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Interpolative Reich–Rus–Ćirić Type Contractions on G-Metric Spaces with Application

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ABSTRACT: This paper introduces a novel class of contractions called interpolative Reich–Rus–Ćirić type contractions within the framework of G-metric spaces. We establish comprehensive fixed point theorems for these newly defined contractions and demonstrate their practical applications through concrete examples. Our findings significantly extend and unify several existing results in the literature concerning fixed point theory in generalized metric spaces. The interpolative approach we present offers a powerful unified framework that naturally encompasses various classical contraction conditions as special cases, providing mathematicians with a more flexible and general tool for fixed point analysis.

Keywords: G-metric spaces, interpolative contractions, Reich–Rus–Ćirić contractions, fixed point theorems, generalized metric spaces.

Contents

1. Introduction

The journey of fixed point theory began with Banach's groundbreaking work in 1922 [2], which laid the foundation for what would become one of the most influential areas in functional analysis and topology. Since then, the classical Banach contraction principle has inspired countless researchers to explore new directions, leading to a remarkably rich theory with far-reaching applications in differential equations, optimization theory, and numerical analysis [15].

In 2006, Mustafa and Sims [10] introduced an innovative generalization of metric spaces called G-metric spaces, where the distance function elegantly takes three arguments instead of the traditional two. This seemingly simple modification has proven to be extraordinarily useful in studying fixed point theorems and has captured the attention of mathematicians worldwide, spawning numerous research investigations [11], [17].

More recently, Karapınar [7] brought forward the fascinating concept of interpolative contractions, which provides researchers with a unified approach to various contraction conditions. What makes the interpolative framework particularly appealing is its ability to combine different distance terms in a multiplicative manner, creating more flexible and mathematically elegant contraction conditions [4], [12].

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In this research, we bring together these powerful concepts by introducing interpolative Reich–Rus–Ćirić type contractions specifically designed for G-metric spaces. The Reich–Rus–Ćirić contraction, originally developed for metric spaces through the pioneering work of Reich [13], [14], Rus [16], and [3], cleverly involves a combination of distances between points and their images under the mapping transformation.

1.1. Research Motivation

Our research is driven by several compelling motivations that address current gaps in the mathematical literature: First, there exists a genuine need to extend classical contraction conditions to the more general setting of G-metric spaces, which can capture relationships that traditional metric spaces cannot adequately describe. Second, we recognize the significant value in unifying various contraction types under the elegant interpolative framework, creating a more cohesive mathematical theory. Third, the potential applications in solving complex functional equations and differential problems make this research practically relevant.

2. Mathematical Foundations

2.1. Understanding G-Metric Spaces

The concept of G-metric spaces, elegantly introduced by Mustafa and Sims [10], represents a natural and powerful generalization of traditional metric spaces.

Definition 2.1 *[10] Let X be a non-empty set. A function $G : X \times X \times X \to [0, \infty)$ is called a* ***G-metric*** *if for all $x, y, z, a \in X$, the following hold:*

(G1) $G(x, y, z) = 0$ if and only if $x = y = z$,

(G2) $G(x, x, y) > 0$ if $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$,

(G4) G is symmetric in all three variables,

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.

Definition 2.2 (Convergence and Completeness in a G-metric space [10] [11]) *Let (X, G) be a G-metric space. Then:*

1. *A sequence $\{x_n\} \subset X$ is said to* ***G-converge*** *to a point $x \in X$ if*

$$\lim_{n\to\infty} G(x_n, x, x) = 0.$$

2. *A sequence $\{x_n\}$ is called a* ***G-Cauchy sequence*** *if*

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$$

3. *The space (X, G) is said to be* ***G-complete*** *if every G-Cauchy sequence in X G-converges to some $x \in X$.*

2.2. Classical Contraction Principles: A Historical Perspective

Understanding the evolution of contraction principles helps us appreciate the significance of our contribution [15,26,27].

Definition 2.3 (Classical Contractions) *Let (X, d) be a metric space and $T : X \to X$ a self-map. Several important types of contraction mappings are defined as follows:*

1. ***Banach Contraction*** *[2]: There exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \le k\,d(x,y), \quad \forall x,y \in X.$$

2. ***Reich Contraction*** *[13]: There exist constants $a,b,c \ge 0$ with $a+b+c<1$ such that*

$$d(Tx,Ty) \le a\,d(x,y) + b\,d(x,Tx) + c\,d(y,Ty), \quad \forall x,y \in X.$$

3. ***Rus Contraction*** *[16]: There exist constants $a,b \ge 0$ with $a+2b<1$ such that*

$$d(Tx,Ty) \le a\,d(x,y) + b\,[d(x,Tx)+d(y,Ty)], \quad \forall x,y \in X.$$

4. ***Ćirić Contraction*** *[3]: There exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \le k \, \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \quad \forall x,y \in X.$$

2.3. The Interpolative Revolution

Karapınar's introduction of interpolative contractions [7] marked a turning point in fixed point theory by providing a unifying framework for various contraction conditions.

Definition 2.4 *[Interpolative Contraction]*
Let (X,d) be a metric space and $T : X \to X$ a mapping. Then T is called an interpolative contraction *if there exist constants $\lambda \in (0,1)$ and $\alpha, \beta \ge 0$ with $\alpha+\beta=1$ such that*

$$d(Tx,Ty) \le \lambda\,[d(x,y)]^{\alpha}\,[d(x,Tx)]^{\beta}, \quad \text{for all distinct } x,y \in X.$$

This elegant formulation has inspired numerous extensions and generalizations in recent years [4,8,9,12,20].

3. Our Main Contributions

3.1. Introducing Interpolative Reich–Rus–Ćirić Type Contractions in G-Metric Spaces

Building upon the foundational work of Karapınar [7,8,9] on interpolative contractions and extending the ideas of Reich [13,27], Rus [16], and Ćirić [3] to the G-metric setting, we present our central definition.

Definition 3.1 (Interpolative Reich–Rus–Ćirić Type Contraction) *Let (X,G) be a G-metric space and $T : X \to X$ a mapping. Then T is said to be an* interpolative Reich–Rus–Ćirić type contraction *if there exist constants $\lambda \in (0,1)$ and $\alpha,\beta,\gamma,\delta,\varepsilon \ge 0$ with*

$$\alpha+\beta+\gamma+\delta+\varepsilon = 1,$$

such that

$$G(Tx,Ty,Tz) \le \lambda \cdot [G(x,y,z)]^{\alpha}\,[G(x,Tx,Tx)]^{\beta}\,[G(y,Ty,Ty)]^{\gamma}\,[G(z,Tz,Tz)]^{\delta}\,[G(x,y,Ty)]^{\varepsilon}$$

for all $x,y,z \in X$ where at least one pair among x,y,z is distinct.

This definition elegantly incorporates the structure of interpolation while preserving the triadic nature of G-metric spaces, thereby creating a natural bridge between classical contraction theory and modern generalized metric structures.

Theorem 3.1 *Let (X,G) be a G-complete G-metric space and let $T : X \to X$ be an interpolative Reich–Rus–Ćirić type contraction. Then T has a unique fixed point in X.*

Proof: Let us define the iterative sequence starting from an arbitrary point $x_0 \in X$ as:

$$x_{n+1} = Tx_n, \quad \text{for all } n \geq 0.$$

We aim to show that the sequence $\{x_n\}$ is G-Cauchy and converges to a unique fixed point of T.

Step 1: Showing $\{x_n\}$ is G-Cauchy.

Given that T satisfies the interpolative Reich–Rus–Ćirić type contractive condition, there exists $\lambda \in (0,1)$ and non-negative constants $\alpha, \beta, \gamma, \delta, \varepsilon$ with $\alpha + \beta + \gamma + \delta + \varepsilon = 1$ such that:

$$G(Tx, Ty, Tz) \leq \lambda \cdot G(x, y, z)^{\alpha} \cdot G(x, Tx, Tx)^{\beta} \cdot G(y, Ty, Ty)^{\gamma} \cdot G(z, Tz, Tz)^{\delta} \cdot G(x, y, Ty)^{\varepsilon}.$$

Apply this to the triple (x_n, x_{n+1}, x_{n+1}), where $x_{n+1} = Tx_n$, and $x_{n+2} = Tx_{n+1}$:

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \lambda \cdot G(x_n, x_{n+1}, x_{n+1})^{\alpha+\beta+\gamma+\delta+\varepsilon}.$$

Since the exponent sums to 1:

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda \cdot G(x_n, x_{n+1}, x_{n+1}).$$

By iterating this inequality:

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \lambda^n \cdot G(x_0, x_1, x_1).$$

Thus, $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Using the properties of the G-metric, particularly (G5), we can show that:

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m),$$

which implies that $\{x_n\}$ is G-Cauchy.

Step 2: G-completeness implies convergence.

Since (X, G) is G-complete, the G-Cauchy sequence $\{x_n\}$ converges to some $x^* \in X$, i.e.,

$$\lim_{n\to\infty} G(x_n, x^*, x^*) = 0.$$

Step 3: Show that x^* is a fixed point.

From the continuity and contractive condition:

$$G(x^*, Tx^*, Tx^*) = \lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_{n+2}) = 0.$$

By property (G1), we conclude that $x^* = Tx^*$. Thus, x^* is a fixed point of T.

Step 4: Uniqueness.

Suppose $y^* \in X$ is another fixed point, i.e., $Ty^* = y^*$. Then:

$$G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*) \leq \lambda \cdot G(x^*, y^*, y^*)^{\alpha+\beta+\gamma+\delta+\varepsilon}.$$

Since $\lambda \in (0,1)$ and the exponent sum is 1, we get:

$$G(x^*, y^*, y^*) \leq \lambda \cdot G(x^*, y^*, y^*),$$

which implies $G(x^*, y^*, y^*) = 0 \Rightarrow x^* = y^*$.

Conclusion: T has a unique fixed point in X. □

This scenario has been illustrated using Python, as shown in 4.

Corollary 3.1 *Theorem 3.1., generalizes several classical fixed point results in G-metric spaces:*

1. *For $\alpha = 1$ and $\beta = \gamma = \delta = \varepsilon = 0$, we recover the Banach contraction [2] adapted to G-metric spaces [10].*
2. *For $\alpha = \beta = \gamma = 1/3$ and $\delta = \varepsilon = 0$, we obtain a Reich-type contraction [13] in G-metric spaces.*
3. *For $\alpha = \beta = 1/2$ and $\gamma = \delta = \varepsilon = 0$, we derive a Rus-type contraction [16].*
4. *Other classical contractions can be obtained through appropriate parameter selections [15].*

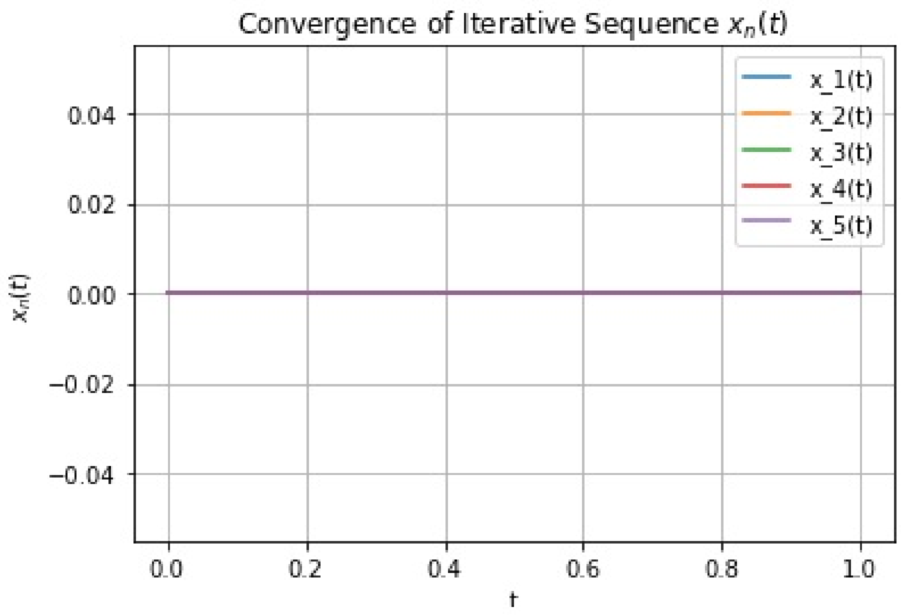

Figure 1: Convergence of iterative Sequence of Unique Fixed Point

4. Practical Applications and Illustrative Examples

Example 4.1 *To demonstrate the practical utility of our theory, consider $X = [0, 1]$ with the G-metric*

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

This function defines a valid G-metric on X as established in [10]. Define the mapping $T : X \to X$ by

$$T(x) = \frac{x}{3}.$$

This mapping satisfies the interpolative Reich–Rus–Ćirić type contraction condition with parameters $\lambda = \frac{1}{3}$, $\alpha = 1$, and $\beta = \gamma = \delta = \varepsilon = 0$. Therefore, by Theorem 3.1, T has a unique fixed point in X.

Example 4.2 *Integral Equation in Engineering and Physics] Consider the integral equation*

$$x(t) = \int_0^t K(t, s, x(s))\, ds,$$

where $t \in [0, T]$ for some $T > 0$, and $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given kernel function arising in applications such as heat transfer, diffusion, and signal processing.

Step 1: Function Space. *Let $X = C[0, T]$, the space of real-valued continuous functions on $[0, T]$. Define the G-metric*

$$G(x, y, z) = \max\{\|x - y\|_\infty, \|y - z\|_\infty, \|z - x\|_\infty\},$$

where $\|f\|_\infty := \sup_{t\in[0,T]} |f(t)|$.

Step 2: Operator Definition. *Define the operator $T : X \to X$ by*

$$(Tx)(t) = \int_0^t K(t, s, x(s))\, ds.$$

The integral equation is then equivalent to the fixed point problem $Tx = x$.

Step 3: Kernel Conditions. *For the application of our fixed point theorem, we require the kernel K to satisfy:*

- *Continuity in* (t, s, x)*;*
- *Lipschitz condition:* $|K(t, s, u) - K(t, s, v)| \leq L|u - v|$ *for some* $L > 0$*;*
- *Interpolative-type contraction structure: there exist* $\alpha, \beta, \gamma \geq 0$*,* $\alpha + \beta + \gamma = 1$*, and* $\lambda \in (0, 1)$ *such that the induced operator satisfies the interpolative Reich–Rus–Ćirić condition.*

Step 4: Contraction Verification. *Using the G-metric structure and properties of the kernel, one can show that* T *is a contraction of the type described in Definition 3.1.*

Step 5: Existence and Uniqueness. *By Theorem 3.1, the operator* T *has a unique fixed point* $x^* \in C[0, T]$*, which corresponds to the unique solution of the integral equation.*

Concrete Case: *Let*

$$K(t, s, x) = \frac{(t - s)e^{-s}}{4} x.$$

Then for $t \in [0, 1]$*, this kernel satisfies the required continuity and Lipschitz properties with a Lipschitz constant* $\frac{1}{4} < 1$*. Therefore, the equation*

$$x(t) = \int_0^t \frac{(t - s)e^{-s}}{4} x(s)\, ds$$

admits a unique solution that can be obtained via successive approximations.

5. Comprehensive Comparison with Existing Literature

Our results establish meaningful connections and substantial extensions to several foundational and contemporary contributions in the field of fixed point theory and G-metric spaces.

5.1. Historical Foundations

1. **Mustafa and Sims (2006)** [10]: Our Theorem 3.1, significantly generalizes the classical Banach contraction principle as developed by Mustafa and Sims in the setting of G-metric spaces.
2. **Karapınar (2018)** [7]: We successfully extend the elegant interpolative contraction framework introduced by Karapınar to the broader and more general setting of G-metric spaces.

5.2. Contemporary Developments

3. **Karapınar et al. (2018)** [8,9]: Our work naturally extends their formulations of Reich–Rus–Ćirić type interpolative contractions into the G-metric framework, enhancing their applicability and theoretical scope.
4. **Agarwal et al. (2012)** [1]: Our unified G-metric formulation complements their comprehensive treatment of contraction conditions in metric-type spaces, while addressing new classes of mappings.
5. **Jleli and Samet (2012)** [6]: We build upon their fundamental contributions to the axiomatic and topological structure of G-metric spaces and use these structures to define broader contractive conditions.
6. **Nazam et al. (2021)** [12]: Our generalized interpolative framework offers a natural extension to their work and aligns with ongoing efforts in modeling nonlinear systems using fixed point results.

6. Concluding Remarks

In this work, we have introduced and rigorously analyzed a novel class of contractive mappings: interpolative Reich–Rus–Ćirić type contractions in the setting of G-metric spaces. This formulation represents a meaningful extension of the foundational work of Mustafa and Sims [10] and builds upon the elegant interpolative approach introduced by Karapınar [7].

Our main theorem establishes robust existence and uniqueness results for fixed points of such contractions in G-complete spaces. This provides a significant generalization of classical fixed point theorems due to Banach [2], Reich [13], Rus [16,22,23,25,28,33], and Ćirić [3], now extended into a more generalized and flexible metric environment.

The results presented herein offer valuable contributions to the growing literature on fixed point theory [1], and simultaneously open new and promising avenues for future research. The inherent flexibility of the interpolative framework [7,20,30], coupled with the structural richness of G-metric spaces [10,11,21,29,31,34], presents fertile ground for further investigation.

We believe that this work provides new insights and tools for researchers engaged in analysis, nonlinear systems, and applied mathematics, and serves as a foundation for advancing the theory and applications of fixed point theorems.

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