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The Cosmological Model Incorporating Viscous Fluid within the Framework of General Relativity

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ABSTRACT: Dark matter and dark energy, elusive elements of the universe, exert gravitational forces of attraction or repulsion, influencing the cosmological model. This phenomenon, known as cosmic acceleration, has been the subject of theoretical and observational studies spanning many decades. In this paper we have presented the cosmological models in presence of viscous fluid. Incorporation of cosmological constant Λ and gravitational constant G makes model more realistic. We solve the model in two phases: with case first, we assume that the viscosity of fluids is a power function of density. While another case comprises that the viscosity of the cosmological model is to be considered as the form of scalar expansion. To get the deterministic model, we have assumed a power law of gravitational constant G. The relationship between viscosity and density; and with viscosity and scalar expansion determines the existence of dark energy. The geometrical aspect of the cosmological models also discussed.

Key Words: Viscosity, cosmological constant, deceleration parameter, scalar expansion.

Contents

1. Introduction

A significant portion of astrophysical data indicates that the structure of our universe is characterized by anisotropy. This anisotropic nature holds crucial implications for the overall behavior of the universe on a large scale. The concept of an anisotropic cosmological model plays a pivotal role in understanding these dynamics. The cosmological term Λ, commonly referred to as the cosmological constant, presents itself as a central enigma within the realm of relativistic cosmology. Initially, this cosmological constant Λ was postulated to account for a static solution in Einstein's field equations. However, recent studies, as referenced in sources [[1], [2], [3], [4]], have shed light on an alternative perspective. These investigations propose that a minute positive value of Λ could serve as a theoretical candidate for dark energy, which in turn drives the accelerated expansion of the universe [5]. This proposition gains support from a diverse array of observations that collectively suggest the existence of a non-zero cosmological constant [6]. Linde [7] further posits that Λ might be contingent on temperature and linked to the process of spontaneous symmetry breaking, potentially making it a function of time within a spatially homogeneous expanding universe [8]. Bulk viscosity emerges as a necessary component for explaining the observed high entropy-to-baryon ratio observed in the universe. Numerous researchers, including Pavon [9], Maartens [10], and Pradhan et al. [11], Kumar [12], have delved into cosmological models incorporating bulk viscosity. Contributions by Klimek [13] and Murphy [14] elucidate how viscosity averts initial singularities, whereas Padmanabhan and Chitre [15] propose that bulk viscosity could lead

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to solutions akin to inflation. Arbab [16], Beesham et al. [17], and Bali and Tinker [18] have extended this investigation to include models with time-varying G and Λ terms, where the viscosity coefficient is selected as a power function of matter density. Bali and Pradhan [19] adopt the assumption that the coefficient of bulk viscosity ξ varies inversely with the expansion rate θ, which in turn is related to the shear σ. Furthermore, the viscosity coefficient ζ is defined as a function of temperature, $\zeta = \alpha T^n$, where α and n stand as constants [13]. While the underlying intricacies of these phenomena may be intricate, a key dissipative effect can be encapsulated in a comprehensive term [20] that encompasses a singular parameter ζ. Notably, in 2000, Capozziello explored how viscosity could interact with alternative gravitational theories to shape the cosmos' evolution. Subsequently, converged on solutions that predict an inflationary phase induced by bulk viscosity [[15]-[17]]. Recent years witnessed surging interest in exploring uniform and anisotropic cosmological models. Viscosity coefficients, studied by researchers like Nawsad Ali [5] and Klimek [13], vary with matter density or expansion scale. Diverse scenarios within the universe's evolution could trigger bulk viscosity, such as strings, magnetic monopoles, particle creation, and interactions with radiation fields. Bulk viscosity's impact on universe evolution, particularly during radiation and matter-dominated eras, exhibits inflation-like parallels [15]. Dunn and Tupper [21] analyzed tilted, viscous models. Mukherjee [22] revealed Bianchi type I models' homogeneous shearing. Pawar and Dagwal [23] explored tilted, symmetric cosmological models with bulk viscous fluid using $\zeta = \zeta_0 \rho^s$. In f(T) gravity, Dagwal and Pawar [30] presented two-fluid models. Berman and Gomide [25] proposed Hubble parameter variation laws. Akarsu and Dereli [26] introduced a generalized deceleration parameter. Singh and Singh [27] found solutions for constant deceleration models. Bali and Pradhan [28] studied Bianchi Type III String Models with Time Dependent Bulk Viscosity. Bali and Yadav [29] examined the viscous fluid cosmological model of Bianchi Type-IX within general relativity, assuming that the shear viscosity is directly proportional to the scalar expansion, denoted as $\eta \propto \theta$. Bali et al. [29] investigated Bianchi type V viscous fluid models with decaying vacuum energy. Chokyia [34] tested generalized second law of thermodynamics in cosmological models with bulk viscosity and modified gravity. In pursuit of a deterministic model, two fundamental assumptions were adopted: the power law governing the gravitational constant and the distinct law dictating the variation of Hubble's parameter, as proposed by Berman in 1983. This particular variation yields a consistent deceleration parameter for the universe, alongside the adoption of a power law for the gravitational constant G. Recent experimental evidence and critical analysis have bolstered the idea of a transition from an anisotropic phase of cosmic expansion to an isotropic one. This development justifies the consideration of universe models featuring an anisotropic foundation. In this context, we present an anisotropic cosmological universe incorporating time-varying G and Λ in conjunction with a viscous fluid. The chosen viscous coefficient is defined as $\zeta = \zeta_0 \rho^s$, and $\zeta = \zeta_0 \theta$, where ζ_0, ζ_1, and s are constants. The presentation of the paper as follows: The metric and Einstein's field equations are established in section 2. Cosmic solution for the both cases of the model is taken in section 3. Geometrical interpretation for all cosmological parameters were plotted and discussed in section 4. In section 5 concludes about outcome of the models.

2. The Metric and Field Equations

The metric of the cosmological model is given by [[30]-[31]]

$$ds^2 = dt^2 - A^2 \left\{ dx^2 + dy^2 + \left(1 + \beta \int \frac{1}{A^3}\, dt\right)^2 dz^2 \right\}, \tag{2.1}$$

Where A is the function of time t and β is constant.

The Einstein field equations with time dependent G and Λ are given by

$$R_i^j - \frac{1}{2} g_i^j R + \Lambda g_i^j = -8\pi G T_i^j\,. \tag{2.2}$$

Energy momentum tensor for bulk viscous fluid [[5], [16]-[18]] is to be considered as

$$T_i^j = (\rho + \bar{p}) v_i v^j + \bar{p} g_i^j\,. \tag{2.3}$$

where $\bar{p}$ is effective pressure and ρ is the matter density for the cosmic fluid

$$\bar{p} = p - \zeta v^i_{;i}\,. \tag{2.4}$$

with satisfying linear equation of state condition

$$p = \omega\rho \tag{2.5}$$

We obtained the following field equations using from equations (2.2)-(2.5) are

$$\frac{A_4^2}{A^2} + 2\frac{A_{44}}{A} - \Lambda = 8\pi G\bar{p}\,. \tag{2.6}$$

$$3\frac{\dot{A}_4^2}{A^2} + \frac{2\beta \dot{A}_4}{A^4\left(1+\beta\int\frac{1}{A^3}dt\right)} - \Lambda = -8\pi G\rho. \tag{2.7}$$

Where the subscript 4 at the symbol A denoted ordinary differentiation with respect to time t.

3. Solution of the Field Equations

To get the deterministic model we have assumed a supplementary condition [31]

$$G = \alpha T^n \tag{3.1}$$

where α and n are constants.
The deceleration parameter q is defined as follows:

$$q = -\frac{aa_{44}}{a_4^2} \tag{3.2}$$

here a is a scale factor of the considered model which is to be defined To get the deterministic model we have assumed a supplementary condition [31] as per equation (2.1) is,

$$A = \left[A^3\left(1+\beta\int\frac{1}{A^3}dt\right)\right]^{1/3} \tag{3.3}$$

From equation (3.2), we get

$$a = (lt+m)^{1/(1+q)} \tag{3.4}$$

Now equating equations (3.3) and (3.4), we get

$$\left[A^3\left(1+\beta\int\frac{1}{A^3}\,dt\right)\right]^{1/3} = (lt+m)^{1/(1+q)}. \tag{3.5}$$

On solving equation (3.5), we obtained

$$A = T^{\frac{1}{1+q}}\cdot e^{n_1 T^{\frac{q-2}{1+q}}}. \tag{3.6}$$

Where, $T = lt+m$, $n_1 = \frac{\beta(1+q)}{3c(2-q)}$ and $c = 1$.
The metric (2.1) reduces to

$$ds^2 = \frac{dt^2}{c^2} - T^{\frac{2}{1+q}}e^{2n_1 T^{\frac{q-2}{1+q}}}(dx^2+dy^2) - T^{\frac{2}{1+q}}e^{-4n_1 T^{\frac{q-2}{1+q}}}dz^2. \tag{3.7}$$

From equations (3.4), (3.6), (3.8) and (3.13) we get (for, $p = \omega\rho$);

$$-2\frac{A_4^2}{A^2} + 2\frac{A_{44}}{A} - \frac{2\beta A_4}{A^4\left(1+\beta\int\frac{1}{A^3}\,dt\right)} = 8\pi G\left(\rho(1+\omega) - \zeta\frac{A_4}{A}\right). \tag{3.8}$$

3.1. Cosmology for Model I: $\zeta=\zeta_0\rho^s$

To determine the coefficient of bulk viscosity is assumed to be a power function of matter density [[5], [9]- [10]], i.e. $\zeta=\zeta_0\rho^s$.
To find the solution, from equation (2.5), (2.6) and (2.7),

$$-2\frac{A_4^2}{A^2}+2\frac{A_{44}}{A}-\frac{2\beta A_4}{A^4\left(1+\beta\int\frac{1}{A^3}\,dt\right)}=8\pi G\left(\rho(1+\omega)-\zeta_0\rho^s\frac{A_4}{A}\right). \tag{3.9}$$

Matter density for $s=1$, we have

$$\rho=\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n\left[1+\omega+\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}\right] \tag{3.10}$$

as from above $p=\omega\rho$, we get

$$\rho=\omega\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n\left[1+\omega+\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}\right] \tag{3.11}$$

The Equilibrium pressure of the fluid is given by

$$\bar{p}=\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n\left[1+\omega+\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}\right]\times\left[\omega-\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]. \tag{3.12}$$

Cosmological constant for the model is,

$$\Lambda=\frac{l^2}{(1+q)^2}T^{-2}+\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}-\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}+\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}$$
$$\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n\left[1+\omega+\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}\right]\times2\left[\omega-\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]. \tag{3.13}$$

From model I with equation (3.10) we get the viscosity for $s=1$ is,

$$\zeta=\zeta_0\rho=\zeta_0\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n\left[1+\omega+\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}\right] \tag{3.14}$$

for $s=2$, density becomes

$$\rho=\frac{-1-\omega\pm\sqrt{(1+\omega)^2+4\left[\frac{\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}}{4\pi\alpha T^n}\right]}}{2\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)} \tag{3.15}$$

and pressure reduces to,

$$p=\frac{\omega\left[-1-\omega\mp\sqrt{(1+\omega)^2+4\cdot\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n}}\right]}{2\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)} \tag{3.16}$$

Effective pressure will be

$$\bar{p}=\frac{\omega\left[-1-\omega\mp\sqrt{(1+\omega)^2+4\cdot\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n}}\right]}{2\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)} -\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right) \tag{3.17}$$

and equation (2.6) and (3.15)

$$\begin{aligned}\Lambda=&\frac{l^2}{(1+q)^2}T^{-2}+\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}-\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}+\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}\\&-\frac{8\pi\alpha T^n\omega\left[-1-\omega\mp\sqrt{(1+\omega)^2+4\cdot\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n}}\right]}{2\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}\\&+8\pi\alpha T^n\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\end{aligned} \tag{3.18}$$

From model I with equation (3.15), we get viscosity of the model represents

$$\zeta=\zeta_0\frac{\left[-1-\omega\mp\sqrt{(1+\omega)^2+4\cdot\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n}}\right]^2}{\left[2\zeta_0\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]^2} \tag{3.19}$$

3.2. Cosmology for Model II: $\zeta=\zeta_1\theta$

To obtain the coefficient of bulk viscosity is proportional to scalar expansion [[5], [32], [33]] i.e. $\zeta=\zeta_1\theta$.

$$\rho=\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)+12\pi\alpha T^n\zeta_1\frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n(1+\omega)} \tag{3.20}$$

From equation (2.5) and (3.18) we get

$$p=\omega\left[\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)+12\pi\alpha T^n\zeta_1\frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n(1+\omega)}\right] \tag{3.21}$$

Equilibrium pressure of the model from equation (2.4) and (3.16)

$$\bar{p}=\omega\left[\frac{\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}}+\frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}}-\frac{ql^2}{(1+q)^2}T^{-2}-\frac{l^2}{(1+q)^2}T^{-2}-\frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}}+\frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}}-\frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}}+\frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right)+12\pi\alpha T^n\zeta_1\frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)}{4\pi\alpha T^n(1+\omega)}\right]$$
$$-3\zeta_1\frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1}-\frac{\beta l}{3c}T^{\frac{1}{1+q}}\right) \tag{3.22}$$

From equations (2.6) and (3.22)

$$\Lambda = \frac{l^2}{(1+q)^2}T^{-2} + \frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}} - \frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}} + \frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}} + \frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}} - \frac{ql^2}{(1+q)^2}T^{-2}$$
$$- \frac{2\omega\left[\left(\frac{\beta l^2}{3(1+q)}T^{\frac{-4-q}{1+q}} + \frac{\beta^2 l^2}{9c^2}T^{\frac{-6}{1+q}} - \frac{ql^2}{(1+q)^2}T^{-2} - \frac{l^2}{(1+q)^2}T^{-2} - \frac{\beta^2 l^2}{9c^2}T^{\frac{2}{1+q}} + \frac{2\beta l^2}{3c(1+q)}T^{\frac{-q}{1+q}} - \frac{2\beta l}{1+q}T^{\frac{-4-q}{1+q}} + \frac{2\beta^2 l}{3c}T^{\frac{-2}{1+q}}\right) + 12\pi\alpha T^n \zeta_1 \frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1} - \frac{\beta l}{3c}T^{\frac{1}{1+q}}\right)\right]}{(1+\omega)}$$
$$-24\pi\alpha T^n \omega \zeta_1 \frac{l}{1+q}T^{-1}\left(\frac{l}{1+q}T^{-1} - \frac{\beta l}{3c}T^{\frac{1}{1+q}}\right) \quad (3.23)$$

For model II with equation (3.20) the viscosity of the model is defined as,

$$\zeta = 3\zeta_1 \frac{l}{1+q}T^{-1}. \quad (3.24)$$

4. Graphical Interpretation of the Model

Model I: $\zeta = \zeta_0 \rho^s$, when $s = 1$

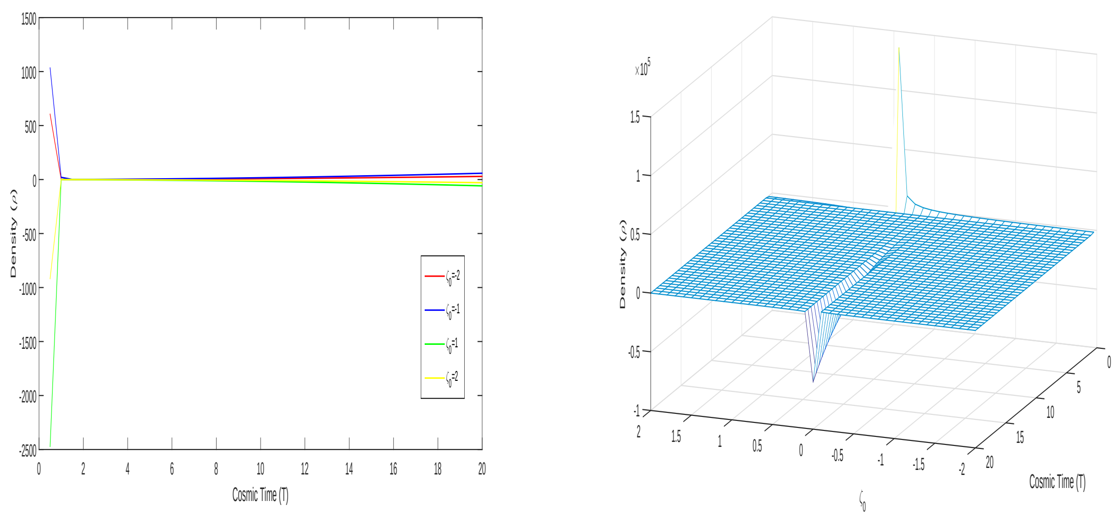

Figure 1: from equation (3.10), variation of density vs cosmic time (in 2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

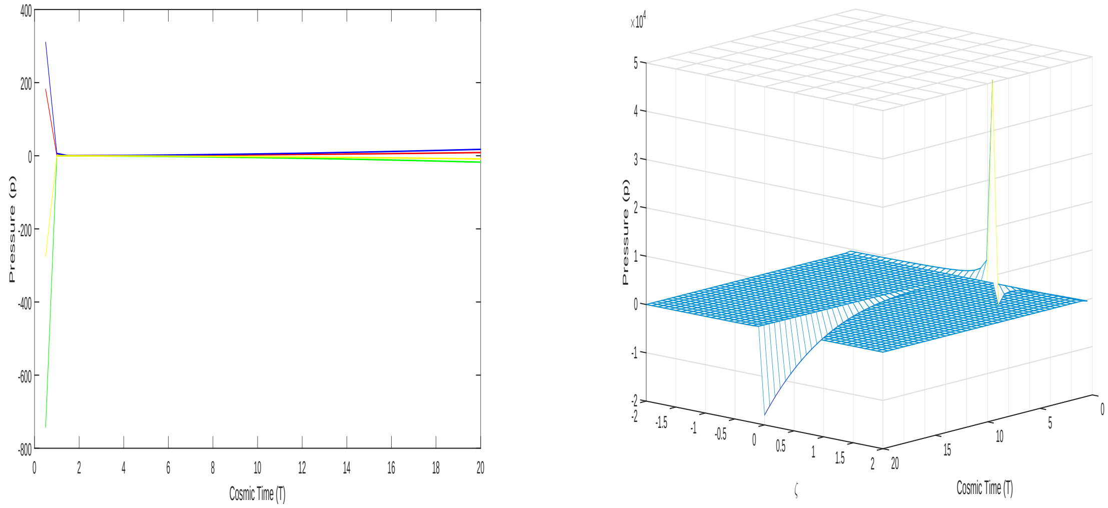

Figure 2: from equation (3.11), variation of pressure vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

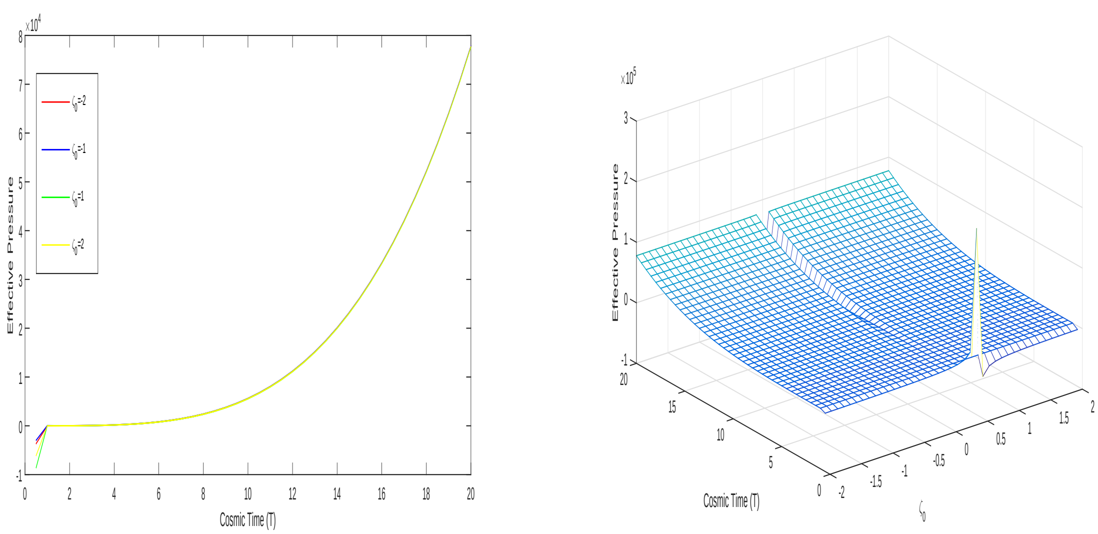

Figure 3: from equation (3.1), variation of effective pressure vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

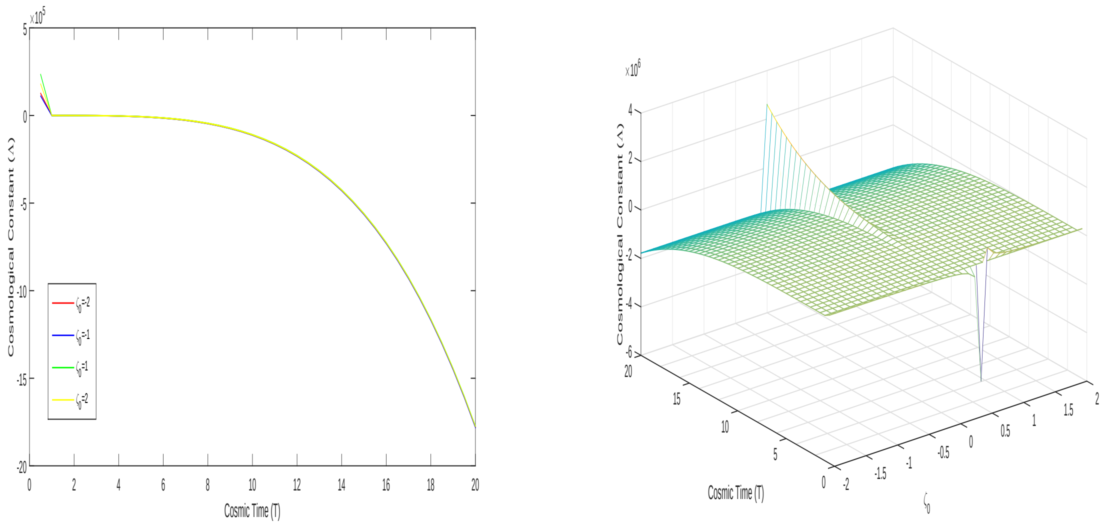

Figure 4: from equation (3.13), variation of cosmological constant vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

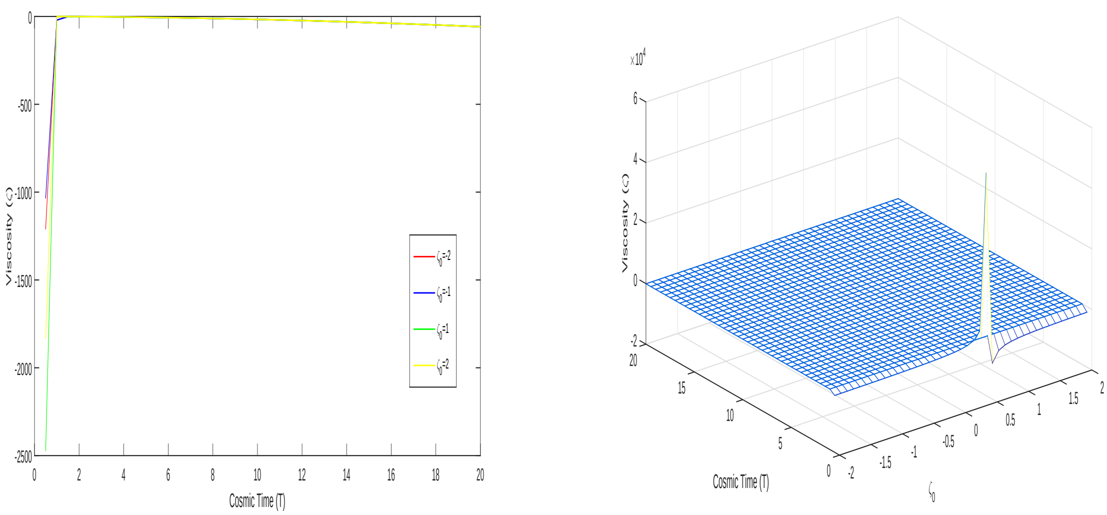

Figure 5: from equation (3.14), variation of effective pressure vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

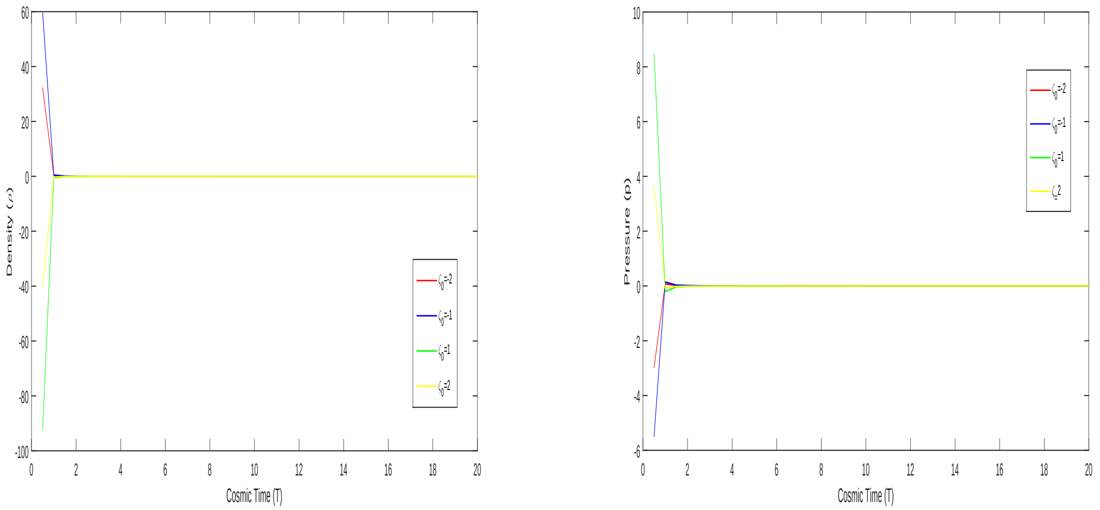

Figure 6: from equation (3.15), and (3.16) variation of cosmological constant vs cosmic time (2D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

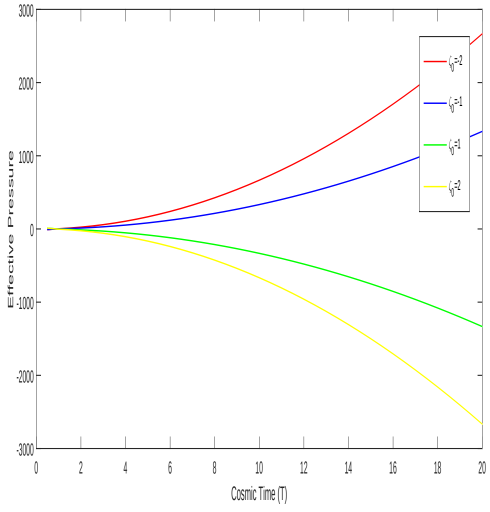

Figure 7: from equation (3.17), variation of effective pressure vs cosmic time (2D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

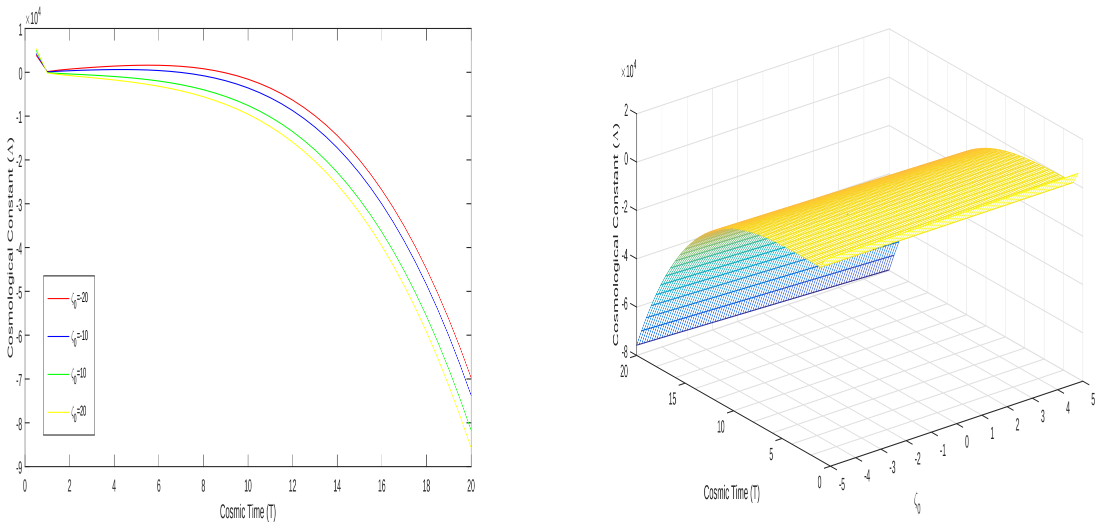

Figure 8: from equation (3.18) variation of cosmological constant vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

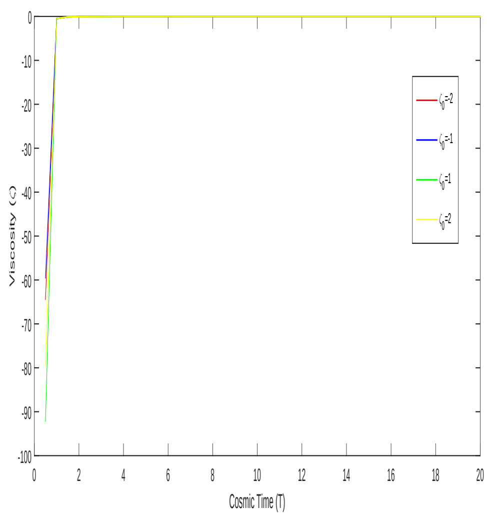

Figure 9: from equation (3.19), variation of effective pressure vs cosmic time (2D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

Model I: $\zeta = \zeta_0 \rho^s$, when $s = 1$

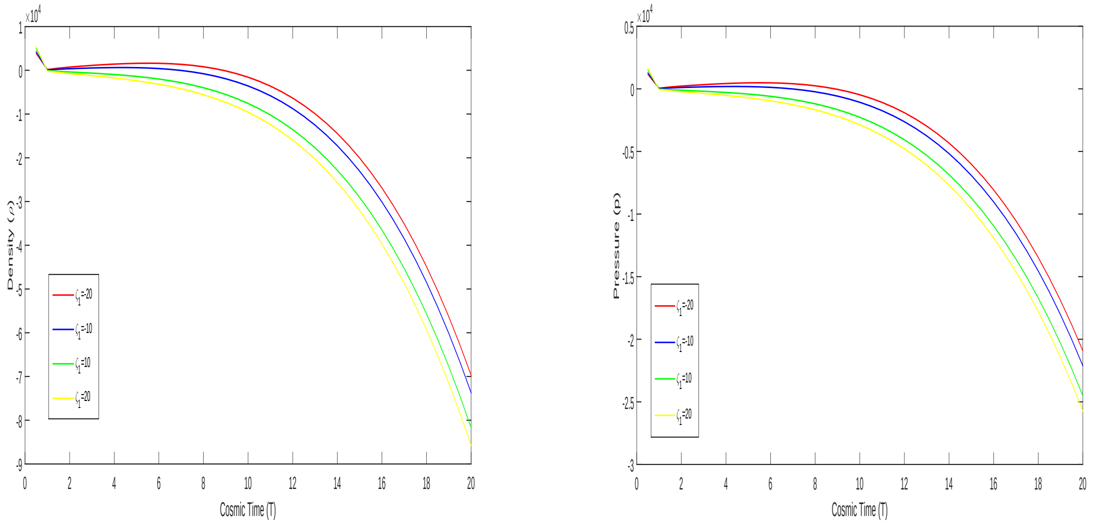

Figure 10: from equation (3.20), and (3.21) variation of cosmological constant vs cosmic time (2D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

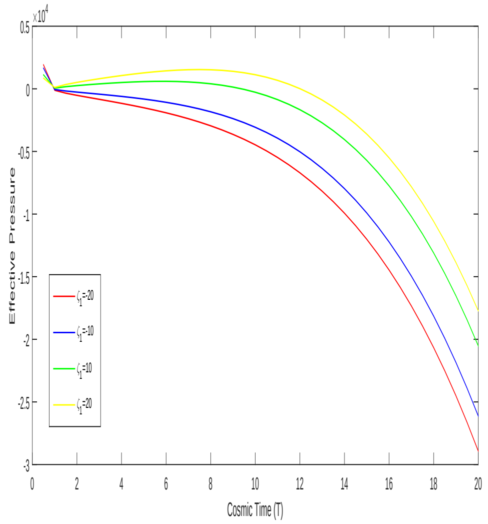

Figure 11: from equation (3.22), variation of effective pressure vs cosmic time (2D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

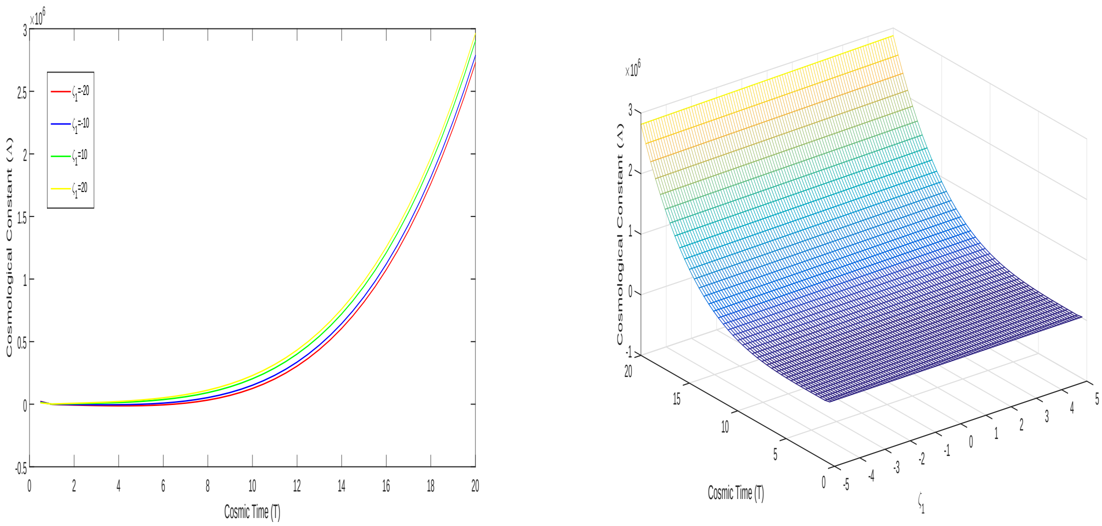

Figure 12: from equation (3.23) variation of cosmological constant vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

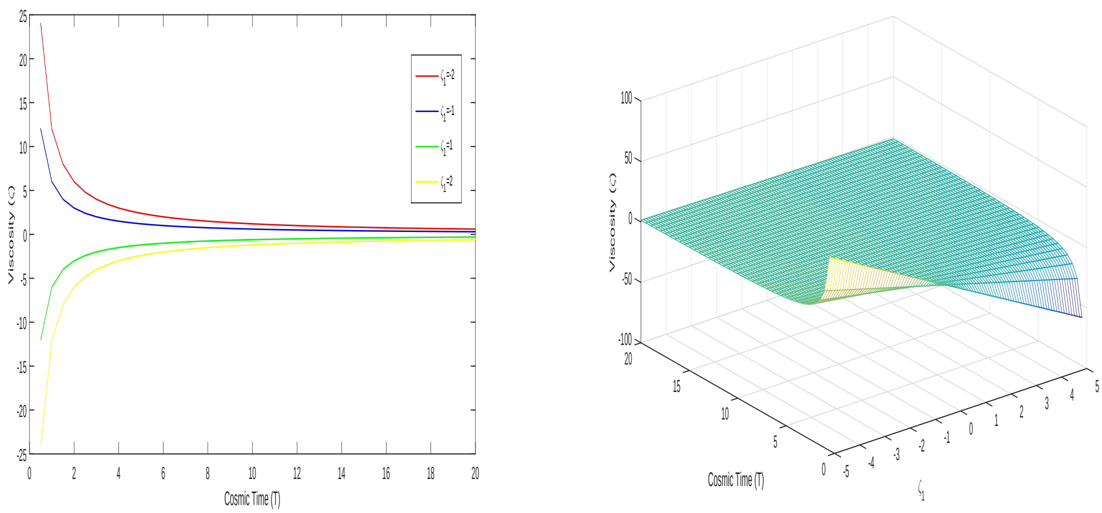

Figure 13: from equation (3.24) variation of cosmological constant vs cosmic time (2D and 3D) for the arbitrary constant values are taken $l = -1, q = -0.5, \alpha = 1, \beta = 10, c = 1, \pi = 3.14, n = 0.2, \omega = 0.3, \zeta_0 = 2$.

5. Conclusion

- From Model I: $\zeta = \zeta_0 \rho^s$, when $s = 1$, we observed from Fig. 1(a) and 1(b) for $\zeta_0 < 0$, initially the density was positive i.e. very dense after some time $\rho \to \theta$ means normal expansion exist. If $\zeta_0 > 0$, then negative density exist which means dark energy exist.

- Since, we have considered $\omega = 0.3$, radiation era hence the pressure for model I also behaving same nature as density behave.

- Effective pressure $\bar{p} = p - \zeta v^i_{;i}$ approaches to infinity for $-2 \leq \zeta_0 \leq 2$, but $p- \to -\infty$ for $\zeta_0 = 0$

shows asymptotic behavior due to existence of dark energy.

- The variation of cosmological constant (Λ) is inversely proportional w.r.t. effective pressure and cosmic time also. When $\zeta_0 = 0$ then $\Lambda \to \infty$ on $t \to \infty$, represents attractive gravity influencing the universe expansion.
- • Viscosity of the model I was highly negative at early stage the transition of viscosity from negative to positive at early times only then after it settled down to homogeneous form which means the model I initially occurs inflation era.
- Fig. 6(a), 6(b), 7(a) and 7(b) represent density and pressure, both are inversely for four different value of ζ_0, which means the universe was highly dense i.e. expanding universe.
- Effective pressure for model I with $s = 2$, for $\zeta_0 < 0 \implies \bar{p} \to \infty$ and $\zeta > 0 \implies \bar{p} \to -\infty$ i.e. symmetrical nature along $\zeta_0 = 0$.
- Cosmological constant for model I with $s = 2, \Lambda \to -\infty$ for all value of ζ_0, which can be seen in Fig. 9(a) and 9(b), represents repulsive gravity, causing cosmic acceleration and driving the universe expansion.
- Initially viscosity was negative as cosmic time varies. The density, pressure and effective pressure for model II i.e. if we assumed $\zeta = \zeta_1\theta$ viscosity is proportional to scalar expansion then both $p \to -\infty, \rho \to -\infty$ for various value of ζ_1.
- But cosmological constant (Λ) is inversely proportional to cosmic time for various value of ζ_1, i.e. attractive gravity influencing the universe expansion.
- In this model we observe that the dependency of viscosity on density and scalar expansion boost to dark energy scenario. The bulk viscosity contributes negatively to the pressure in an expanding universe, just like cosmological constant [15].
- The viscosity for case II also highly negative at early times later on approaches to zero. The obtained solution resembles with previous solution [14] The solutions obtained lead to inflationary phase [[15], [18]].

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