilizing condition (3.5), we have

$$\mathcal{I}_3 \leq C\delta^{1-\gamma}\int_{\{u_n\leq\delta\}} f\,dx + \int_{\{u_n>\delta\}} f_n h_n(u_n)V_j(u_n)\,(T_k(u_n)-T_k(u))\,dx. \tag{4.26}$$

For the first term on the right-hand side of (4.26), if $h(0)<\infty$ (i.e., $\gamma=0$), then $\delta^{1-\gamma}\int_{\{u_n\leq\delta\}} f$ converges to zero as $n\to\infty$ and $\delta\to 0^+$. If $h(0)=\infty$ (i.e., $0<\gamma\leq 1$), then by Fatou's lemma and (4.17), we see that $fh(u)\in L^1_{loc}(\Omega)$, which implies $\{u=0\}\subset\{f=0\}$ up to a set of zero Lebesgue measure. Thus, in both cases,

$$\limsup_{\delta\to 0^+}\limsup_{n\to\infty}\left(\delta^{1-\gamma}\int_{\{u_n\leq\delta\}} f\right) = \limsup_{\delta\to 0^+}\left(\delta^{1-\gamma}\int_{\{u\leq\delta\}} f\right) = 0. \tag{4.27}$$

For the second term on the left-hand side of (4.26), using the Lebesgue convergence theorem, we have

$$\lim_{n\to\infty}\int_{\{u_n>\delta\}} f_n h_n(u_n)V_j(u_n)\,(T_k(u_n)-T_k(u))\,dx = 0. \tag{4.28}$$

Combining (4.26), (4.27) and (4.28), we conclude that

$$\limsup_{\delta\to 0^+}\limsup_{j\to\infty}\limsup_{n\to\infty}\mathcal{I}_3 \leq 0. \tag{4.29}$$

Finally, for the fourth term $\mathcal{I}_4$, by applying Lemma 4.2, condition (3.8), Hardy's inequality, and the Lebesgue convergence theorem, we obtain

$$a(x, T_k(u_n), \nabla T_k(u)) \to a(x, T_k(u), \nabla T_k(u)) \text{ strongly in } \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^{1-p'}) \tag{4.30}$$

and since $T_k(u_n) - T_k(u) \rightharpoonup 0$ weakly in X, it follows that

$$\lim_{n\to\infty} \mathcal{I}_4 = 0. \tag{4.31}$$

Bringing together (4.23), (4.25), (4.29) and (4.31) in (4.22), we deduce

$$\lim_{n\to\infty} \int_\Omega \left(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))\right) \cdot \nabla \left(T_k(u_n) - T_k(u)\right) dx = 0. \tag{4.32}$$

Thus, by Lemma 3.2 in [3] and using the strict monotonicity condition (3.9), we conclude that

$$T_k(u_n) \to T_k(u) \text{ strongly in } X \text{ as } n \to \infty. \tag{4.33}$$

In particular, there exists a subsequence still labelled with n such that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{4.34}$$

□

Step 4: Proof of Theorem 3.1

First, in Lemma 4.2, we established that the limit u of the approximate sequence u_n from (4.1) is finite almost everywhere in Ω and that, for all $k > 1$, $T_k(u) \in W_0^{1,p}(\Omega, \omega)$, verifying assertion (3.11).

Next, using Lemma 4.2, (4.33), (4.16) and noting that a is a Carathéodory function, we conclude that $a(x, T_k(u), \nabla T_k(u)) \in \prod_{i=1}^{N} L^{p'}(\Omega, \omega_i^{1-p'})$, hence verifying assertion (3.12).

We now proceed to show assertion (3.13). Let $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ be fixed and set $M = \|\varphi\|_{L^\infty(\Omega)} + k$. Taking $T_k(u_n - \varphi)^+$ as a test function in (4.1), we obtain

$$\begin{aligned}
\int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx &= \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \varphi)^+ \, dx \\
&\le \int_{\{\varphi < u_n < \varphi + M\}} |a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla (T_M(u_n) - \varphi)| \, dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx \le& \int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla \varphi| \, dx \\
&+ \int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)| \, dx.
\end{aligned} \tag{4.35}$$

For the first term on the right-hand side of (4.35), using (3.8), the Hardy's inequality and Young's inequality, we find

$$\begin{aligned}
\int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla \varphi| \, dx \le& \sum_{i=1}^{N} \int_\Omega \omega_i(x)^{\frac{1}{p}} \Big(k(x) + \sigma(x)^{\frac{1}{p'}} |T_M(u_n)|^{\frac{q}{p'}} \\
&+ C \sum_{j=1}^{N} \omega_j(x)^{\frac{1}{p'}} |\partial_j T_M(u_n)|^{p-1} \Big) |\partial_i \varphi| \, dx \\
\le& C \left(\|k\|_{L^{p'}(\Omega)} + \|T_M(u_n)\|_X + \|\varphi\|_X \right) \\
\le& C,
\end{aligned} \tag{4.36}$$

where C is a constant independent of n by Lemma 4.2.

Similarly, for the second term in (4.35), we find

$$\int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)|\, dx \le C. \tag{4.37}$$

Combining (4.35), (4.36) and (4.37), we obtain

$$\int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx \le C.$$

Applying Fatou's lemma to the above inequality yields that $fh(u)T_k(u-\varphi)^+ \in L^1(\Omega)$. A similar argument with $T_k(u_n - \varphi)^-$ shows that $fh(u)T_k(u-\varphi)^- \in L^1(\Omega)$, so $fh(u)T_k(u-\varphi) \in L^1(\Omega)$.

To establish assertion (3.13), we now take $T_k(u-\varphi)$ as a test function in (4.1), where $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, giving us

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \varphi)\, dx = \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)\, dx. \tag{4.38}$$

Our objective is to take the limit as $n \to \infty$ in this expression. For the left-hand side, note that $\nabla T_k(u_n - \varphi)$ is nonzero only on the set $\{|u_n - \varphi| < k\}$. Setting $M = \|\varphi\|_{L^\infty(\Omega)} + k$, Lemma 4.3 shows that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } X \text{ as } n \to \infty.$$

Hence, by (3.8),

$$a(x, T_M(u_n), \nabla T_M(u_n)) \to a(x, T_M(u), \nabla T_M(u)) \quad \text{strongly in } \prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'}) \text{ as } n \to \infty. \tag{4.39}$$

Thus,

$$\lim_{n\to\infty} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \varphi)\, dx = \int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi)\, dx. \tag{4.40}$$

For the right-hand side, if $h(0) < \infty$, the limit follows by the Dominated Convergence Theorem. If $h(0) = \infty$, we decompose the right-hand side of (4.38) as

$$\int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)\, dx = \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx + \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^- \, dx.$$

Now, we pass to the limit in the first term on the right-hand side of the previous inequality, and we can treat the second term in a similar manner.

Let β be small enough such that $\beta \notin \{\eta > 0 : \text{meas}(\{u = \eta\}) > 0\}$, which is at most a countable set. By taking $(u_n - \varphi)^+ V_\beta(u_n)$ as a test function in (4.1) and disregarding the positive terms, we obtain

$$\int_{\{u_n \le \beta\}} f_n h_n(u_n)(u_n - \varphi)^+ \, dx \le \int_\Omega V_\beta(u_n) a(x, T_n(u_n), \nabla u_n) \cdot \nabla (u_n - \varphi)^+ \, dx. \tag{4.41}$$

Using (3.8) and Lemma 4.2, it is straightforward to show that $V_\beta(u_n) a(x, T_n(u_n), \nabla u_n)$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$. Hence, $V_\beta(u_n) a(x, T_n(u_n), \nabla u_n)$ converges weakly to $V_\beta(u) a(x, u, \nabla u)$ in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^{1-p'})$.

Moreover, $T_{2\beta}(u_n)$ converges strongly to $T_{2\beta}(u)$ in X as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \int_{\{u_n \le \beta\}} f_n h_n(u_n)(u_n - \varphi)^+ \, dx \le \int_\Omega V_\beta(u) a(x, u, \nabla u) \cdot \nabla (u - \varphi)^+ \, dx = C_\beta, \tag{4.42}$$

where

$$\lim_{\beta\to 0^+} C_\beta = \int_{\{u=0\}} a(x,u,\nabla u)\cdot\nabla(u-\varphi)^+\,dx = 0, \tag{4.43}$$

since $a(x,0,0)=0$ for almost every $x\in\Omega$.

We decompose further as follows

$$\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx = \int_{\{u_n\le\beta\}} f_n h_n(u_n)(u_n-\varphi)^+\,dx + \int_{\{u_n>\beta\}} f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx \tag{4.44}$$

for $k>\beta$. Regarding the second term on the right-hand side of the previous inequality, we find that

$$f_n h_n(u_n)T_k(u_n-\varphi)^+ \le k \sup_{s\in(\beta,\infty)} h(s)\,f,$$

by the Lebesgue Theorem, we deduce

$$\lim_{n\to\infty}\int_{\{u_n>\beta\}} f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx = \int_{\{u>\beta\}} f h(u)T_k(u-\varphi)^+\,dx.$$

Furthermore, we have already shown that $fh(u)T_k(u-\varphi)^+\in L^1(\Omega)$. Thus, a second application of the Lebesgue Theorem yields

$$\lim_{\beta\to 0^+}\lim_{n\to\infty}\int_{\{u_n>\beta\}} f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx = \int_{\{u>\beta\}} f h(u)T_k(u-\varphi)^+\,dx, \tag{4.45}$$

since it follows from $fh(u)T_k(u-\varphi)\in L^1(\Omega)$ that $\{u=0\}\subset\{f=0\}$.

Taking the limits as $n\to\infty$ and then $\beta\to 0^+$ in (4.44), by using (4.42), (4.43), and (4.45), we obtain

$$\lim_{n\to\infty}\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^+\,dx = \int_\Omega f h(u)T_k(u-\varphi)^+\,dx.$$

Finally, by reasoning in the same manner as before, we conclude that

$$\lim_{n\to\infty}\int_\Omega f_n h_n(u_n)T_k(u_n-\varphi)^-\,dx = \int_\Omega f h(u)T_k(u-\varphi)^-\,dx,$$

which is sufficient to pass to the limit on the right-hand side of (4.38), thus proving assertion (3.13).

So, with this last step the proof of Theorem 3.1 is concluded.

5. Example

In this final section, we will provide an illustrative example of our problem. To do so, we will take $\Omega = B_{\mathbb{R}^N}(0,1)$, where $B_{\mathbb{R}^N}(0,1)$ is the unit ball of $\mathbb{R}^N$. Consider the Carathéodory function:

$$a_i(x,s,\xi) = b(s)w_i\,|\xi_i|^{p-1}\operatorname{sgn}(\xi_i) \qquad \text{for } i=1,\dots,N$$

where,

$$b(s) = \frac{1}{(1+|s|)^{p-1}}$$

and $w_i(x)$ are a given weight functions strictly positive almost everywhere in Ω.

We will assume that the weight functions fulfill the following condition:

$$w_i(x) = w(x), \quad x\in\Omega, \qquad \text{for all } i=0,\dots,N.$$

Then we can consider the Hardy inequality (3.3) in the form,

$$\left(\int_\Omega |u(x)|^q\sigma(x)dx\right)^{1/q} \le c\left(\int_\Omega |\nabla u(x)|^p w\right)^{1/p}.$$

It is straightforward to demonstrate that the $a_i(x,s,\xi)$ are Carathéodory functions that satisfy the growth condition (3.8) and the degenerate coercivity (3.6). On the other hand, the monotonicity condition (3.9)is verified. In fact,

$$\sum_{i=1}^{N}\Big(a_i(x,s,\xi)-a_i(x,s,\hat{\xi})\Big)\cdot\Big(\xi_i-\hat{\xi}_i\Big)=b(s)w\sum_{i=1}^{N}\Big(|\xi_i|^{p-1}\operatorname{sgn}\xi_i-\Big|\hat{\xi}_i\Big|^{p-1}\operatorname{sgn}\hat{\xi}_i\Big)\cdot\Big(\xi_i-\hat{\xi}_i\Big)$$
$$>0$$

for almost all $x\in\Omega$ and for all $\xi,\hat{\xi}\in\mathbb{R}^N$ with $\xi\neq\hat{\xi}$, since $w>0$ a.e. in Ω and $b(s)>0$. In particular, let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x)=\operatorname{dist}(x,\partial\Omega)$ and set

$$w(x)=d^{\varepsilon}(x),\quad \sigma(x)=d^{\kappa}(x).$$

In this case, the Hardy inequality reads

$$\left(\int_\Omega |u(x)|^q d^{\kappa}(x)dx\right)^{1/q}\leq c\left(\int_\Omega |\nabla u(x)|^p d^{\varepsilon}(x)dx\right)^{1/p}.$$

The corresponding embedding (3.4) is compact if:

(i) For $1<p\leq q<\infty$,

$$\varepsilon<p-1,\quad \frac{N}{q}-\frac{N}{p}+1\geq 0,\quad \frac{\kappa}{q}-\frac{\varepsilon}{p}+\frac{N}{q}-\frac{N}{p}+1>0. \tag{5.1}$$

(ii) For $1\leq q<p<\infty$,

$$\varepsilon<p-1,\quad \frac{\kappa}{q}-\frac{\varepsilon}{p}+\frac{1}{q}-\frac{1}{p}+1>0. \tag{5.2}$$

(iii) For $q>1$,

$$\kappa\left(q'-1\right)<1. \tag{5.3}$$

Remark 5.1 *Condition* (5.1) *or* (5.2) *are sufficient for the compact embedding* (3.4) *to hold; [see for example [11] or [24]], while condition* (5.2) *is sufficient for* (3.2) *to hold* [*[17], pp. 40-41*].

Finally, if we take, for example,

$$f(x)=\exp^{-|x|^2}\quad\text{and}\quad h(s)=\frac{1}{\sqrt{s}}\quad\text{for}\quad s>0,$$

then, all hypotheses of Theorem3.1 are satisfied. Therefore, problem (1.1) has at least one solution.

Acknowledgment

The authors would like to thank the anonymous reviewer for offering valuable suggestions to improve this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability Statement

The manuscript has no associated data.

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