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Stochastic Fubini-Tonelli Theorem for the Itô-Henstock Integral*

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ABSTRACT: In this paper, we formulate a version of stochastic Fubini-Tonelli theorem for the Itô-Henstock integral of a Hilbert-Schmidt-valued stochastic process driven by a Hilbert space-valued Q-Wiener process.

Key Words: Itô-Henstock integral, Hilbert-Schmidt-valued process, Q-Wiener process, stochastic Fubini-Tonelli.

Contents

1. Introduction

The Riemann integral, which is the first integral covered in basic calculus, is conceivably the most well-known integral. Some functions, however, cannot be integrated in the Riemann sense. Some of the problems with the Riemann integral are fixed by the definition of the Lebesgue integral, which has gained popularity and is almost entirely used by numerous mathematicians. However, understanding Lebesgue integration requires an extensive study of measure theory. It also fails to integrate highly oscillatory functions.

One may prove that the derivative of the function $f(x) = x^2 \sin x^{-2}$ if $x \neq 0$ and $f(0) = 0$ is not Lebesgue integrable on $[0, 1]$. Fortunately, the Henstock integral can be used to integrate the function f on $[0, 1]$, see [5, Theorem 9.6]. The Henstock integral or Henstock-Kurzweil integral is one of the noteworthy integrals that were introduced, which can integrate highly oscillatory functions and is, in some sense, more general than the Lebesgue integral. The Henstock method to integration, also known as the generalized Riemann or Henstock-Kurzweil approach, is a technique of presenting the integral that is similar to how the Riemann integral is presented.

A real-valued function f, which is not assumed to be measurable, is said to be *Henstock integrable* [13, Definition 2.2] to $A \in \mathbb{R}$ on $[a, b]$ if for every $\epsilon > 0$, there is a function $\delta(\xi) > 0$ such that whenever a division D given by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ satisfies $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$ we have $|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A| < \epsilon$. The Henstock integral of f on $[a, b]$ is given by A and it is uniquely determined. The Henstock integral, which is a Riemann-type definition of an integral, differs from the Lebesgue integral in the sense that the definition is more explicit and does not need a thorough understanding of measure theory. It is known that the Henstock integral includes the improper Riemann, Lebesgue, and Newton integrals. Several authors have since developed an interest in this integral, see [5,6,8,9,13,14,15,17,18]. The Henstock approach to integration has also been used to deal with stochastic integration for finite and infinite-dimensional spaces, see [1,10,11,12,19,20,23,24,27,28,29,30,31].

In the study of iterated integrals in mathematical analysis, the conclusion known as Fubini's theorem [26, Theorem 6.6] identifies conditions under which it is possible to compute a double integral using iterated integrals. As a consequence, it allows the order of integration to be changed in iterated integrals.

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Fubini's theorem is succeeded by Tonelli's theorem [26, Theorem 6.7], in which the assumptions are different, but the theorem is almost the same. Tonelli's theorem states that on the product of two σ-finite measure spaces, a product measure integral can be evaluated by way of an iterated integral for nonnegative measurable functions, whether or not they have a finite integral.

In this paper, we introduce two versions of stochastic Fubini-Tonelli theorem for the Itô-Henstock integral of a Hilbert-Schmidt-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process defined in [10, Definition 2.4].

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a sequence $\{\mathcal{F}_t : 0 \leq t \leq T\}$ of σ-subfields of $\mathcal{F}$ such that $\mathcal{F}_t \subseteq \mathcal{F}$ and $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for $t_1 < t_2$, called a *filtration*. A probability space together with $\{\mathcal{F}_t : 0 \leq t \leq T\}$, or simply $\{\mathcal{F}_t\}$, is called a *filtered probability space* and is denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

Let U and V be separable Hilbert spaces, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from Ω to V. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an orthonormal basis (abbrev. as ONB) $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, $\{\lambda_j\} \in \ell^1$, and $\lambda_j \to 0$ as $j \to \infty$, see [22, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an eigensequence defined by Q. The subspace U_Q of U equipped with the inner product $\langle u, v\rangle_{U_Q} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{Ker}Q^{1/2}]^{\perp}$ is a separable Hilbert space with $\left\{\sqrt{\lambda_j}e_j\right\}$ as its ONB, see [2, p.90], [4, p.23]. The space $L_2(U_Q, V)$ of all Hilbert-Schmidt operators from U_Q to V is a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$, see [21, p.112]. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [2, p.418], [21, p.111]. We note that $L(U,V)$ is properly contained in $L_2(U_Q, V)$, see [4, p.25]. We also note that $L_2(U_Q, V)$ contains genuinely unbounded linear operators from U to V.

Let $B = \{B_t\}_{0 \leq t \leq T}$ be an adapted real-valued process. Then B is called a *Brownian motion* (abbrev. as BM) or *real-valued Wiener process* if the following properties are satisfied:

(i) $B(0, \omega) = 0$ for all $\omega \in \Omega$;

(ii) for $0 \leq s < t \leq T$, the increment $B_t - B_s$ is Gaussian with $\mathcal{L}(B_t - B_s) \sim \mathcal{N}(0, t-s)$;

(iii) for $0 \leq s < t \leq T$, $B_t - B_s$ is independent of $\mathcal{F}_s$; and

(iv) $B(\cdot, \omega) : [0,T] \to \mathbb{R}$ is continuous for almost all $\omega \in \Omega$.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by Q, and $\{B_j\}$ be a sequence of independent BM defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. A process $\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j$, is called a Q-Wiener process in U, with the series converging in $L^2(\Omega, U)$. For every $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u\rangle_U$, with the series converging in $L^2(\Omega, \mathbb{R})$. Then there exists a U-valued process W, known as a U-valued Q-Wiener process, such that $\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u\rangle_U$ $\mathbb{P}$-almost surely (abbrev. as $\mathbb{P}$-a.s.). It should be noted that the process W is a multi-dimentional BM, and if we assume that $\lambda_j > 0$ for all j, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \ldots$, is a sequence of real-valued BM defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, see [2, p.87].

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if (i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and (ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T]$. A Q-Wiener process W_t, $t \in [0,T]$ is called a *Q-Wiener process with respect to a filtration* $\{\mathcal{F}_t\}$ if (i) W_t is adapted to $\{\mathcal{F}_t\}$, $t \in [0,T]$ and (ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$. A U-valued Q-Wiener process $W(t)$, $t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration, see [21, p.16]. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

A stochastic process $M : [0,T] \times \Omega \to V$ is said to be a *martingale* if (i) M is adapted; (ii) for all $t \in [0,T]$, M_t is Bochner integrable, i.e. $\mathbb{E}[\|M_t\|_V] < \infty$; and (iii) for any $0 \le s \le t \le T$, $\mathbb{E}\left[M_t \middle| \mathcal{F}_s\right] = M_s$ $\mathbb{P}$-a.s.. A martingale $M : [0,T] \times \Omega \to V$ is said to be *square-integrable* if $M_T \in L^2(\Omega, V)$. It is known [21, p.21] that the space of all continuous square-integrable martingales $\mathcal{M}_T^2 := \mathcal{M}_T^2(V)$ is a Banach space with norm $\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|M_t\|_V^2\right]\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\|M_T\|_V^2\right]\right)^{\frac{1}{2}}$, and the Q-Wiener process $W \in \mathcal{M}_T^2(U)$.

In [10], using a generalized Riemann approach, an alternative definition of the Itô integral of a Hilbert-Schmidt-valued stochastic process was introduced [10, Theorem 2.12], known as the Itô-Henstock integral. It was also shown [10, Theorem 2.6] that the Itô-Henstock integral belongs to the space of all continuous square-integrable martingales.

From now onwards, assume that U and V are separable Hilberts spaces, $Q : U \to U$ is a symmetric non-negative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and $W_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_t^{(j)} e_j$ is a U-valued Q-Wiener process. A stochastic process $f : [0,T] \times \Omega \to L_2(U_Q, V)$ means a process measurable as mappings from $([0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F})$ to $(L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$, or we simply say that a process f is *measurable*, if no confusion arises. The given closed interval $[0,T]$ is nondegenerate, i.e. $0 < T$ and can be replaced with any closed interval $[a,b]$. If no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection D.

Definition 2.1 *Let δ be a positive function on $[0,T]$. A finite collection $D = \{([\xi_i, v_i], \xi_i)\}_{i=1}^n$ of inteval-point pairs is a* δ-fine belated partial division *of $[0,T]$ if $\{[\xi_i, v_i]\}_{i=1}^n$ is a collection of non-overlapping subintervals of $[0,T]$ and each $[\xi_i, v_i]$ is δ-fine belated, i.e. $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.*

We note that the term *partial division* is used in Definition 2.1, to emphasize that the finite collection of non-overlapping intervals of $[0,T]$ may not cover the entire interval $[0,T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 2.2 *Given $\eta > 0$, a given δ-fine belated partial division $D = \{([\xi, v], \xi)\}$ of $[0,T]$ is said to be a* (δ, η)-fine belated partial division *of $[0,T]$ if it fails to cover $[0,T]$ by at most length η, that is,*

$$\left| T - (D)\sum (v - \xi) \right| \le \eta.$$

Before we proceed with Definition 2.3, we need to note [10, Lemma 2.3], which states that if $f : [0,T] \times \Omega \to L_2(U_Q, V)$ is an adapted process, then for $0 \le \xi \le v \le T$, $\sum_{j=1}^{\infty} (B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \in L^2(\Omega, V)$.

Definition 2.3 *Let $f : [0,T] \times \Omega \to L_2(U_Q, V)$ be an adapted process. Then f is said to be* Itô-Henstock integrable*, or $\mathcal{IH}$-integrable, on $[0,T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\epsilon > 0$, there is a positive function δ on $[0,T)$ and a number $\eta > 0$ such that for any (δ, η)-fine belated partial division $D = \{([\xi_i, v_i], \xi_i)\}_{i=1}^n$ of $[0,T]$, we have*

$$\mathbb{E}\left[\|S(f, D, \delta, \eta) - A\|_V^2\right] < \epsilon,$$

where

$$S(f, D, \delta, \eta) := (D)\sum \left\{ \sum_{j=1}^{\infty} (B_v^{(j)} - B_\xi^{(j)}) f_\xi(\sqrt{\lambda_j} e_j) \right\} := \sum_{i=1}^{n} \left\{ \sum_{j=1}^{\infty} (B_{v_i}^{(j)} - B_{\xi_i}^{(j)}) f_{\xi_i}(\sqrt{\lambda_j} e_j) \right\}.$$

In this case, f is $\mathcal{IH}$-integrable to A on $[0,T]$ and A is called the $\mathcal{IH}$-integral of f which will be denoted by $(\mathcal{IH})\int_0^T f_t\, dW_t$ or $(\mathcal{IH})\int_0^T f\, dW$. We shall denote $(\mathcal{IH})\int_0^0 f\, dW$ by the zero random variable $\mathbf{0}$ from Ω to V and denote by $\Lambda_{\mathcal{IH}}(U_Q, V)$, the collection of all Itô-Henstock integrable processes on $[0,T]$. In view of [10, Lemma 2.3], $S(f, D, \delta, \eta) \in L^2(\Omega, V)$.

It is worth noting that the $\mathcal{IH}$-integral possesses some of the standard properties of an integral namely, uniqueness of the integral, linearity, integrability on every subinterval of $[0,T]$, the Sequential definition, the Cauchy criterion, and the Saks-Henstock Lemma. One may refer to [10, p.376] for the statement of these results, and the proofs of these properties are standard in Henstock-Kurzweil integration, see [13].

Theorem 2.1 (Sequential definition) *A process $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$ if and only if there exist $A \in L^2(\Omega, V)$, a decreasing sequence of positive functions $\{\delta_n\}$ on $[0,T)$, and a decreasing sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) = A \quad \text{in } L^2(\Omega, V),$$

where $D_n = \{([\xi^{(n)}, v^{(n)}], \xi^{(n)})\}$ is any (δ_n, η_n)-fine belated partial division of $[0,T]$. In this case, we write

$$(\mathcal{IH})\int_0^T f_t\, dW_t = A.$$

The following result [10, Theorem 2.10] is known as a version of Itô-isometry for the $\mathcal{IH}$-integral. The Lebesgue integral is denoted by $(\mathcal{L})\int$.

Theorem 2.2 (Itô-isometry) *Let $f \in \Lambda_{\mathcal{IH}}(U_Q, V)$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and*

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\, dW_t\right\|_V^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\, dt < \infty.$$

Before we proceed with the main results of this study, we will first investigate a specific norm for the space $\Lambda_{\mathcal{IH}}(U_Q, V)$ and transform it into a Hilbert space.

Let (X, Σ, μ) be a measure space and B be a Banach space. A measurable function $\varphi : X \to B$ is called a *simple function* if there exist distinct elements $c_1, c_2, \ldots, c_n \in X$ and disjoint sets $E_1, E_2, \ldots, E_n \in \Sigma$, where $E_k = \{x \in X : \varphi(x) = c_k\}$ and $\bigcup_{k=1}^n E_k = X$, such that $\varphi(x) = \sum_{k=1}^n c_k \chi_{E_K}(x)$, where χ_{E_k} is the characteristic function of E_k. The *integral* of a simple function φ is defined as

$$\int_X \varphi(x)\, d\mu(x) = \sum_{k=1}^n c_k \mu(E_k).$$

A measurable function $f : X \to B$ is said to be *Bochner integrable* on X, [3, Definition 1, p.44] if there exists a sequence of simple functions $\{f_n\}$ on X such that $\lim_{n\to\infty} (\mathcal{L})\int_X \|f_n(t) - f(t)\|_B\, dt < \infty$. In this case, the *Bochner integral* of f is defined by

$$(\mathcal{B})\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$

Theorem 2.3 [3, Theorem 2, p.45] *Let (X, Σ, μ) be a measure space and B be a Banach space. A measurable function $f : X \to B$ is said to be Bochner integrable if and only if $(\mathcal{L})\int_X \|f(x)\|_B\, d\mu(x) < \infty$.*

Denote by $L^p(X,B)$, $1 \le p < \infty$, the collection of all Bochner integrable functions from X to B such that $(\mathcal{L})\int_X \|f(x)\|_B^p \, d\mu(x) < \infty$. Then $L^p(X,B)$ equipped with the norm

$$\|f\|_p = \left((\mathcal{L})\int_X \|f(x)\|_B^p \, d\mu(x) \right)^{1/p}$$

is a Banach space, see [3, p.97].

Theorem 2.4 *For $f, g \in \Lambda_{\mathcal{IH}}(U_Q, V)$, define*

$$\langle f, g\rangle_{\Lambda_{\mathcal{IH}}} := (\mathcal{L})\int_0^T \mathbb{E}\left[\langle f, g\rangle_{L_2(U_Q,V)}\right] dt < \infty.$$

Then $(\Lambda_{\mathcal{IH}}(U_Q,V), \langle\cdot,\cdot\rangle_{\Lambda_{\mathcal{IH}}})$ is an inner product space and the norm $\|\cdot\|_{\Lambda_{\mathcal{IH}}}$ induced by the inner product $\langle\cdot,\cdot\rangle_{\Lambda_{\mathcal{IH}}}$ is complete, that is, $(\Lambda_{\mathcal{IH}}(U_Q,V), \|\cdot\|_{\Lambda_{\mathcal{IH}}})$ is a Hilbert space.

Proof: Since $L_2(U_Q,V)$ is a Hilbert space [21, p.112] and the Lebesgue integral of $[0,T]$ and the expectation (integral) on Ω are linear and symmetric, $\langle\cdot,\cdot\rangle_{\Lambda_{\mathcal{IH}}}$ is also linear and symmetric. Moreover, if $\langle f, f\rangle_{\Lambda_{\mathcal{IH}}} = 0$, then $\|f\|^2_{\Lambda_{\mathcal{IH}}} = 0$ a.e. so that $f = 0$ a.e.. Thus, $(\Lambda_{\mathcal{IH}}(U_Q,V), \langle\cdot,\cdot\rangle_{\Lambda_{\mathcal{IH}}})$ is an inner product space.

Now, consider the norm $\|\cdot\|_{\Lambda_{\mathcal{IH}}}$ induced by the inner product $\langle\cdot,\cdot\rangle_{\Lambda_{\mathcal{IH}}}$. Let $\{f_n\}$ be a Cauchy sequence in $\Lambda_{\mathcal{IH}}(U_Q,V)$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \ge N$,

$$\|f_n - f_m\|_{\Lambda_{\mathcal{IH}}} = \left((\mathcal{L})\int_0^T \mathbb{E}\left[\|f_n - f_m\|^2_{L_2(U_Q,V)}\right] dt \right)^{1/2} < \epsilon.$$

We note that if a process $f(\cdot,\cdot) : [0,T]\times\Omega \to L_2(U_Q,V)$ is measurable, then the process $\|f(\cdot,\cdot)\|^2_{L_2(U_Q,V)} : [0,T]\times\Omega \to \mathbb{R}$ is also measurable and in view of [26, p.435], we may define the integral on $\Omega\times[0,T]$ by

$$\int_{[0,T]\times\Omega} \|f(t,\omega)\|^2_{L_2(U_Q,V)} \, dt \times d\mathbb{P}(\omega).$$

By Tonelli's theorem [26, Theorem 6.7],

$$\begin{aligned}\int_{[0,T]\times\Omega} \|f(t,\omega)\|^2_{L_2(U_Q,V)} \, dt \times d\mathbb{P}(\omega) &= (\mathcal{L})\int_0^T \mathbb{E}\left[\|f(t,\cdot)\|^2_{L_2(U_Q,V)}\right] dt \\ &= \mathbb{E}\left[(\mathcal{L})\int_0^T \|f(t,\cdot)\|^2_{L_2(U_Q,V)} \, dt\right].\end{aligned}$$

Hence, $\{f_n\}$ is a Cauchy sequence in $L^2([0,T]\times\Omega, L_2(U_Q,V))$ so that there exists a process $g(\cdot,\cdot) : [0,T]\times\Omega \to L_2(U_Q,V)$, measurable as mappings from $([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F})$ to $(L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$ such that $\lim_{n\to\infty} f_n = g$ in $L^2([0,T]\times\Omega, L_2(U_Q,V))$. Moreover,

$$\|g\|_2 = \left((\mathcal{L})\int_0^T \mathbb{E}\left[\|g(t,\cdot)\|^2_{L_2(U_Q,V)}\right] dt \right)^{1/2} < \infty.$$

We may assume that g is adapted so that $g \in \Lambda_{\mathcal{IH}}(U_Q,V)$ and $\lim_{n\to\infty} f_n = g$ in $\Lambda_{\mathcal{IH}}(U_Q,V)$, by [10, Theorem 2.11]. Thus, $\Lambda_{\mathcal{IH}}(U_Q,V)$ is a Hilbert space. □

3. Stochastic Fubini-Tonelli Theorem

This section formulates the two versions of stochastic Fubini-Tonelli theorem for the $\mathcal{IH}$-integral. In the first version, the integral on a finite measure space is in the sense of Bochner.

We will need the following known result later.

Theorem 3.1 *[27, Theorem 5] A function $f : [0, T] \to \mathbb{R}$ is Lebesgue integrable to $A \in \mathbb{R}$ if and only if there exist a decreasing sequence of positive functions $\{\delta_n\}$ on $[0, T)$ and a decreasing sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n\to\infty} (D_n) \sum f(\xi^{(n)})(v^{(n)} - \xi^{(n)}) = A,$$

where $D_n = \{([\xi^{(n)}, v^{(n)}], \xi^{(n)})\}$ is any (δ_n, η_n)-fine belated partial division of $[0, T]$. In this case, we write $(\mathcal{L}) \int_0^T f(t)\, dt = A$.

Theorem 3.2 (Stochastic Fubini-Tonelli Theorem I) *Let $(G, \mathcal{G}, \mu)$ be a finite measure space and $f : [0, T] \times \Omega \times G \to L_2(U_Q, V)$ be a function such that for all $x \in G$, $f(\cdot, \cdot, x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$ and for all $(t, \omega) \in [0, T] \times \Omega$, $f(t, \omega, \cdot)$ is Bochner integrable on G and assume that $(\mathcal{B}) \int_G f(\cdot, \cdot, x) : [0, T] \times \Omega \to L_2(U_Q, V)$ is an adapted process. Suppose that*

$$\mathbb{E}\left[(\mathcal{L}) \int_G \|f(\cdot, \cdot, x)\|^2_{L_2(U_Q,V)}\, d\mu(x)\right]$$

is Lebesgue integrable on $[0, T]$. Then the following hold:

(i) *The function $G \to L^2(\Omega, V)$, $x \mapsto (\mathcal{IH}) \int_0^T f(t, \cdot, x)\, dW_t$ is Bochner integrable on G, that is,*

$$(\mathcal{L}) \int_G \left(\mathbb{E}\left[\left\|(\mathcal{IH}) \int_0^T f(t, \cdot, x)\, dW_t\right\|^2_V\right]\right)^{1/2} d\mu(x) < \infty.$$

(ii) *$(\mathcal{B}) \int_G f(\cdot, \cdot, x)\, d\mu(x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$ and*

$$(\mathcal{IH}) \int_0^T \left((\mathcal{B}) \int_G f(t, \cdot, x)\, d\mu(x)\right) dW_t = (\mathcal{B}) \int_G \left((\mathcal{IH}) \int_0^T f(t, \cdot, x)\, dW_t\right) d\mu(x)$$

for almost all $\omega \in \Omega$.

Proof: First, we show that the function $G \to L^2(\Omega, V)$, $x \mapsto (\mathcal{IH}) \int_0^T f(t, \cdot, x)\, dW_t$ is Bochner integrable on G. Since $\mathbb{E}\left[(\mathcal{L}) \int_G \|f(\cdot, \cdot, x)\|^2_{L_2(U_Q,V)}\, d\mu(x)\right]$ is Lebesgue integrable on $[0, T]$,

$$(\mathcal{L}) \int_0^T \left(\mathbb{E}\left[(\mathcal{L}) \int_G \|f(t, \cdot, x)\|^2_{L_2(U_Q,V)}\, d\mu(x)\right]\right) dt < \infty.$$

Using the classical Tonelli theorem [26, Theorem 6.7] and [10, Lemma 2.9], we have

$$\begin{aligned}\infty &> (\mathcal{L})\int_0^T \left(\mathbb{E}\left[(\mathcal{L})\int_G \|f(t,\cdot,x)\|^2_{L_2(U_Q,V)}\,d\mu(x)\right]\right)dt\\ &= (\mathcal{L})\int_G \left((\mathcal{L})\int_0^T \left(\int_\Omega \|f(t,\omega,x)\|^2_{L_2(U_Q,V)}\,d\mathbb{P}(\omega)\right)dt\right)d\mu(x)\\ &= (\mathcal{L})\int_G \left(\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right\|^2_V\right]\right)d\mu(x)\\ &= (\mathcal{L})\int_G \left\|(\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right\|^2_{L^2(\Omega,V)} d\mu(x).\end{aligned}$$

Note that $\left\|(\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right\|_{L^2(\Omega,V)} \leq 1 + \left\|(\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right\|^2_{L^2(\Omega,V)}$. Since $(G,\mathcal{G},\mu)$ is a finite measure space,

$$(\mathcal{L})\int_G \left\|(\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right\|_{L^2(\Omega,V)} d\mu(x) < \infty.$$

Next, we show that $(\mathcal{B})\int_G f(\cdot,\cdot,x)\,d\mu(x) \in \Lambda_{\mathcal{IH}}(U_Q,V)$. Let $\epsilon > 0$ be given. Since

$$\mathbb{E}\left[(\mathcal{L})\int_G \|f(\cdot,\cdot,x)\|^2_{L_2(U_Q,V)}\,d\mu(x)\right]$$

is Lebesgue integrable on $[0,T]$, by Theorem 3.1, there exist a decreasing sequence of positive functions $\{\delta_n\}$ on $[0,T)$ and a decreasing sequence of positive constants $\{\eta_n\}$ such that

$$\begin{aligned}&\lim_{n\to\infty}(D_n)\sum \mathbb{E}\left[(\mathcal{L})\int_G \left\|f(\xi^{(n)},\cdot,x)\right\|^2_{L_2(U_Q,V)} d\mu(x)\right](v^{(n)}-\xi^{(n)})\\ =\;& (\mathcal{L})\int_0^T \left(\mathbb{E}\left[(\mathcal{L})\int_G \|f(t,\cdot,x)\|^2_{L_2(U_Q,V)}\,d\mu(x)\right]\right)dt < \infty,\end{aligned}$$

where $D_n = \{([\xi^{(n)},v^{(n)}],\xi^{(n)})\}$ is any (δ_n,η_n)-fine belated partial division of $[0,T]$. Since $(\mathcal{B})\int_G f(\cdot,\cdot,x) : [0,T]\times\Omega \to L_2(U_Q,V)$ is adapted, by [10, Lemma 2.9] and [3, Theorem 6, p.47], we have

$$\begin{aligned}&(D_n)\sum \mathbb{E}\left[(\mathcal{L})\int_G \left\|f(\xi^{(n)},\cdot,x)\right\|^2_{L_2(U_Q,V)} d\mu(x)\right](v^{(n)}-\xi^{(n)})\\ =\;& (D_n)\sum (v^{(n)}-\xi^{(n)})\mathbb{E}\left[(\mathcal{L})\int_G \left\|f(\xi^{(n)},\cdot,x)\right\|^2_{L_2(U_Q,V)} d\mu(x)\right]\\ =\;& (D_n)\sum (v^{(n)}-\xi^{(n)})\mathbb{E}\left[\left\|(\mathcal{B})\int_G f(\xi^{(n)},\cdot,x)\,d\mu(x)\right\|^2_{L_2(U_Q,V)}\right]\\ =\;& \mathbb{E}\left[\left\|(D_n)\sum\left\{\sum_{j=1}^\infty (B^{(j)}_{v^{(n)}} - B^{(j)}_{\xi^{(n)}})\left((\mathcal{B})\int_G f(\xi^{(n)},\cdot,x)\,d\mu(x)\right)(\sqrt{\lambda_j}e_j)\right\}\right\|^2_V\right].\end{aligned}$$

Moreover, using the classical Tonelli theorem [26, Theorem 6.7] and [10, Lemma 2.9], we have

$$\begin{aligned}\infty &> (\mathcal{L})\int_0^T \left(\mathbb{E}\left[(\mathcal{L})\int_G \|f(t,\cdot,x)\|^2_{L_2(U_Q,V)}\,d\mu(x)\right]\right)dt\\ &= \int_\Omega \left((\mathcal{L})\int_G \left\|(\mathcal{IH})\int_0^T f(t,\omega,x)\,dW_t\right\|_V^2 d\mu(x)\right)d\mathbb{P}(\omega).\end{aligned}$$

Since the function $G \to L^2(\Omega, V)$, $x \mapsto (\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t$ is Bochner integrable on G, using [3, Theorem 6, p.47], we have

$$\begin{aligned}&\int_\Omega \left((\mathcal{L})\int_G \left\|(\mathcal{IH})\int_0^T f(t,\omega,x)\,dW_t\right\|_V^2 d\mu(x)\right)d\mathbb{P}(\omega)\\ &= \mathbb{E}\left[\left\|(\mathcal{B})\int_G \left((\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right)d\mu(x)\right\|_V^2\right].\end{aligned}$$

Hence,

$$\lim_{n\to\infty} S\left((\mathcal{B})\int_G f(\cdot,\cdot,x)\,d\mu(x), D_n, \delta_n, \eta_n\right) = (\mathcal{B})\int_G \left((\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right)d\mu(x)$$

in $L^2(\Omega, V)$, where

$$\begin{aligned}&S\left((\mathcal{B})\int_G f(\cdot,\cdot,x)\,d\mu(x), D_n, \delta_n, \eta_n\right)\\ &= (D_n)\sum\left\{\sum_{j=1}^\infty (B^{(j)}_{v^{(n)}} - B^{(j)}_{\xi^{(n)}})\left((\mathcal{B})\int_G f(\xi^{(n)},\cdot,x)\,d\mu(x)\right)(\sqrt{\lambda_j}e_j)\right\}.\end{aligned}$$

By the sequential definition of the $\mathcal{IH}$-integral (Theorem 2.1),

$$(\mathcal{B})\int_G f(\cdot,\cdot,x)\,d\mu(x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$$

and

$$(\mathcal{IH})\int_0^T \left((\mathcal{B})\int_G f(t,\cdot,x)\,d\mu(x)\right)dW_t = (\mathcal{B})\int_G \left((\mathcal{IH})\int_0^T f(t,\cdot,x)\,dW_t\right)d\mu(x)$$

in $L^2(\Omega, V)$. The assertion holds for some subsequence. □

Next, we present a different version of stochastic Fubini-Tonelli theorem, where the integral in a measure space is defined using Henstock approach. In this case, we need to consider the Henstock integral on a measure space defined in [16].

One may refer to [16] for the discussions of the following concepts.

Let $(\mathcal{X}, \mathcal{S}, \ell)$ be a measure space, where $\mathcal{S}$ is a σ-algebra of subsets of $\mathcal{X}$ and ℓ is a measure on $\mathcal{S}$, endowed with a locally compact Hausdorff topology $\mathscr{T} \subseteq \mathcal{S}$. Let $\mathscr{T}_1$ be a basis for $\mathscr{T}$ consisting of relatively compact open sets. Since $\mathscr{T}$ is locally compact, $\mathscr{T}_1$ always exists, see [16, p.3]. From now onwards, we shall assume that for all $U \in \mathscr{T}_1$, we have $\ell(U) > 0$ if $U \neq \varnothing$, and $\ell(U) = \ell(\overline{U})$. This means that $\ell(\partial U) = 0$ for all $U \in \mathscr{T}_1$. Let

$$\mathcal{I}_0 = \{\overline{U_1} \setminus \overline{U_2} : U_1, U_2 \in \mathscr{T}_1 \text{ where } U_1 \nsubseteq U_2 \text{ and } U_2 \nsubseteq U_1\},$$

$$\mathcal{I}_1 = \{\cap_{i\in\Lambda} V_i \neq \varnothing : V_i \in \mathcal{I}_0 \text{ and } \Lambda \text{ is a finite index set}\}.$$

Observe that $\mathcal{I}_0$ includes all sets of the form $\overline{U}$, where $U \in \mathscr{T}_1$, and $\mathcal{I}_0 \subseteq \mathcal{I}_1$. One can verify that $\mathcal{I}_1$ is closed under finite intersections, if the intersection is nonempty, and elements of $\mathcal{I}_1$ are measurable since $\mathcal{S}$ is a σ-algebra. An element $I \in \mathcal{I}_1$ is called a *generalized interval*, or simply *interval*, when no confusion arises. Note that generalized intervals are relatively compact, though not necessarily closed or compact. Moreover, generalized intervals are connected. Since $\ell(U) = \ell(\overline{U})$ for all $U \in \mathscr{T}_1$, it follows that $\ell(I) = \ell(\overline{I})$ for all generalized interval I. If $\mathcal{X} = \mathbb{R}$ with the usual topology $\mathscr{T}$ on $\mathbb{R}$, then the generalized intervals are the usual bounded intervals, see [16, Example 1.1].

A finite union E of (possibly just one) mutually disjoint intervals is called an *elementary set*. Hence, generalized intervals are elementary sets. Note also that $\ell(E) = \ell(\overline{E})$ for all elementary set E since $\ell(I) = \ell(\overline{I})$ for all generalized interval I. An elementary set E is said to have a *finite measure* if $\ell(E) < \infty$. Throughout the following discussions, assume that an elementary set has a finite measure. If a subset E_0 of E is an elementary set, then E_0 is said to be an *elementary subset* of E. If $I \subseteq E$ and I is an interval, then we call I a *subinterval* of E.

Definition 3.1 *Let $\Delta : \overline{E} \to \mathscr{T}_1$ be a function such that for every $x \in \overline{E}$, we have $x \in \Delta(x) \in \mathscr{T}_1$. We call Δ a gauge on E. A finite collection $D = \{(I_{\zeta_i}, \zeta_i)\}_{i=1}^n$, or simply $D = \{(I_\zeta, \zeta)\}$, of inteval-point pairs is a Δ-fine division of E if $\{I_{\zeta_i}\}_{i=1}^n$ are mutually disjoint subintervals of E such that $E = \cup_{i=1}^n I_{\zeta_i}$ and for each $i \in \{1, 2, \dots, n\}$, $\zeta_i \in \overline{I_{\zeta_i}}$ and $I_{\zeta_i} \subseteq \Delta(\zeta_i)$.*

We note that given a gauge Δ on E, a Δ-fine division of E exists. A constructive proof of this result is presented in [16, Theorem 1.1 (Cousin's lemma)]. The following definition is analogous to [16, Definition 1.1].

Definition 3.2 *Let $(\mathcal{X}, \mathcal{S}, \ell)$ be a measure space, E be an elementary set in $\mathcal{X}$, and H be a Banach space. A function $f : \overline{E} \to H$ is said to be* Henstock integrable*, or $\mathcal{H}$-integrable, on E if there exists $A \in H$ such that for every $\epsilon > 0$, there is a gauge Δ on E such that for any Δ-fine division $D = \{(I_\zeta, \zeta)\}$ of E, we have*

$$\left\| (D) \sum f(\zeta)\ell(I_\zeta) - A \right\|_H < \epsilon.$$

In this case, f is $\mathcal{H}$-integrable to A on E and A is called the $\mathcal{H}$-integral of f which will be denoted by $(\mathcal{H}) \displaystyle\int_E f$.

The proofs of the standard properties of the $\mathcal{H}$-integral namely, uniqueness of the integral, linearity, integrability on every subinterval of $[0, T]$, and the Cauchy criterion are found in [16, p.22-37].

We shall now prove the second version of stochastic Fubini-Tonelli theorem.

Theorem 3.3 (Stochastic Fubini-Tonelli Theorem II) *Let $(\mathcal{X}, \mathcal{S}, \ell)$ be a measure space and $f : [0, T] \times \Omega \times \overline{E} \to L_2(U_Q, V)$ be a function such that for all $x \in \overline{E}$, the process $f(\cdot, \cdot, x) : [0, T] \times \Omega \to L_2(U_Q, V)$ is $\mathcal{IH}$-integrable on $[0, T]$ and the function $\overline{E} \to \Lambda_{\mathcal{IH}}(U_Q, V)$, $x \mapsto f(\cdot, \cdot, x)$ is $\mathcal{H}$-integrable on E. Suppose that for all $(t, \omega) \in [0, T] \times \Omega$, $f(t, \omega, \cdot)$ is $\mathcal{H}$-integrable on E and $(\mathcal{H}) \displaystyle\int_E f(\cdot, \cdot, x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$. Then the function $\overline{E} \to L^2(\Omega, V)$, $x \mapsto (\mathcal{IH}) \displaystyle\int_0^T f(t, \cdot, x)\, dW_t$ is $\mathcal{H}$-integrable on E and*

$$(\mathcal{H}) \int_E \left((\mathcal{IH}) \int_0^T f(t, \cdot, x)\, dW_t \right) = (\mathcal{IH}) \int_0^T \left((\mathcal{H}) \int_E f(t, \cdot, x) \right) dW_t$$

for almost all $\omega \in \Omega$.

Proof: We will show that the function $\overline{E} \to L^2(\Omega, V)$, $x \mapsto (\mathcal{IH}) \displaystyle\int_0^T f(t, \cdot, x)\, dW_t$ is $\mathcal{H}$-integrable on E. Since $(\mathcal{H}) \displaystyle\int_E f(\cdot, \cdot, x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$, let

$$B := (\mathcal{IH}) \int_0^T \left((\mathcal{H}) \int_E f(t, \cdot, x) \right) dW_t \in L^2(\Omega, V).$$

Since the function $\overline{E} \to \Lambda_{\mathcal{IH}}(U_Q, V)$, $x \mapsto f(\cdot,\cdot,x)$ is $\mathcal{H}$-integrable on E, there is a gauge Δ on E such that for any Δ-fine division $D = \{(I_\zeta, \zeta)\}$ of E, we have

$$\left\| (D) \sum f(\cdot,\cdot,\zeta)\ell(I_\zeta) - (\mathcal{H}) \int_E f(\cdot,\cdot,x) \right\|_{\Lambda_{\mathcal{IH}}} < \epsilon.$$

Using Theorem 2.2, we have

$$\begin{aligned}
&\left\| (D) \sum \left((\mathcal{IH}) \int_0^T f(t,\cdot,\zeta)\, dW_t \right) \ell(I_\zeta) - B \right\|_{L^2(\Omega,V)} \\
= &\left(\mathbb{E}\left[\left\| (D) \sum \left((\mathcal{IH}) \int_0^T f(t,\cdot,\zeta)\, dW_t \right) \ell(I_\zeta) - B \right\|_V^2 \right] \right)^{1/2} \\
= &\left(\mathbb{E}\left[\left\| (\mathcal{IH}) \int_0^T \left((D) \sum f(t,\cdot,\zeta)\ell(I_\zeta) - (\mathcal{H}) \int_E f(t,\cdot,x) \right) dW_t \right\|_V^2 \right] \right)^{1/2} \\
= &\left(\mathbb{E}\left[(\mathcal{L}) \int_0^T \left\| (D) \sum f(t,\cdot,\zeta)\ell(I_\zeta) - (\mathcal{H}) \int_E f(t,\cdot,x) \right\|_{L_2(U_Q,V)}^2 dt \right] \right)^{1/2} \\
= &\left\| (D) \sum f(\cdot,\cdot,\zeta)\ell(I_\zeta) - (\mathcal{H}) \int_E f(\cdot,\cdot,x) \right\|_{\Lambda_{\mathcal{IH}}} < \epsilon.
\end{aligned}$$

This completes the proof of the theorem. □

4. Conclusion and Recommendations

In this paper, we formulate two versions of stochastic Fubini-Tonelli theorem for the Itô-Henstock integral of a Hilbert-Schmidt-valued stochastic process driven by a Hilbert space-valued Q-Wiener process, defined in [10, Definition 2.4]. The first version of the theorem uses the Bochner integral for the integral on a finite measure space while the second version uses the Henstock integral defined in [16, Definition 1.1], for the integral on a measure space in which the measure is endowed with a locally compact Hausdorff topology. We observe that in the second version of the stochastic Fubini-Tonelli theorem, the integral on a measure space is interpreted in the sense of Henstock, which, in certain aspects, is more powerful than the Bochner integral [25, p.275]. The proof is both simple and straightforward. However, the assumption used is quite strong, as it requires that $(\mathcal{H}) \int_E f(\cdot,\cdot,x) \in \Lambda_{\mathcal{IH}}(U_Q, V)$. This condition could be weakened if a Fubini-Tonelli type theorem for the Henstock integral can be established, and formulate a result that preserves the Henstock integral when using the inner product. Moreover, a worthwhile direction for further investigation is to use Henstock-Kurzweil approach to deal with stochastic partial differential equations.

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