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Fourth Hankel Determinant for Functions Associated with Exponential Function with Respect to Symmetric Points

Govinda Raju R.* and Ruby Salestina M.

ABSTRACT: In this paper, we investigate the co-efficient bound of the fourth Hankel determinant for a specific class of functions associated with exponential function.

Key Words: Subordination, Feketo-Szego inequality, coefficient bounds, Hankel determinant, symmetric points.

Contents

1. Introduction

The class of analytic functions with the normalizing conditions $f(0) = 0$ and $f'(0) = 1$ in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$ and denote it by $\mathcal{A}$. This class contains the functions of the form given below.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathcal{U}. \tag{1.1}$$

The subclass of $\mathcal{A}$ in which all its members are one-one is the class of univalent functions and denote it by $\mathcal{S}$. Another subclass of $\mathcal{A}$ is the class $\mathcal{P}$ of functions with positive real part.The members of this class are of the form.

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n, \ z \in \mathcal{U}. \tag{1.2}$$

Let the functions f and $g \in \mathcal{A}$. The function f is said to be subordinate to g, written as $f \prec g$, if there exists a Schwartz function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| \leq 1$ such that $f(z) = g(w(z))$, $z \in \mathcal{U}$.

We note that $f(z) \prec g(z) \implies f(\mathcal{U}) \prec g(\mathcal{U})$. Furthermore if the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$).

Feketo-Szegö inequalities of functions of various subclasses of $\mathcal{A}$ have been studied extensively by various authors in ([1], [6], [10], [17], [19]). Such an inequality is concerned about finding the greatest value of the quantity $|a_3 - \mu a_2^2|$, where μ is either real or complex. This is called as Hankel determinant for that class given as follows.

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}.$$

* Corresponding author.
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For a function $f \in \mathcal{A}$ the q^{th} Hankel determinant $H_q(n)$, with $q \geq 1$ and $n \geq 1$, was studied by Noonan and Thomas [20] and it is defined by Pommerenke [22] as follows.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In 2007, Babalola considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of univalent functions. Many authors have studied the Hankel determinant $H_4(1)$ where $q = 4$ and $n = 1$ in the above definition is the fourth Hankel determinant for some subclasses of univalent functions given by the following.

$$H_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}$$

$$\begin{aligned} H_4(1) = &\left(a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2)\right) \\ &-a_2\left(a_2(a_5a_7 - a_6^2) - a_4(a_3a_7 - a_4a_6) + a_5(a_3a_6 - a_4a_5)\right) \\ &+a_3\left(a_2(a_4a_7 - a_5a_6) - a_3(a_3a_7 - a_4a_6) + a_5(a_3a_5 - a_4^2)\right) \\ &-a_4\left(a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2)\right). \end{aligned}$$

Several authors ([4], [21]) have worked on the upper bounds of Hankel determinants of various subclasses of $\mathcal{A}$. Mendiratta [18] introduced and studied the class of starlike functions associated with exponential functions $S_e^* = S^*(e^z)$. Hai-Yan Zhang et al. [12] very recently investigated the upper bound of the third Hankelde terminal for the function class S_e^* associated with exponential function in 2018.

In their work [16], Ganesh Koride et al. have defined the following subclass associated with the exponential function.

Definition 1.1 *[18] A function $f \in \mathbb{S}^*(e^z)$ if and only if*

$$\frac{2[zf'(z)]}{f(z) - f(-z)} \prec e^z, \text{ for all } z \in \mathcal{U}. \tag{1.3}$$

The non-sharp bounds for the third Hankel determinant of starlike and convex functions related to the exponential function, as established in [[16]]. In our work, we have find the upper bound for fourth Hankel determinant of the same class defined above.

2. Preliminaries

To prove our main results we need the following lemmas.

Lemma 2.1 *[22] If $p \in \mathcal{P}$ and has the above form, then*

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1^2|}{2}.$$

Lemma 2.2 *[22] If $p \in \mathcal{P}$ then $|p_{n+k} - \mu p_n p_k| < 2$ for $0 \leq \mu \leq 1$.*

Lemma 2.3 *[22] If $p \in \mathcal{P}$ then $|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2(|J| + |K - 2J| + |J - K + L|)$.*

To find the bound of fourth hankel determinant, we need to have bounds of some inequalities such as $|a_5a_7 - a_6^2|$, $|a_4a_7 - a_5a_6|$, $|a_4a_6 - a_5^2|$, $|a_3a_7 - a_4a_6|$, $|a_3a_6 - a_4a_5|$ and $|a_3a_5 - a_4^2|$. In this section we find bounds of these inequalities.

3. Main Results

Theorem 3.1 *If $f \in \mathcal{S}_s^*(e^z)$, then*

I. $|a_5a_7 - a_6^2| \leq \dfrac{12395}{55296}$

II. $|a_4a_7 - a_5a_6| \leq \dfrac{11339}{23040}$

III. $|a_4a_6 - a_5^2| \leq \dfrac{11}{64}$

IV. $|a_3a_7 - a_4a_6| \leq \dfrac{13417}{34560}$

V. $|a_3a_6 - a_4a_5| \leq \dfrac{355}{1152}$

VI. $|a_3a_5 - a_4^2| \leq \dfrac{49}{192}$

Proof:

Since $f \in \mathcal{S}_s^*(e^z)$, with the help of principle of subordination we have

$$\frac{2[zf'(z)]}{f(z)-f(-z)} = e^{w(z)}. \tag{3.1}$$

Let $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$
which is analytic in the given domain with $p(0) = 0$ and maps unit disc onto the right half of the w-plane. By simple computations we can write $w(z)$ in terms of $p(z)$.

$$\begin{aligned} w(z) = \frac{p_1z}{2} &+ \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)z^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8} - \frac{p_1p_2}{2}\right)z^3 + \left(\frac{p_4}{2} - \frac{p_2^2}{4} - \frac{p_1^4}{16} + \frac{3p_1^2p_2}{8} - \frac{p_1p_3}{2}\right)z^4 \\ &+ \left(\frac{p_5}{2} - \frac{p_1p_4}{2} + \frac{3p_1p_2^2}{8} + \frac{3p_1^2p_3}{8} - \frac{p_1^3p_2}{4} + \frac{p_1^5}{32} - \frac{p_2p_3}{2}\right)z^5 \\ &+ \left(\frac{p_6}{2} - \frac{p_1p_5}{2} + \frac{3p_1^2p_4}{8} + \frac{3p_1p_2p_3}{8} - \frac{p_1^3p_3}{4} - \frac{3p_1^2p_2^2}{8} + \frac{5p_1^4p_2}{32} - \frac{p_1^6}{64} - \frac{p_2p_4}{2} + \frac{p_2^3}{8} - \frac{p_3^2}{4}\right)z^6 + \cdots. \end{aligned}$$

Consider, $e^{w(z)} = 1 + w(z) + \dfrac{(w(z))^2}{2!} + \dfrac{(w(z))^3}{3!} + \dfrac{(w(z))^4}{4!} + \dfrac{(w(z))^5}{5!} + \dots$.
After substituting and equating corresponding coefficients, we get

$1 + 2a_2z + 2a_3z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 + (6a_6 - 4a_3a_4 + 2a_2a_3^2 - 2a_2a_5)z^5 + (6a_7 - 6a_3a_5 +$

$2a_3^3)z^6 + \dots$

$$\begin{aligned}
=&1 + \frac{1}{2}p_1 z + \left(\frac{1}{2}p_2 - \frac{1}{8}p_1^2\right) z^2 + \left(-\frac{1}{4}p_1p_2 + \frac{1}{2}p_3 + \frac{1}{48}p_1^3\right) z^3 \\
&+ \left(-\frac{1}{4}p_1p_3 + \frac{1}{2}p_4 - \frac{1}{8}p_2^2 + \frac{1}{16}p_2p_1^2 + \frac{1}{384}p_1^4\right) z^4 \\
&+ \left(-\frac{1}{4}p_1p_4 + \frac{1}{16}p_1p_2^2 - \frac{19}{3840}p_1^5 + \frac{1}{2}p_5 - \frac{1}{4}p_3p_2 + \frac{1}{16}p_3p_1^2 + \frac{1}{96}p_2p_1^3\right) z^5 \\
&+ \left(\frac{1}{8}p_1p_3p_2 + \frac{1}{64}p_1^2p_2^2 + \frac{1}{16}p_1^2p_4 - \frac{1}{4}p_1p_5 + \frac{1}{48}p_2^3 + \frac{151}{46080}p_1^6 + \frac{1}{2}p_6\right. \\
&\qquad \left. -\frac{1}{4}p_4p_2 - \frac{1}{8}p_3^2 + \frac{1}{96}p_3p_1^3 - \frac{19}{768}p_2p_1^4\right) z^6 \\
&+ \left(\frac{1}{32}p_3p_1^2p_2 + \frac{1}{8}p_1p_4p_2 - \frac{19}{384}p_1^3p_2^2 + \frac{1}{96}p_1^3p_4 + \frac{1}{16}p_1^2p_5 + \frac{1}{96}p_1p_2^3\right. \\
&\qquad + \frac{1}{16}p_1p_3^2 - \frac{1}{4}p_1p_6 - \frac{1091}{645120}p_1^7 + \frac{1}{2}p_7 - \frac{1}{4}p_2p_5 + \frac{1}{16}p_3p_2^2 \\
&\qquad \left. + \frac{151}{7680}p_2p_1^5 - \frac{1}{4}p_3p_4 - \frac{19}{768}p_3p_1^4\right) z^7 + \dots
\end{aligned}$$

which implies,

$$a_1 = 1 \tag{3.2}$$

$$a_2 = \frac{p_1}{4} \tag{3.3}$$

$$a_3 = \frac{p_2}{4} - \frac{p_1^2}{16} \tag{3.4}$$

$$a_4 = -\frac{p_1^3}{384} - \frac{p_1p_2}{32} + \frac{p_3}{8} \tag{3.5}$$

$$a_5 = \frac{p_1^4}{384} - \frac{p_1p_3}{16} + \frac{p_4}{8} \tag{3.6}$$

$$a_6 = \frac{-19p_1^5}{23040} + \frac{p_1^3p_2}{192} - \frac{p_1p_4}{32} - \frac{p_2p_3}{48} + \frac{p_5}{12} \tag{3.7}$$

$$a_7 = \frac{257p_1^6}{552960} - \frac{41p_1^4p_2}{9216} + \frac{13p_1^3p_3}{2304} + \frac{5p_1^2p_2^2}{768} - \frac{p_2^3}{576} + \frac{p_1^2p_4}{384} - \frac{p_3^2}{48} - \frac{p_1p_5}{24} - \frac{p_2p_4}{96} + \frac{p_6}{12} + \frac{p_1p_2p_3}{192} \tag{3.8}$$

By using Lemmas 2.1, 2.2 and 2.3, we get

$$\begin{aligned}
&|a_2| \le \frac{1}{2}, && |a_3| \le \frac{1}{2}, && |a_4| \le \frac{22}{96}, \\
&|a_5| \le \frac{7}{24}, && |a_6| \le \frac{259}{720}, && |a_7| \le \frac{367}{432}.
\end{aligned}$$

From equations (3.6), (3.7) and (3.8), we can write

$$
\begin{aligned}
(a_5a_7-a_6^2)=&\left(\frac{p_1^4}{384}-\frac{p_1p_3}{16}+\frac{p_4}{8}\right)\left(\frac{257p_1^6}{552960}-\frac{41p_1^4p_2}{9216}+\frac{13p_1^3p_3}{2304}+\frac{5p_1^2p_2^2}{768}-\frac{p_2^3}{576}+\frac{p_1^2p_4}{384}-\frac{p_3^2}{48}\right.\\
&\left.-\frac{p_1p_5}{24}-\frac{p_2p_4}{96}+\frac{p_6}{12}+\frac{p_1p_2p_3}{192}\right)-\left(\frac{-19p_1^5}{23040}+\frac{p_1^3p_2}{192}-\frac{p_1p_4}{32}-\frac{p_2p_3}{48}+\frac{p_5}{12}\right)^2.\\
=&-\frac{1}{98304}p_1^6p_2^2+\frac{1}{34560}p_1^5p_5-\frac{1}{221184}p_1^4p_2^3+\frac{1}{4608}p_1^4p_6-\frac{53}{17694720}p_1^8p_2-\frac{1}{144}p_5^2\\
&-\frac{1}{2304}p_2^2p_3^2-\frac{127}{8847360}p_1^7p_3+\frac{1}{768}p_1p_3^3-\frac{5}{12288}p_1^4p_3^2-\frac{1}{1536}p_1^2p_4^2-\frac{1}{4608}p_4p_2^3\\
&+\frac{1}{96}p_4p_6+\frac{59}{4423680}p_4p_1^6-\frac{1}{768}p_4^2p_2-\frac{1}{384}p_4p_3^2+\frac{563}{1061683200}p_1^{10}+\frac{1}{9216}p_1p_3p_2^3\\
&-\frac{1}{3072}p_1^2p_3^2p_2+\frac{5}{6144}p_4p_1^2p_2^2-\frac{19}{73728}p_4p_2p_1^4-\frac{7}{36864}p_1^3p_3p_2^2+\frac{569}{2211840}p_1^5p_3p_2\\
&-\frac{1}{192}p_1p_3p_6+\frac{1}{288}p_5p_2p_3+\frac{5}{9216}p_1^3p_3p_4+\frac{1}{384}p_1^2p_3p_5-\frac{1}{1152}p_5p_1^3p_2.\\
=&p_1^2p_5\left(\frac{1}{34560}p_1^3-\frac{1}{1152}p_1p_2+\frac{1}{384}p_3\right)-\frac{p_3^2}{384}\left(p_4-\frac{384}{768}p_1p_3\right)-\frac{p_1^2p_4^4}{1536}\\
&+p_1^7\left(\frac{563}{1061683200}p_1^3-\frac{53}{17694720}p_1p_2-\frac{127}{8847360}p_3\right)+\frac{p_4}{96}\left(p_6-\frac{96}{768}p_2p_4\right)\\
&-p_2^2p_3\left(\frac{7}{36864}p_1^3-\frac{1}{9216}p_1p_2+\frac{1}{2304}p_3\right)-\frac{1}{144}p_5\left(p_5-\frac{144}{288}p_2p_3\right)-\frac{p_1p_3p_6}{192}\\
&-\frac{5}{12288}p_1^4p_3\left(p_3-\frac{569}{900}p_1p_2\right)-\frac{p_1^6p_2^2}{98304}-\frac{p_1^4p_2^3}{221184}+\frac{p_1^4p_6}{4608}-\frac{p_2^3p_4}{4608}-\frac{p_1^2p_3^2p_2}{3072}\\
&+p_4p_1^3\left(\frac{59}{4423680}p_1^3-\frac{19}{73728}p_1p_2+\frac{5}{9216}p_3\right)+\frac{5}{6144}p_1^2p_2^2p_4
\end{aligned}
$$

Now by applying triangular inequality to the above equation we obtain

$$
\begin{aligned}
|a_5a_7-a_6^2|\leq&\left|p_1^2p_5\right|\left|\frac{1}{34560}p_1^3-\frac{1}{1152}p_1p_2+\frac{1}{384}p_3\right|+\left|\frac{p_3^2}{384}\right|\left|p_4-\frac{384}{768}p_1p_3\right|+\left|\frac{p_1^2p_4^4}{1536}\right|\\
&+\left|p_1^7\right|\left|\frac{563}{1061683200}p_1^3-\frac{53}{17694720}p_1p_2-\frac{127}{8847360}p_3\right|+\left|\frac{p_4}{96}\right|\left|p_6-\frac{96}{768}p_2p_4\right|\\
&+\left|p_2^2p_3\right|\left|\frac{7}{36864}p_1^3-\frac{1}{9216}p_1p_2+\frac{1}{2304}p_3\right|+\left|\frac{1}{144}p_5\right|\left|p_5-\frac{144}{288}p_2p_3\right|+\left|\frac{p_1p_3p_6}{192}\right|\\
&+\left|\frac{5}{12288}p_1^4p_3\right|\left|p_3-\frac{569}{900}p_1p_2\right|+\left|\frac{p_1^6p_2^2}{98304}\right|+\left|\frac{p_1^4p_2^3}{221184}\right|+\left|\frac{p_1^4p_6}{4608}\right|+\left|\frac{p_2^3p_4}{4608}\right|+\left|\frac{p_1^2p_3^2p_2}{3072}\right|\\
&+\left|p_4p_1^3\right|\left|\frac{59}{4423680}p_1^3-\frac{19}{73728}p_1p_2+\frac{5}{9216}p_3\right|+\left|\frac{5}{6144}p_1^2p_2^2p_4\right|
\end{aligned}
$$

By using Lemma 2.3, we can simplify the inequalities and we get

$$
\left|\frac{1}{34560}p_1^3-\frac{1}{1152}p_1p_2+\frac{1}{384}p_3\right|\leq 2\left(\frac{1}{34560}+\left|\frac{1}{1152}-\frac{2}{34560}\right|+\left|\frac{1}{34560}-\frac{1}{1152}+\frac{1}{384}\right|\right)\leq\frac{1}{192}
$$

$$
\begin{aligned}
\left|\frac{59}{4423680}p_1^3-\frac{19}{73728}p_1p_2+\frac{5}{9216}p_3\right|&\leq 2\left(\frac{59}{4423680}+\left|\frac{19}{73728}-\frac{118}{4423680}\right|+\left|\frac{59}{442368}-\frac{19}{73728}+\frac{5}{9216}\right|\right)\\
&\leq\frac{977}{737280}
\end{aligned}
$$

$$\begin{aligned}\Big|\frac{563}{1061683200}p_1^3-\frac{53}{17694720}p_1p_2-\frac{127}{8847360}p_3\Big| &\le 2\Bigg(\frac{563}{1061683200}+\Big|\frac{53}{17694720}-\frac{1126}{1061683200}\Big|\\ &\quad+\Big|\frac{563}{1061683200}-\frac{53}{17694720}-\frac{127}{8847360}\Big|\Bigg)\\ &\le \frac{1}{203}\end{aligned}$$

$$\Big|\frac{7}{36864}p_1^3-\frac{1}{9216}p_1p_2+\frac{1}{2304}p_3\Big| \le \left(\frac{7}{36864}+\Big|\frac{7}{36864}-\frac{2}{9216}\Big|+\Big|\frac{7}{36864}-\frac{1}{9216}+\frac{1}{2304}\Big|\right) \le \frac{3}{2048}$$

By using Lemma 2.2, we can simplify the inequalities and we get

$\Big|p_4-\frac{384}{768}p_1p_3\Big| \le 2$

$\Big|p_6-\frac{96}{768}p_2p_4\Big| \le 2$

$\Big|p_5-\frac{144}{288}p_2p_3\Big| \le 2$

$\Big|p_3-\frac{569}{900}p_1p_2\Big| \le 2$

Hence by using the Lemma 2.1, we get

$$|(a_5a_7-a_6^2)| \le \frac{19}{70}.$$

Now lets find the bound of second inequality:

$$\begin{aligned}(a_4a_7-a_5a_6) &= \left(-\frac{p_1^3}{384}-\frac{p_1p_2}{32}+\frac{p_3}{8}\right)\left(\frac{257p_1^6}{552960}-\frac{41p_1^4p_2}{9216}+\frac{13p_1^3p_3}{2304}+\frac{5p_1^2p_2^2}{768}-\frac{p_2^3}{576}+\frac{p_1^2p_4}{384}-\frac{p_3^2}{48}-\frac{p_1p_5}{24}\right.\\ &\quad\left.-\frac{p_2p_4}{96}+\frac{p_6}{12}+\frac{p_1p_2p_3}{192}\right)-\left(\frac{p_1^4}{384}-\frac{p_1p_3}{16}+\frac{p_4}{8}\right)\left(\frac{-19p_1^5}{23040}+\frac{p_1^3p_2}{192}-\frac{p_1p_4}{32}-\frac{p_2p_3}{48}+\frac{p_5}{12}\right).\\ &=\frac{1}{768}p_1^2p_2p_5-\frac{1}{384}p_1p_2p_6+\frac{1}{768}p_2p_4p_3-\frac{3}{8192}p_1^4p_2p_3+\frac{1}{3072}p_1p_2^2p_4+\frac{1}{1536}p_1^2p_2^2p_3\\ &\quad-\frac{13}{18432}p_1^3p_2p_4-\frac{5}{3072}p_3p_1^2p_4-\frac{1}{384}p_3^3-\frac{127}{8847360}p_1^7p_3+\frac{1}{768}p_1p_3^3-\frac{5}{12288}p_1^4p_3^2\\ &\quad-\frac{1}{1536}p_1^2p_4^2-\frac{1}{4608}p_4p_2^3+\frac{59}{4423680}p_4p_1^6+\frac{1}{96}p_4p_6+\frac{199}{212336640}p_1^9-\frac{11}{55296}p_1^3p_2^3\\ &\quad+\frac{1}{18432}p_1p_2^4-\frac{73}{4423680}p_1^7p_2+\frac{1}{8192}p_1^5p_2^2-\frac{1}{4608}p_3p_2^3-\frac{1}{122880}p_3p_1^6+\frac{1}{96}p_3p_6\\ &\quad+\frac{7}{9216}p_3^2p_1^3+\frac{131}{737280}p_1^5p_4-\frac{1}{9216}p_1^4p_5-\frac{1}{4608}p_1^3p_6+\frac{1}{256}p_1p_4^2-\frac{1}{96}p_4p_5.\\ &=\frac{p_1^2p_2}{768}\left(p_5-\frac{768}{1536}p_2p_3\right)+p_6\left(-\frac{1}{4608}p_1^3-\frac{1}{384}p_1p_2+\frac{1}{96}p_3\right)+\frac{1}{256}p_1p_4^2\\ &\quad-\frac{1}{96}p_4\left(p_5-\frac{96}{768}p_2p_3\right)-p_1^3p_3\left(\frac{1}{122880}p_1^3-\frac{3}{8192}p_1p_2+\frac{7}{9216}p_3\right)-\frac{1}{384}p_3^3\\ &\quad+p_1^2p_4\left(\frac{131}{737280}p_1^3-\frac{13}{18432}p_1p_2-\frac{5}{3072}p_3\right)+\frac{59p_1^6}{4423680}\left(p_4-\frac{127\cdot 4423680\cdot p_1p_3}{8847360\cdot 59}\right)\\ &\quad-\frac{73p_1^7}{4423680}\left(p_2-\frac{199\cdot 4423680\cdot p_1^2}{73\cdot 212336640}\right)+p_2^3\left(-\frac{11}{55296}p_1^3+\frac{1}{18432}p_1p_2-\frac{1}{4608}p_3\right)\\ &\quad+\frac{1}{3072}p_1p_2^2p_4+\frac{1}{768}p_1p_3^3-\frac{5}{12288}p_1^4p_3^2-\frac{1}{1536}p_1^2p_4^2-\frac{1}{4608}p_4p_2^3+\frac{1}{96}p_4p_6\\ &\quad+\frac{1}{8192}p_1^5p_2^2-\frac{1}{9216}p_1^4p_5\end{aligned}$$

With the help of Lemmas 2.1, 2.2, 2.3, we lead to,

$$|a_4a_7 - a_5a_6| \leq \frac{11339}{23040}.$$

We will calculate the bound of third inequality now:
With the help of equations (3.5), (3.6) and (3.7), we can write

$$\begin{aligned}(a_4a_6 - a_5^2) &= \left(-\frac{p_1^3}{384} - \frac{p_1p_2}{32} + \frac{p_3}{8}\right)\left(\frac{-19p_1^5}{23040} + \frac{p_1^3p_2}{192} - \frac{p_1p_4}{32} - \frac{p_2p_3}{48} + \frac{p_5}{12}\right) - \left(\frac{p_1^4}{384} - \frac{p_1p_3}{16} + \frac{p_4}{8}\right)^2 \\ &= -\frac{1}{8847360}\left(p_1^3 + 12p_1p_2 - 48p_3\right)\left(-19p_1^5 + 120p_1^3p_2 - 720p_1p_4 - 480p_2p_3 + 1920p_5\right) \\ &\quad - \frac{1}{147456}\left(p_1^4 - 24p_1p_3 + 48p_4\right)^2.\end{aligned}$$

Hence, we arrive at

$$|a_4a_6 - a_5^2| \leq \frac{11}{64}.$$

To find the bound of fourth inequality we proceed as follows:
By using equations (3.4), (3.5), (3.7), and (3.8), we have

$$\begin{aligned}(a_3a_7 - a_4a_6) &= \left(\frac{p_2}{4} - \frac{p_1^2}{16}\right)\left(\frac{257p_1^6}{552960} - \frac{41p_1^4p_2}{9216} + \frac{13p_1^3p_3}{2304} + \frac{5p_1^2p_2^2}{768} - \frac{p_2^3}{576} + \frac{p_1^2p_4}{384} - \frac{p_3^2}{48} - \frac{p_1p_5}{24} - \frac{p_2p_4}{96}\right. \\ &\quad \left. + \frac{p_6}{12} + \frac{p_1p_2p_3}{192}\right) - \left(-\frac{p_1^3}{384} - \frac{p_1p_2}{32} + \frac{p_3}{8}\right)\left(\frac{-19p_1^5}{23040} + \frac{p_1^3p_2}{192} - \frac{p_1p_4}{32} - \frac{p_2p_3}{48} + \frac{p_5}{12}\right) \\ &= -\frac{1}{2304}p_2^4 - \frac{23}{737280}p_1^8 + \frac{1}{576}p_1^2p_2^3 + \frac{169}{442368}p_2p_1^6 + \frac{1}{48}p_2p_6 - \frac{1}{384}p_4p_2^2 - \frac{1}{384}p_2p_3^2 \\ &\quad - \frac{25}{18432}p_2^2p_1^4 - \frac{1}{4096}p_1^4p_4 + \frac{13}{4608}p_1^3p_5 - \frac{1}{192}p_1^2p_6 + \frac{1}{768}p_1^2p_3^2 - \frac{23}{92160}p_1^5p_3 - \frac{1}{96}p_3p_5 \\ &\quad + \frac{1}{1536}p_1p_3p_2^2 + \frac{7}{18432}p_2p_3p_1^3 + \frac{1}{256}p_1p_3p_4 + \frac{1}{3072}p_2p_1^2p_4 - \frac{1}{128}p_1p_2p_5. \\ &= -\frac{1}{2304}p_2^3\left(p_2 - \frac{2304}{576}p_1^2\right) + \frac{169}{442368}p_1^6\left(p_2 - \frac{23x442368}{737280x169}p_1^2\right) + \frac{p_2}{48}\left(p_6 - \frac{48}{384}p_2p_4\right) \\ &\quad - \frac{p_2p_3}{384}\left(p_3 - \frac{384}{1536}p_1p_2\right) - \frac{23}{92160}p_1^5p_3 - p_1p_4\left(\frac{1}{4096}p_1^3 - \frac{1}{3072}p_1p_2 - \frac{1}{256}p_3\right) \\ &\quad + p_5\left(\frac{13}{4608}p_1^3 - \frac{1}{128}p_1p_2 - \frac{1}{96}p_3\right) - \frac{1}{192}p_1^2\left(p_6 - \frac{192}{768}p_3^2\right) + \frac{7}{18432}p_2p_1^3\left(p_3 - p_1p_2\right)\end{aligned}$$

By using Lemma 2.1, Lemma 2.2 and Lemma 2.3, we can simplify the inequality and we get,

$$|a_3a_7 - a_4a_6| \leq \frac{13417}{34560}.$$

Now we calculate the bound value of fifth inequality:
From equations (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}(a_3a_6 - a_4a_5) &= \left(\frac{p_2}{4} - \frac{p_1^2}{16}\right)\left(\frac{-19p_1^5}{23040} + \frac{p_1^3p_2}{192} - \frac{p_1p_4}{32} - \frac{p_2p_3}{48} + \frac{p_5}{12}\right) - \left(-\frac{p_1^3}{384} - \frac{p_1p_2}{32} + \frac{p_3}{8}\right) \\ &\quad \left(\frac{p_1^4}{384} - \frac{p_1p_3}{16} + \frac{p_4}{8}\right).\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{368640}\left(p_1^2-4p_2\right)\left(-19p_1^5+120p_1^3p_2-720p_1p_4-480p_2p_3+1920p_5\right)\\
&\quad+\frac{1}{147456}\left(p_1^3+12p_1p_2-48p_3\right)\left(p_1^4-24p_1p_3+48p_4\right).\\
&= -\frac{1}{256}p_1p_2p_4-\frac{83}{184320}p_2p_1^5+\frac{1}{48}p_2p_5-\frac{1}{192}p_3p_2^2+\frac{1}{768}p_2^2p_1^3+\frac{7}{3072}p_1^3p_4+\frac{43}{737280}p_1^7\\
&\quad-\frac{1}{192}p_1^2p_5-\frac{1}{1536}p_1^2p_2p_3+\frac{1}{128}p_1p_3^2-\frac{1}{64}p_3p_4-\frac{1}{2048}p_3p_1^4\\
&=p_4\left(\frac{7}{3072}p_1^3-\frac{1}{256}p_1p_2-\frac{1}{64}p_3\right)-p_1^2p_2\left(\frac{83}{184320}p_1^3-\frac{1}{768}p_1p_2+\frac{1}{1536}p_3\right)\\
&\quad+\frac{p_2}{48}\left(p_5-\frac{48}{192}p_2p_3\right)+\frac{43}{737280}p_1^7-\frac{1}{192}p_1^2p_5+\frac{1}{128}p_1p_3^2-\frac{1}{2048}p_3p_1^4
\end{aligned}
$$

By using Lemma 2.1, Lemma 2.2 and Lemma 2.3, we can simplify the inequality and we get

$$|a_3a_6-a_4a_5|\le\frac{355}{1152}.$$

Now we find the last inequality which is required to find the bound fourth hankel determinant: By using equations (3.4), (3.5) and (3.6), we can write

$$
\begin{aligned}
(a_3a_5-a_4^2)&=\left(\frac{p_2}{4}-\frac{p_1^2}{16}\right)\left(\frac{p_1^4}{384}-\frac{p_1p_3}{16}+\frac{p_4}{8}\right)-\left(-\frac{p_1^3}{384}-\frac{p_1p_2}{32}+\frac{p_3}{8}\right)^2\\
&=\frac{-1}{128}p_1p_2p_3+\frac{1}{32}p_2p_4+\frac{1}{2048}p_1^4p_2+\frac{7}{1536}p_1^3p_3-\frac{1}{128}p_1^2p_4-\frac{25}{147456}p_1^6-\frac{1}{1024}p_1^2p_2^2\\
&\quad-\frac{1}{64}p_3^2\\
&=p_3\left(\frac{7}{1536}p_1^3-\frac{1}{128}p_1p_2-\frac{1}{64}p_3\right)+\frac{p_1^4}{2048}\left(p_2-\frac{25x2048}{147456}p_1^2\right)+\frac{p_4}{32}\left(p_2-\frac{32}{128}p_1^2\right)\\
&\quad-\frac{1}{1024}p_1^2p_2^2
\end{aligned}
$$

By using Lemmas 2.1, 2.2, 2.3 we obtain,

$$|a_3a_5-a_4^2|\le\frac{49}{192}.$$

□

In the next theorem, we find the bound of fourth hankel determinant for which we use the results of above theorem.

Theorem 3.2 *If $f\in\mathcal{S}_s^*(e^z)$, then* $|H_4(1)|\le\frac{29}{19}$.

Proof: By definition of fourth hankel determinant, we have

$$
\begin{aligned}
H_4(1)=&\left(a_3(a_5a_7-a_6^2)-a_4(a_4a_7-a_5a_6)+a_5(a_4a_6-a_5^2)\right)\\
&-a_2\left(a_2(a_5a_7-a_6^2)-a_4(a_3a_7-a_4a_6)+a_5(a_3a_6-a_4a_5)\right)\\
&+a_3\left(a_2(a_4a_7-a_5a_6)-a_3(a_3a_7-a_4a_6)+a_5(a_3a_5-a_4^2)\right)\\
&-a_4\left(a_2(a_4a_6-a_5^2)-a_3(a_3a_6-a_4a_5)+a_4(a_3a_5-a_4^2)\right)
\end{aligned}
$$

Then,

$$\begin{aligned}|H_4(1)| \leq &\left(|a_3||(a_5a_7-a_6^2)|+|a_4||(a_4a_7-a_5a_6)|+|a_5||(a_4a_6-a_5^2)|\right)\\ &+|a_2|\left(|a_2||(a_5a_7-a_6^2)|+|a_4||(a_3a_7-a_4a_6)|+|a_5||(a_3a_6-a_4a_5)|\right)\\ &+|a_3|\left(|a_2||(a_4a_7-a_5a_6)|+|a_3||(a_3a_7-a_4a_6)|+|a_5||(a_3a_5-a_4^2)|\right)\\ &+|a_4|\left(|a_2||(a_4a_6-a_5^2)|+|a_3||(a_3a_6-a_4a_5)|+|a_4||(a_3a_5-a_4^2)|\right)\end{aligned}$$

By substituting the bounds of $|a_2|$, $|a_3|$,$|a_4|$, $|a_5|$ and using the above Theorem, we obtain

$$H_4(1)| \leq \frac{29}{19}.$$

4. Conclusion

In this study, we have derived the initial coefficient bounds and established a non-sharp bound for the fourth Hankel determinant for a specific subclass of analytic functions associated with the exponential function. The results obtained extend and complement existing findings in the literature concerning coefficient inequalities and Hankel determinants for various classes of analytic and univalent functions. Although the derived bound for the fourth Hankel determinant is not sharp, it provides a useful estimate that may serve as a foundation for obtaining sharper bounds in future investigations. Further research could focus on improving these estimates or exploring similar bounds for higher-order Hankel determinants within this and related subclasses of analytic functions.

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Govinda Raju R.,
Department of Mathematics,
Vidyavardhaka college of Engineering, Mysuru,
India.
E-mail address: govindarajurycm@gmail.com

and

Ruby Salestina M.,
Department of Mathematics,
Yuvaraja's College, Mysuru,
India.
E-mail address: ruby.salestina@gmail.com