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On S-Weakly Quasi n-Absorbing Submodules and their Extensions

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ABSTRACT: Let R be a commutative ring with identity, S a multiplicative subset of R and n a positive integer. This paper aims to introduce the concept of S-weakly quasi n-absorbing submodules as a natural generalization of weakly quasi n-absorbing submodules. Specifically, a submodule N of an R-module M with $(N :_R M) \cap S = \emptyset$ is termed S-weakly quasi n-absorbing submodule if there exists an (fixed) $s \in S$ such that for some $a \in R$ and $m \in M$, whenever $0 \neq a^n m \in N$, it holds that either $sa^n \in (N :_R M)$ or $sa^{n-1}m \in N$. This element $s \in S$ is referred to as an S-weakly element of N. In addition to establishing various properties, characterizations, and illustrative examples of the concept, this study examines the behavior of S-weakly quasi n-absorbing submodules within a variety of algebraic framework, including localizations, homomorphic images, idealizations, and amalgamations.

Keywords: S-weakly quasi n-absorbing submodules, S-weakly quasi n-absorbing ideals, trivial extension, amalgamated algebra.

Contents

1. Introduction

In this article, we assume that all rings are commutative with a non-zero identity, and that all modules are unital. We begin with some definitions and notations that will be used throughout the paper. Let R be a commutative ring. $U(R)$ (resp., $Z(R) = R \setminus \mathrm{Reg}(R)$) denotes the set of (resp., zero-divisors) units elements of R. Recall that a nonempty subset S of R is said to be a multiplicative set if it satisfies the following conditions: $0 \notin S$, $1 \in S$, $s_1 s_2 \in S$, for all $s_1, s_2 \in S$. An ideal I of R is said to be proper if $I \neq R$. Let M be an R-module and N a submodule of M. We define $(N :_R M) := \{x \in R : xm \in N \text{ for all } m \in M\}$. This set is an ideal of R, called the residual ideal of N in M. We recall from [3] that an ideal I is said to be weakly prime if for all $a, b \in R$ with $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. On the other hand, from [21] an ideal I disjoint from a multiplicative set S is S-weakly prime if there exists $s \in S$ such that for all $a, b \in R$, if $0 \neq ab \in I$, then $sa \in I$ or $sb \in I$.

In 2022, Hani A. Khashan and Ece Yetkin Celikel introduced the concept of weakly S-prime submodules as a generalization of weakly prime submodules. Following [20], a submodule N of an R-module M with $(N :_R M) \cap S = \emptyset$ is called weakly S-prime submodule if there exists (a fixed) $s \in S$ such that for $a \in R$ and $m \in M$, whenever $0 \neq am \in N$, then either $sa \in (N :_R M)$ or $sm \in N$.

In [16], the authors introduced and investigated the concept of weakly quasi n-absorbing submodules where n is a positive integer. They defined a submodule N of M to be weakly quasi n-absorbing submodule if whenever $a \in R$ and $m \in M$ with $0 \neq a^n m \in N$, then either $a^n \in (N :_R M)$ or $a^{n-1}m \in N$. Furthermore, a proper submodule N of M is called semiprime if whenever $r \in R$ and $m \in M$ with $r^2 m \in N$, then $rm \in N$. For more details about the concept of prime, n-absorbing and related notions, we refer the reader to [22,5,14].

This paper consists of three sections, including the introduction. In Section 2, we introduce the concept of an S-weakly quasi n-absorbing submodule as a generalization of weakly quasi n-absorbing submodule. Let R be a ring, S a multiplicative subset of R, M an R-module, and N a submodule of M with $(N :_R M) \cap S = \emptyset$. We say that N is an S-weakly quasi n-absorbing submodule if there exists

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an (fixed) $s \in S$ such that for some $a \in R$ and $m \in M$ if $0 \neq a^n m \in N$ then either $sa^n \in (N :_R M)$ or $sa^{n-1}m \in N$. In particular, if we take $M = R$ as an R-module, we obtain the definition of an S-weakly quasi n-absorbing ideal as follows. An ideal I of R disjoint from S is said to be an S-weakly quasi n-absorbing if there exists an (fixed) $s \in S$ such that whenever $0 \neq a^n b \in I$ for some $a, b \in R$, then either $sa^n \in I$ or $sa^{n-1}b \in I$. This fixed element $s \in S$ is called an S-weakly element of N (and of I). We begin by illustrating an example of an S-weakly quasi n-absorbing submodule that does not qualify as an weakly quasi n-absorbing submodule (cf. Examples 2.1, 2.2). Subsequently, we explore several properties of S-weakly quasi n-absorbing submodules, demonstrating that they mirror many of the key characteristics weakly quasi n-absorbing submodules (see Theorem 2.1). Our study extends to various contexts within commutative ring theory, covering topics such as localization and homomorphic image (refer to Propositions 2.2 and Proposition 2.3, respectively). In the final section, we delve deeper into the structure of trivial ring extensions $R \ltimes M$, (Theorem 3.1 and Proposition 3.1) and amalgamation rings $A \bowtie^f J$, (see Theorems 3.2, 3.3), with a particular focus on the form of S-weakly quasi n-absorbing ideals. Any notation or terminology left undefined can be found in [25].

2. Main Results

In this section, we discuss the core properties of S-weakly quasi n-absorbing submodules and investigate their behavior across various classes of ring, highlighting some of their key characteristics. Additionally, we examine their relationships with other types of submodules (ideals).

Definition 2.1 Let S be a multiplicative set of a ring R, M an R-module and n be a positive integer. A submodule N of M with $(N :_R M) \cap S = \emptyset$ is called an S-weakly quasi n-absorbing submodule if there exists an (fixed) $s \in S$ such that for all $a \in R$ and $m \in M$ if $0 \neq a^n m \in N$, it follows that either $sa^n \in (N :_R M)$ or $sa^{n-1}m \in N$. This fixed element $s \in S$ is referred to as an S-weakly element of N.

In particular, by taking $M = R$ considered as an R-module, we obtain the corresponding definition of an S-weakly quasi n-absorbing ideal, as given below:

Definition 2.2 Let S be a multiplicative subset of a ring R, and let n be a positive integer. An ideal I of R disjoint from S is said to be an S-weakly quasi n-absorbing if there exists an (fixed) $s \in S$ such that whenever $0 \neq a^n b \in I$ for some $a, b \in R$, then either $sa^n \in I$ or $sa^{n-1}b \in I$. This fixed element $s \in S$ is called an S-weakly element of I.

It is evident from the preceding definitions that every weakly quasi n-absorbing submodule N with $(N :_R M) \cap S = \emptyset$ is clearly an S-weakly quasi n-absorbing. In particular when S consists entirely of units elements, then the converse also holds. However, the converse of this implication does not hold in general, as illustrated by the following examples.

Example 2.1 *Let* $M := \mathbb{Z}/2^4.3\mathbb{Z}$ *be a* $\mathbb{Z}$*-module, and consider the submodule* $N = 2^2.3M$*. Let us take the multiplicative subset* $S = \{2^n : n \in \mathbb{N}\}$*. It is straightforward to verify that* $(N :_{\mathbb{Z}} M) = 12\mathbb{Z}$ *and* $(N :_{\mathbb{Z}} M) \cap S = \emptyset$*. Let* $a \in \mathbb{Z}$ *and* $m \in M$ *such that* $0 \neq a^2 m \in N$*. Then, for* $s = 2^3$*, we obtain* $sam = 0 \in N$*. Therefore,* N *is an* S*-weakly quasi* 2*-absorbing submodule of* M*. However,* N *is not a weakly quasi* 2*-absorbing in the usual sense. Indeed, for* $a = 2 \in \mathbb{Z}$ *and* $m = \overline{3} \in M$*, although* $a^2 m = \overline{2^2 \cdot 3} \in N \setminus \{0\}$*, we observe that neither* $am = \overline{2 \cdot 3} \in N$ *nor* $a^2 = 2^2 \in (N :_{\mathbb{Z}} M) = 12\mathbb{Z}$*.*

Example 2.2 *Consider the ring* $R = \mathbb{Z}[X]$ *and the multiplicative set* $S = \left\{2^k : k > 0\right\} \subseteq R$*. Let* $I = 2^n X\mathbb{Z}$*. Then,* I *is an* S*-weakly quasi* n*-absorbing ideal, with* $s = 2^n \in S$ *an* S*-weakly element of* I*. Let* $a, b \in R$ *such that* $0 \neq a^n b \in I \subseteq X\mathbb{Z}[X]$*. Since* $X\mathbb{Z}[X]$ *is prime ideal of* R*, it follows that either* $a \in X\mathbb{Z}[X]$ *or* $b \in X\mathbb{Z}[X]$*. In the former case, we have* $sa \in X\mathbb{Z}[X]$*, and hence* $sa^n \in I$*. In the letter case, we conclude* $sb \in sX\mathbb{Z}[X] = I$*, and consequently* $sa^{n-1}b \in I$*. Therefore,* I *satisfies the condition of being an* S*-weakly quasi* n*-absorbing ideal. However,* I *is not a weakly quasi* n*-absorbing ideal. For instance,* $0 \neq 2^n X \in I$*, but neither* $2^{n-1}X \in I$ *nor* $2^n \in I$*.*

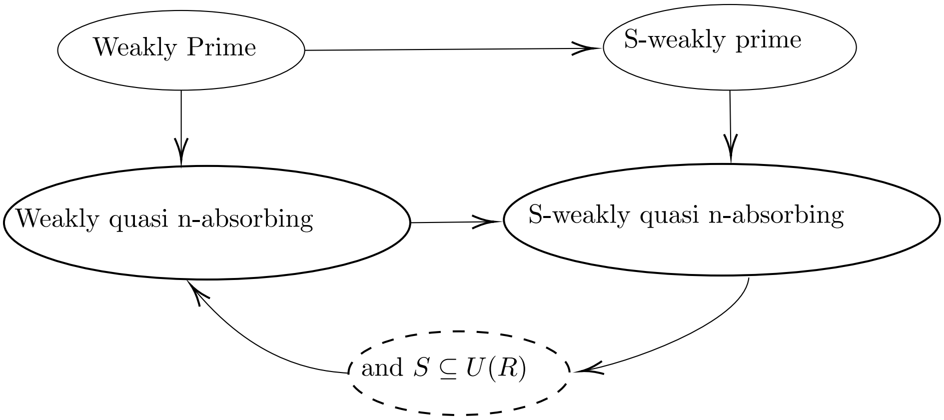
Recall from [27] that an R-module M is said to be u-S-torsion(u-always abbreviates uniformly), if there exists an element s of S such that $sM = 0$.

Remark 2.1 (1) *Every submodule N of a u-S-torsion module M with $(N :_R M) \cap S = \emptyset$ is an S-weakly quasi n-absorbing submodule.*

(2) *For $n = 1$, the concepts of S-weakly quasi n-absorbing submodules and weakly S-prime submodules are equivalent. Thus, the notion of S-weakly quasi n-absorbing submodule can be regarded as a natural generalization of weakly S-prime submodules. In what follows, we consider the case where $n \geq 2$. Under this assumption, every weakly S-prime submodule is an S-weakly quasi n-absorbing submodule. By the next example, we show that this generalization is proper.*

Example 2.3 *Let $M = \mathbb{Z} \times \mathbb{Z}_6$ be a module over the ring $\mathbb{Z}$, $S = \{1\}$ a multiplicative subset of $\mathbb{Z}$. Consider the submodule $N = 2\mathbb{Z} \times \langle \bar{3} \rangle \subseteq M$. Then N is a S-weakly quasi 2-absorbing submodule which is not S-weakly prime. To verify this argument, let $a \in \mathbb{Z}$ and $(m_1, m_2) \in M$ such that $0 \neq a^2(m_1, m_2) \in N$. If $a \in 6\mathbb{Z}$, then $a^2 \in (N :_{\mathbb{Z}} M)$, we are done. So, assume that $a \notin 6\mathbb{Z}$. We have the following two cases: Case I: Let $a \in 2\mathbb{Z}$ but $a \notin 3\mathbb{Z}$. Since $a^2 m_2 \in \langle \bar{3} \rangle$ and $a^2 \notin (\langle \bar{3} \rangle :_{\mathbb{Z}} \mathbb{Z}_6) = 3\mathbb{Z}$, we have $m_2 \in \langle \bar{3} \rangle$ and hence, $a(m_1, m_2) \in N$. Case II. Let $a \in 3\mathbb{Z}$ but $a \notin 2\mathbb{Z}$. Now, since $a^2 m_1 \in 2\mathbb{Z}$ and $a^2 \notin (2\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 2\mathbb{Z}$, we have $m_1 \in 2\mathbb{Z}$ and so, $a(m_1, m_2) \in N$, as required. On the other hand, it is not a S-weakly prime submodule as $0 \neq 2(1, 3) \in N$ but $2 \notin (N :_{\mathbb{Z}} M) = 6\mathbb{Z}$ and $(1, 3) \notin N$.*

We now illustrate the position of the concept of S-weakly quasi n-absorbing ideals within the hierarchy of related ideal-theoretic notions through the following diagram:



Let R be a ring, and let $S \subseteq R$ be a multiplicative subset. We define the saturation of S, denoted S^*, as the set $\{r \in R : \frac{r}{1}$ is a unit in $S^{-1}R\}$. It is clear that $S \subseteq S^*$, and that S^* itself forms a multiplicative subset of R. The set S is said to be saturated if it coincides with its saturation, that is, if $S^* = S$ (see [12]). For an R-module M, we denote $U_M(R)$ as the set $\{r \in R : rM = M\}$, which contains $U(R)$, and is a saturated multiplicative subset of R.

Proposition 2.1 *Let S be a multiplicative subset of a ring R, and let N be a submodule of an R-module M. Then the following assertions hold:*

(1) *Let $T \subseteq S$ be two multiplicative subsets of R. If N is a T-weakly quasi n-absorbing submodule such that $(N :_R M) \cap S = \emptyset$, then N is an S-weakly quasi n-absorbing submodule. Conversely, if for each $s \in S$, there exists an element $t \in T$ such that $st \in T$, and N is an S-weakly quasi n-absorbing submodule of M, then N is a T-weakly quasi n-absorbing submodule of M.*

(2) *N is an S-weakly quasi n-absorbing submodule of M if and only if N is an S^*-weakly quasi n-absorbing submodule.*

(3) *Let $M = Rm_1$ be a cyclic R-module, $S \subseteq U_M(R)$ and N be a submodule of M with $(N :_R M) \cap S = \emptyset$. Then N is a weakly quasi n-absorbing submodule if and only if N is an S-weakly quasi n-absorbing submodule.*

Proof:

(1) Straightforward.

(2) Suppose that N is an S-weakly quasi n-absorbing submodule. It is clear that $(N :_R M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, it follows from (1) that N is an S^*-weakly quasi n-absorbing submodule. For the converse, let $0 \neq a^n m \in N$ for some $a \in R$ and $m \in M$. Since N is an S^*-weakly quasi n-absorbing submodule, there exists $s \in S^*$ such that $sa^n \in (N :_R M)$ or $sa^{n-1}m \in N$. Since $\frac{s}{1}$ is a unit in $S^{-1}R$, there exists $t \in S$ and $r \in R$ such that $str \in S$. Set $s' = str \in S$. Then, we have $s'a^n \in (N :_R M)$ or $s'a^{n-1}m \in N$, as needed.

(3) Suppose that N is an S-weakly quasi n-absorbing submodule of $M = Rm_1$. Let $0 \neq a^n m \in N$ for some $a \in R$ and $m = rm_1 \in M$. Then, there exists $s \in S$ such that $sa^n \in (N :_R M)$ or $sa^{n-1}rm_1 \in N$. If $sa^{n-1}rm_1 \in N$, then since $S \subseteq U_M(R)$, it follows that $sa^{n-1}Rm_1 = a^{n-1}M \subseteq N$, which implies $a^{n-1}m = a^{n-1}rm_1 \in N$. Now, suppose $sa^n \in (N :_R M)$. Then $sa^n M \subseteq N$. Since $s \in S \subseteq U_M(R)$, we have $sM = M$, hence $a^n M = sa^n M \subseteq N$, and thus $a^n \in (N :_R M)$. Therefore, N is a weakly quasi n-absorbing submodule. The converse is clear.

□

Note that the condition "$S \subseteq U_M(R)$" in Proposition 2.1 is crucial. (see Example 2.1) The following three results provide useful characterizations of the S-weakly quasi n-absorbing submodule property.

Theorem 2.1 *Let S be a multiplicative subset of a ring R. For a submodule N of an R-module M satisfying $(N :_R M) \cap S = \emptyset$, the following statements are equivalent:*

(1) *N is an S-weakly quasi n-absorbing submodule of M.*

(2) *There exists $s \in S$ such that $(N :_M a^n) = (0 :_M a^n)$ or $(N :_M a^n) \subseteq (N :_M sa^{n-1})$ for each $a \in R$ with $a^n \notin (N :_R sM)$.*

(3) *There exists $s \in S$ such that for any $a \in R$ and for any submodule K of M, if $0 \neq a^n K \subseteq N$, then $sa^n \in (N :_R M)$ or $sa^{n-1}K \subseteq N$.*

Proof: (1) ⇒ (2) Let s be an S-weakly element of N, and $a^n \notin (N :_M sM)$. Let $m \in (N :_M a^n)$. If $a^n m = 0$, then clearly $m \in (0 :_M a^n)$. If $0 \neq a^n m \in N$, then since $sa^n \notin (N :_R M)$ and N is an S-weakly quasi n-absorbing submodule in M, we conclude $sa^{n-1}m \in N$. Thus, $m \in (N :_M sa^{n-1})$, and so $(N :_M a^n) \subseteq (0 :_M a^n) \cup (N :_M sa^{n-1})$. Therefore, $(N :_M a^n) \subseteq (0 :_M a^n)$ which implies $(N :_M a^n) = (0 :_M a^n)$ or $(N :_M a) \subseteq (N :_M sa^{n-1})$.

(2) ⇒ (3) Choose $s \in S$ as in (2), and suppose $0 \neq a^n K \subseteq N$ and $sa^n \notin (N :_R M)$ for some $a \in R$ and a submodule K of M. Then $K \subseteq (N :_M a^n) \setminus (0 :_M a^n)$, and by (2), we have $K \subseteq (N :_M a^n) \subseteq (N :_M sa^{n-1})$. Hence, $sa^{n-1}K \subseteq N$, as required.

(3) ⇒ (1) Let $a \in R$ and $m \in M$ be such that $0 \neq a^n m \in N$. Choose $s \in S$ as in (3), and set $K = Rm$ in (3). It then follows that the desired conclusion holds.

□

This theorem establishes a structural relationship between the S-weakly quasi n-absorbing submodule and its annihilator ideal in a faithful module, and conversely in the cyclic case.

Theorem 2.2 *Let S be a multiplicative set of a ring R, M a faithful R-module, and N a submodule of M. If N is a S-weakly quasi n-absorbing submodule of M, then $(N :_R M)$ is an S-weakly quasi n-absorbing ideal of R. The converse holds if M is a cyclic faithful R-module.*

Proof:

Let $0 \neq a^n b \in (N :_R M)$ for some $a, b \in R$. Since M is a faithful R-module, we have $0 \neq a^n(bM) \subseteq N$. According to Theorem 2.1, there exists $s \in S$ such that either $sa^{n-1}(bM) \subseteq N$, or $sa^n \in (N :_R M)$.

Hence, $(N :_R M)$ is an S-weakly quasi n-absorbing ideal of R. Conversely, assume that $(N :_R M)$ is an S-weakly quasi n-absorbing ideal of R and that $M = Rm$ is a cyclic faithful R-module. Let $a \in R$ and $x := bm \in M$ such that $0 \neq a^n x = a^n bm \in N$. Therefore, $0 \neq a^n b \in (N :_R m) = (N :_R M)$. Since $(N :_R M)$ is an S-weakly quasi n-absorbing ideal of R, we have either $sa^n \in (N :_R M)$ or $sa^{n-1}b \in (N :_R M)$ for some $s \in S$. Thus, we have either $sa^n \in (N :_R M)$ or $sa^{n-1}bm = sa^{n-1}x \in N$ and N is an S-weakly quasi n-absorbing submodule of M. □

Let S be a multiplicative subset of a ring R. We say that S satisfies the maximal multiple condition if there exists an $s \in S$ such that every $t \in S$ divides s, [2]. For instance, if S is finite, then S satisfies the maximal multiple condition. Also $U(R)$ is a multiplicative subset that satisfies the maximal multiple condition. The following proposition explores how S-weakly quasi n-absorbing submodules behave under localization.

Proposition 2.2 *Let N be a submodule of an R-module M, and let S be a multiplicative subset of the ring R such that $Z(M) \cap S = \emptyset$. Then the following assertions hold:*

1. *If N is an S-weakly quasi n-absorbing submodule of M, then $S^{-1}N$ is a weakly quasi n-absorbing of $S^{-1}M$ and there exists an $s \in S$ such that $(N :_M t^n) \subseteq (N :_M st^{n-1})$ for all $t \in S$.*
2. *If S satisfies the maximal multiple condition and $S^{-1}N$ is a weakly quasi n-absorbing $S^{-1}R$-submodule of $S^{-1}M$, then N is an S-weakly quasi n-absorbing submodule of M.*

Proof:

1. Assume that N is an S-weakly quasi n-absorbing submodule of M and suppose that $\frac{0}{1} \neq \left(\frac{a}{s_1}\right)^n \left(\frac{m}{s_2}\right) \in S^{-1}N$, where $a \in R$ and $m \in M$. Then there exists an element u of S such that $0 \neq ua^n m \in N$. Hence, $0 \neq (ua)^n m \in N$, then by hypothesis there exists $s \in S$ such that either $s(ua)^n \in (N :_R M)$ or $s(ua)^{n-1}m \in N$. This yields, $\left(\frac{a}{s_1}\right)^n = \frac{su^n a^n}{su^n s_1^n} \in S^{-1}(N :_R M) = \left(S^{-1}N :_{S^{-1}R} S^{-1}M\right)$ or $\left(\frac{a}{s_1}\right)^{n-1}\left(\frac{m}{s_2}\right) = \frac{su^{n-1}a^{n-1}m}{su^{n-1}s_1^{n-1}s_2} \in S^{-1}N$, hence $S^{-1}N$ is a weakly quasi n-absorbing submodule of $S^{-1}M$. Now, let $t \in S$ and $m \in (N :_M t^n)$. It follows that $0 \neq t^n m \in N$ as $Z(M) \cap S = \emptyset$, and so either $st^n \in (N :_R M)$ or $st^{n-1}m \in N$. Since the first case gives a contradiction, we have $m \in (N :_M st^{n-1})$. Thus, $(N :_M t^n) \subseteq (N :_M st^{n-1})$ for all $t \in S$.
2. Let $s_{max} \in S$ be the element that satisfies the maximal multiple condition. Suppose that $0 \neq a^n m \in N$ for some $a \in R$ and $m \in M$. Then $\frac{a^n}{1}\frac{m}{1} \in S^{-1}N$, which implies that either $\frac{a^n}{1} \in \left(S^{-1}N :_{S^{-1}R} S^{-1}M\right) = S^{-1}(N :_R M)$ or $\frac{a^{n-1}}{1}\frac{m}{1} \in S^{-1}N$. Therefore, there exist $u, v \in S$ such that $ua^n \in (N :_R M)$ or $va^{n-1}m \in N$. Since both u and v divide s_{max}, it follows that $s_{max}a^n \in (N :_R M)$ or $s_{max}a^{n-1}m \in N$, as desired.

□

Lemma 2.1 *[26] Let $f : M_1 \longrightarrow M_2$ be an R-module epimorphism. Then the following statements hold:*

1. *If N is a submodule of M_1, then $(N :_R M_1) \subseteq (f(N) :_R M_2)$.*
2. *If K is a submodule of M_2, then $(K :_R M_2) \subseteq \left(f^{-1}(K) :_R M_1\right)$.*

In the following proposition, we note some facts concerning the stability of S-weakly quasi n-absorbing property under homomorphism.

Proposition 2.3 *Let $f : M_1 \longrightarrow M_2$ be an R-module homomorphism, and let S be a multiplicative subset of R. Then the following statements hold:*

1. *If f is an epimorphism and N is an S-weakly quasi n-absorbing submodule of M_1 containing* $\ker(f)$, *such that $(f(N) :_R M_2) \cap S = \emptyset$, then $f(N)$ is an S-weakly quasi n-absorbing submodule of M_2.*

(2) *If f is a monomorphism and K is an S-weakly quasi n-absorbing submodule of M_2 such that $\big(f^{-1}(K) :_R M_1\big) \cap S = \emptyset$, then $f^{-1}(K)$ is an S-weakly quasi n-absorbing submodule of M_1.*

Proof:

(1) Let $a \in R$ and $m \in M$ with $0 \neq a^n m \in f(N)$. Say $m = f(m_1)$ for some $m_1 \in M_1$. Since $a^n f(m_1) \in f(N)$ and $\ker(f) \subseteq N$, we have $a^n m_1 \in N$ which implies that there exists an $s \in S$ such that $sa^n \in (N :_R M_1)$ or $sa^{n-1} m_1 \in N$. Thus, $sa^n \in (N :_R M_1) \subseteq (f(N) :_R M_2)$ or $sa^{n-1} f(m_1) \in f(N) \subseteq M_2$ by Lemma 2.1, as needed.

(2) First, we show that $\big(f^{-1}(K) :_R M_1\big) \cap S = \emptyset$. Suppose that $r \in \big(f^{-1}(K) :_R M_1\big) \cap S$, then $rM_1 \subseteq f^{-1}(K)$ implies $rM_2 = rf(M_1) \subseteq f(f^{-1}(K)) \subseteq K$, hence $r \in (K :_R M_2)$, a contradiction. Let $a \in R$ and $m \in M_1$ with $0 \neq a^n m \in f^{-1}(K)$. Since $\ker(f) = \{0\}$, we get $0 \neq a^n f(m) \in K$ which implies that there exists $s \in S$ such that $sa^n \in (K :_R M_2)$ or $sa^{n-1} f(m) \in K$. Then, we obtain $sa^n \in (K :_R M_2) \subseteq \big(f^{-1}(K) :_R M_1\big)$ or $sa^{n-1} m \in f^{-1}(K)$, by Lemma 2.1. Therefore, $f^{-1}(K)$ is an S-weakly quasi n-absorbing submodule of M_1.

□

Let R be a commutative ring, and let M be an R-module. Recall from [1] that M is called a multiplication module if every submodule $N \subseteq M$ is of the form $N = IM$ for some ideal I of R. As a consequence of the previous proposition, we give the following explicit result.

Corollary 2.1 *Let S be a multiplicatively closed subset of a ring R and N, K, L be submodules of R-module M. Then the following assertions hold:*

(1) *If N is an S-weakly quasi n-absorbing submodule of M and K is a submodule of M with $K \subseteq N$, then N/K is an S-weakly quasi n-absorbing submodule of M/K.*

(2) *If N is an S-weakly quasi n-absorbing submodule of M such that $(K :_R N) \cap S = \emptyset$, then $K \cap N$ is an S-weakly quasi n-absorbing submodule of N.*

(3) *Let N be an S-weakly quasi n-absorbing submodule of M and for any submodule K of M with $(K :_R M) \cap S \neq \emptyset$, then $N \cap K$ is an S-weakly quasi n-absorbing submodule of M. Additionally, if M is a multiplication module, then NK is an S-weakly quasi n-absorbing submodule of M.*

(4) *Let $K \subseteq N$. If K is an S-weakly quasi n-absorbing submodule of M, and N/K is an S-weakly quasi n-absorbing submodule of M/K, then N is an S-weakly quasi n-absorbing submodule of M.*

(5) *Let N be an S-weakly quasi n-absorbing submodule of M and K is an S-weakly quasi n-absorbing submodule of M such that $((N+K) :_R M) \cap S = \emptyset$, then $N+K$ is also an S-weakly quasi n-absorbing submodule of M.*

Proof:

(1) It is clear that $(N/K :_R M/K) \cap S = \emptyset$. Consider the canonical homomorphism $\pi : M \longrightarrow M/K$ defined by $\pi(m) = m + K$ for all $m \in M$. It follows from Proposition 2.3(1) that N/K is an S-weakly quasi n-absorbing submodule of M/K.

(2) Consider that the injection $i : K \longrightarrow M$ defined by $i(k) = k$ for all $k \in K$. It is easy to see that $\big(i^{-1}(N) :_R N\big) \cap S = \emptyset$. So by applying the Proposition 2.3(2), $i^{-1}(N) = N \cap K$ is an S-weakly quasi n-absorbing submodule of N.

(3) Observe that $(N \cap K :_R M) \cap S = \emptyset$, since $(N :_M M) \cap S = \emptyset$. Let $s \in S$ be an S-weakly element of N, and let $0 \neq a^n m \in N \cap K \subseteq N$. Then $sa^n \in (N :_R M)$ or $sa^{n-1} m \in N$. Choose $s' \in (K :_R M) \cap S$. Then $ss'a^n \in (N :_R M) \cap (K :_R M) = (N \cap K :_R M)$ or $ss'a^{n-1} m \in N \cap (K :_R M)M = N \cap K$. Therefore, $N \cap K$ is an S-weakly quasi n-absorbing submodule of M, with an S-weakly element $t = ss'$. Now, taking into consideration that $NK = (N :_R M)(K :_R M)M$, the remainder of the proof unfolds in an entirely analogous manner.

(4) Suppose that K is an S-weakly quasi n-absorbing submodule of M, and N/K is an S-weakly quasi n-absorbing submodule of R-module M/K with respect to s_1 and s_2, respectively. Let $a \in R$ and $m \in M$ with $0 \neq a^n m \in N$. If $a^n m \in K$, then since K is S-weakly quasi n-absorbing, we have either $s_1 a^{n-1} m \in K \subseteq N$ or $s_1 a^n \in (K :_R M) \subseteq (N :_R M)$. Now, suppose that $a^n m \notin K$. Hence $0 \neq a^n(m + K) \in N/K$, we conclude that either $s_2 a^{n-1}(m + K) \in N/K$ or $s_2 a^n \in (N/K :_R M/K)$ and therefore, $s_2 a^{n-1} m \in N$ or $s_2 a^n \in (N :_R M)$. Thus, N is an S-weakly quasi n-absorbing submodule of M with S-weakly element $s := s_1 s_2$.

(5) Assume that N and K are two S-weakly quasi n-absorbing submodules of M. By Corollary 2.1(1), the quotient $N/(N \cap K)$ inherits the property of being an S-weakly quasi n-absorbing submodule of the module $M/(N \cap K)$. Now, by the well-known module isomorphism $\frac{N}{N \cap K} \cong \frac{N+K}{K}$. It follows that $(N + K)/K$ is an S-weakly quasi n-absorbing submodule of M/K. Consequently, by (4), we conclude that the sum $N + K$ is itself an S-weakly quasi n-absorbing submodule of M.

□

At the end of this section, we discuss S-weakly quasi n-absorbing submodules of cartesian products of modules.

Theorem 2.3 *Let S_1 and S_2 be multiplicatively closed subsets of rings R_1, R_2 and N_1, N_2 be proper submodules of R_i-modules M_i, respectively such that $(N_i :_R M_i) \cap S_i = \emptyset$ where $i = 1, 2$. Then the following statements hold:*

(1) *N_1 is an S_1-weakly quasi n-absorbing submodule of M_1 if and only if $N_1 \times 0$ is an $(S_1 \times \{0\})$-weakly quasi n-absorbing submodule of $M_1 \times M_2$.*

(2) *N_2 is an S_2-weakly quasi n-absorbing submodule of M_2 if and only if $0 \times N_2$ is an $(\{0\} \times S_2)$-weakly quasi n-absorbing submodule of $M_1 \times M_2$.*

Proof: (1) Suppose that N_1 is an S-weakly quasi n-absorbing submodule of M_1 and let s_1 be an S_1-element of N_1. Let (a, b) be an element of $R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ with $(0, 0) \neq (a, b)^n(m_1, m_2) \in N_1 \times 0$, then $0 \neq a^n m_1 \in N_1$ and so either $s_1 a^n \in (N_1 :_R M_1)$ or $s_1 a^{n-1} m_1 \in N_1$. Then, we get either $(s_1, 0)(a, b)^n \in (N_1 \times 0 :_R M_1 \times M_2)$ or $(s_1, 0)(a, b)^{n-1}(m_1, m_2) \in N_1 \times 0$. Hence $N_1 \times 0$ is an $(S \times \{0\})$-weakly quasi n-absorbing submodule of $M_1 \times M_2$. Conversely, assume that $N_1 \times 0$ is an $(S_1 \times \{0\})$-weakly quasi n-absorbing submodule of $M_1 \times M_2$, let $(s_1, 0)$ be a $(S_1 \times \{0\})$-element of $N_1 \times 0$. Let a be an element of R and $m_1 \in M_1$ with $0 \neq a^n m_1 \in N_1$, then $(0, 0) \neq (a, 1)^n(m_1, 0) \in N_1 \times 0$. So $(s_1, 0)(a, 1)^n \in (N_1 \times 0 :_R M_1 \times M_2)$ or we have $(s_1, 0)(a, 1)^{n-1}(m_1, 0) \in N_1 \times 0$. Then we conclude that $s_1 a^n \in (N_1 : M_1)$ or $s_1 a^{n-1} m_1 \in N_1$. Hence N_1 is an S_1-weakly quasi n-absorbing submodule of M_1.

(2) Similar to the proof of assertion (1) above. □

We conclude this section by showing that the intersection of a family of S-weakly quasi n-absorbing submodule is itself an S-weakly quasi n-absorbing submodule.

Proposition 2.4 *Let S be a multiplicative subset of a ring R and M be a multiplication R-module. If $\{N_i\}_{i=1}^k$ is a family of S-weakly quasi n-absorbing submodules of M, then $\bigcap_{i=1}^k N_i$ is S-weakly quasi n-absorbing submodules of M.*

Proof: Let $s_i \in S$ be an S-element of N_i for each $i = 1, \ldots, k$. First, note that $\left(\bigcap_{i=1}^k N_i\right) \cap S = \emptyset$. Put $s = s_1 \cdots s_k$. Suppose that $0 \neq a^n m \in \bigcap_{i=1}^k N_i$ but $s a^n \notin \left(\bigcap_{i=1}^k N_i : M\right)$ for some a of R and $m \in M$. Then $s a^n \notin (N_j : M)$ for some $j = 1, \ldots, k$. Since N_j is S-weakly quasi n-absorbing and $0 \neq a^n m \in N_j$, we get $s_j m \in N_j$ and so $sm \in \left(\bigcap_{i=1}^k N_i\right)$, we are done. □

3. Idealizations and Amalgamations

Let M be a unitary R-module. Recall from [4], the idealization $R \ltimes M$ of M is the direct sum $R \ltimes M = R \oplus M$ which is a commutative ring with componentwise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. The basic properties of trivial ring extensions are summarized in the books [13,15]. Trivial ring extensions have been particularly useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. For instance, see [6,18,19,23].

Let S be a multiplicative subset of R. Then it is easy to verify that $S \oplus M = \{(s, m) : s \in S$ and $m \in M\}$ and $S \ltimes 0 = \{(s, 0) : s \in S\}$ are multiplicative subsets of the ring $R \ltimes M$. It is well known that if N is a submodule of M with $IM \subseteq N$, then $I \ltimes N$ is an ideal of $R \ltimes M$. Now, present some properties of S-weakly quasi n-absorbing ideals in the trivial extension.

Lemma 3.1 *Let S be a multiplicative subset of a ring R and I be a proper ideal of a ring R and let N be a proper submodule of an R-module M. If $I \ltimes N$ is a $(S \ltimes M)$-weakly quasi n-absorbing ideal of $R \ltimes M$, then I is a S-weakly quasi n-absorbing ideal of R.*

Proof: Let (s, m) be an $(S \ltimes M)$-weakly element of $I \ltimes N$. Assume that $0 \neq a^n b \in I$ for some elements a, b in R. Then, $0 \neq (a, 0)^n (b, 0) = (a^n, 0)(b, 0) \in I \ltimes N$, which is an $(S \ltimes M)$-weakly quasi n-absorbing ideal of $R \ltimes M$, where $(a, 0)$, $(b, 0)$ are elements of $R \ltimes M$. Therefore, there exists an element $(s, m) \in S \ltimes M$ such that either $(s, m)(a, 0)^n = (sa^n, a^n m) \in I \ltimes N$ or $(s, m)(a, 0)^{n-1}(b, 0) = (sa^{n-1}b, a^{n-1}bm) \in I \ltimes N$. This implies that either $sa^n \in I$ or $sa^{n-1}b \in I$. Hence, I is an S-weakly quasi n-absorbing ideal of R and this completes the proof.

□

Theorem 3.1 *Let M be an R-module, S a multiplicative subset of R and I a proper ideal of R. Then the following assertions are equivalent:*

1. *$I \ltimes M$ is a $(S \ltimes 0)$-weakly quasi n-absorbing ideal of $R \ltimes M$;*
2. *$I \ltimes M$ is a $(S \ltimes M)$-weakly quasi n-absorbing ideal of $R \ltimes M$;*
3. *I is a S-weakly quasi n-absorbing ideal of R associated to $s \in S$ and if whenever $a^n b = 0$ for some $a, b \in R$ but $sa^n \notin I$ and $sa^{n-1}b \notin I$, then $a^n \in \operatorname{ann}_R(M)$ and $a^{n-1}b \in \operatorname{ann}_R(M)$.*

Proof: (1) ⇒ (2) Follows from Remark 2.1(1) since $S \ltimes 0 \subseteq S \ltimes M$.

(2) ⇒ (3) Suppose that $I \ltimes M$ is an $S \ltimes M$-weakly quasi n-absorbing ideal of $R \ltimes M$. Then by Lemma 3.1, I is an S-weakly quasi n-absorbing ideal of R. Now, suppose that $a^n b = 0$ for some $a, b \in R$ but $sa^n \notin I$ and $sa^{n-1}b \notin I$. Now, we will show that $a^n \in \operatorname{ann}_R(M)$. Assume that $a^n \notin \operatorname{ann}_R(M)$. Then there exists $m \in M$ such that $a^n m \neq 0$ and so we have $(0, 0) \neq (a, 0)^n (b, m) \in I \ltimes M$. As $I \ltimes M$ is an $(S \ltimes M)$-weakly quasi n-absorbing ideal and $(a^n, 0)$, (b, m) are the elements in $R \ltimes M$ so there exists (s, m_1) an $(S \ltimes M)$-weakly element of $I \ltimes M$ such that $(s, m_1)(a, 0)^n \in I \ltimes M$ or $(s, m_1)(a, 0)^{n-1}(b, m) \in I \ltimes M$. Then we get $sa^n \in I$ or $sa^{n-1}b \in I$, which is a contradiction. Thus, $a^n \in \operatorname{ann}_R(M)$. Similarly, we conclude $a^{n-1}b \in \operatorname{ann}_R(M)$.

(3) ⇒ (1) Suppose that $(0, 0) \neq (a, m_1)^n (b, m_2) \in I \ltimes M$, for some elements (a, m_1), (b, m_2) of $R \ltimes M$. If $a^n b \neq 0$, then $sa^n \in I$ or $sa^{n-1}b \in I$ and hence $(s, 0)(a, m_1)^n \in I \ltimes M$ or $(s, 0)(a, m_1)^{n-1}(b, m_2) \in I \ltimes M$. Next, assume that $a^n b = 0$, but $sa^n \notin I$ and $sa^{n-1}b \notin I$. From our assumption (3), $a^n, a^{n-1}b \in \operatorname{ann}_R(M)$. Consequently, we get $(a, m_1)^n (b, m_2) = (a^n b, a^n m_2 + na^{n-1}bm_1) = (0, 0)$, a contradiction.

□

An additional result has been obtained regarding the trivial extension.

Proposition 3.1 *Let S be a multiplicative subset of R, and let I be an ideal of R disjoint from S. Suppose that N is a submodule of an R-module M such that $IM \subseteq N$. Consider the following statements:*

(1) *$I \ltimes N$ is an $(S \ltimes 0)$-weakly quasi n-absorbing ideal of $R \ltimes M$.*

(2) *$I \ltimes N$ is an $(S \ltimes M)$-weakly quasi n-absorbing ideal of $R \ltimes M$.*

(3) *I is an S-weakly quasi n-absorbing ideal of R.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof: $(1) \Rightarrow (2)$ Clear since $S \ltimes 0 \subseteq S \ltimes M$.

$(2) \Rightarrow (3)$ Let $0 \neq a^n b \in I$ for some elements a, b in R. Then, $0 \neq (a, 0)^n (b, 0) = (a^n, 0)(b, 0) \in I \ltimes N$, which is an $(S \ltimes M)$-weakly quasi n-absorbing ideal of $R \ltimes M$, where $(a, 0)$, $(b, 0)$ are elements of $R \ltimes M$. Therefore, there exists an element $(s, m) \in S \ltimes M$ such that either $(s, m)(a, 0)^n = (sa^n, a^n m) \in I \ltimes N$ or $(s, m)(a, 0)^{n-1}(b, 0) = (sa^{n-1}b, a^{n-1}bm) \in I \ltimes N$. This implies that either $sa^n \in I$ or $sa^{n-1}b \in I$. Hence, I is an S-weakly quasi n-absorbing ideal of R and this completes the proof.

□

Let R_1, R_2 be commutative rings, J an ideal of R_2 and $f : R_1 \to R_2$ be a ring homomorphism. Then we can define the following subring of $R_1 \times R_2$:

$$R_1 \bowtie^f J = \{(a, f(a) + j) \mid a \in R_1, j \in J\}.$$

$R_1 \bowtie^f J$ is called the amalgamation of R_1 with R_2 along J with respect to f. This construction is defined and examined by D'Anna et al. which is a generalization of the amalgamated duplication of a ring along an ideal (cf., for instance [7,9,17]). They have studied these constructions in the framework of pullbacks, which allowed them to establish numerous results on the transfer of various ideal and ring-theoretic properties from R_1 and $f(R_1) + J$ to $R_1 \bowtie^f J$. The concept of amalgamation is important and interesting, and it has received a considerable attention during the last few decades. Motivations and additional applications of the amalgamations are discussed in detail in [10] and extensively investigated in different papers [8,24,11].

Let I be an ideal of R_1, S a multiplicative set of R_1 and K an ideal of $f(R_1) + J$, set:

$$\begin{aligned} S' &:= \{(s, f(s)) \mid s \in S\}, \\ I \bowtie^f J &:= \{(i, f(i) + j) \mid i \in I, j \in J\}, \\ S \bowtie^f J &:= \{(s, f(s) + j) \mid s \in S, j \in J\}. \end{aligned}$$

It is easy to verify that $S \bowtie^f J$ and S' are multiplicative subsets of $R_1 \bowtie^f J$, and $I \bowtie^f J$ is an ideal of $R_1 \bowtie^f J$.

Our next result characterizes S-weakly quasi n-absorbing ideals of the form $I \bowtie^f J$ and K^f of the amalgamation $R_1 \bowtie^f J$.

Theorem 3.2 *Consider the amalgamation of rings R_1, R_2 along the ideal J of R_2 with respect to a homomorphism f. Let S be a multiplicative set of R_1 and I be an ideal of R_1 disjoint with S. Then the following assertions are equivalent:*

(1) *$I \bowtie^f J$ is an S'-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$;*

(2) *$I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$;*

(3) *I is an S-weakly quasi n-absorbing ideal of R_1, and for $a, b \in R_1$ with $a^n b = 0$ but $sa^n \notin I$ and $sa^{n-1}b \notin I$, we have:*

$$\sum_{k=0}^{n-1} \binom{n}{k} f(a^k b) i^{n-k} + j \sum_{k=0}^{n} \binom{n}{k} f(a^k) i^{n-k} = 0 \text{ for all } i, j \in J.$$

Proof:

First, note that $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$ if and only if $S' \cap (I \bowtie^f J) = \emptyset$, which holds if and only if $S \cap I = \emptyset$.

(1) ⇒ (2) This follows from Proposition 2.1(1) since $S' \subseteq (S \bowtie^f J)$.

(2) ⇒ (3) Suppose that $I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$ and choose a $(S \bowtie^f J)$-weakly element $(s, f(s)+j)$ of $I \bowtie^f J$. Let $a, b \in R_1$ such that $0 \neq a^n b \in I$ and $sa^{n-1}b \notin I$. Then $(0,0) \neq (a, f(a))^n (b, f(b)) \in I \bowtie^f J$ and we have $(s, f(s)+j)(a^{n-1}b, f(a^{n-1}b)) \notin I \bowtie^f J$. Hence $(s, f(s)+j)(a, f(a))^n \in I \bowtie^f J$ and so $sa^n \in I$. Therefore I is an S-weakly quasi n-absorbing ideal of R_1. Now, let s be an S-weakly element of I and $a, b \in R_1$ with $a^n b = 0$, $sa^n \notin I$ and $sa^{n-1}b \notin I$. Then for every $i, j \in J$, we have $(a, f(a)+i)^n (b, f(b)+j) \in I \bowtie^f J$, $(s, f(s)+i)(a, f(a)+i)^n \notin I \bowtie^f J$ and $(s, f(s)+i)(a, f(a)+i)^n (b, f(b)+j) \notin I \bowtie^f J$, we get:

$$(f(a)+i)^n (f(b)+j) = \sum_{k=0}^{n-1} \binom{n}{k} f(a^k b) i^{n-k} + j \sum_{k=0}^{n} \binom{n}{k} f(a^k) i^{n-k} = 0.$$

(3) ⇒ (1) Let $(a, f(a)+i), (b, f(b)+j) \in R_1 \bowtie^f J$ such that $(0,0) \neq (a, f(b)+i)^n (b, f(b)+j) = (a^n b, (f(a)+i)^n (f(b)+j)) \in I \bowtie^f J$. If $0 \neq a^n b$, then $sa^n \in I$ or $sa^{n-1}b \in I$ for some S-weakly element s of I. Hence, $(s, f(s))(a, f(a)+i)^n \in I \bowtie^f J$ or $(s, f(s))(a, f(a)+i)^{n-1}(b, f(b)+j) \in I \bowtie^f J$ as required. Now, suppose $a^n b = 0$. Then, $(f(a)+i)^n (f(b)+j) \neq 0$, and by assumption, either $sa^n \in I$ or $sa^{n-1}b \in I$. Therefore, $(s, f(s))(a, f(a)+i)^n \in I \bowtie^f J$ or $(s, f(s))(a, f(a)+i)^{n-1}(b, f(b)+j) \in I \bowtie^f J$. Thus, $I \bowtie^f J$ is an S'-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$. □

By Theorem 3.2, we get the following particular case.

Corollary 3.1 *Let R_1, R_2, J, f and S be as in Theorem 3.2. Then any $(S \bowtie^f J)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$ containing $\{0\} \times J$ is of the form $I \bowtie^f J$ where I is a S-weakly quasi n-absorbing ideal of R_1.*

Proof: From Theorem 3.2, $I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$ for any S-weakly quasi n-absorbing ideal of R_1. Let K be an $(S \bowtie^f J)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\psi : R_1 \bowtie^f J \to R_1$ defined by $\psi(a, f(a)+j) = a$ for all $(a, f(a)+j) \in R_1 \bowtie^f J$. Since $\ker(\psi) = \{0\} \times J \subseteq K$ and so $I = \psi(K)$ is an S-weakly quasi n-absorbing ideal of R_1 by using Proposition 2.3. Since $\{0\} \times J \subseteq K$, we conclude that $K = I \bowtie^f J$. □

Let I be a proper ideal of R_1. The (amalgamated) duplication of R_1 along I is a special amalgamation given by

$$R_1 \bowtie I = \{(a, a+i) \mid a \in R_1, i \in I\}.$$

The next corollary is an immediate consequence of the previous Theorem on the transfer of S-weakly quasi n-absorbing ideal property to duplication.

Corollary 3.2 *Let R_1 be a ring and I, K be ideals of R_1 with $K \cap S = \emptyset$. Then the following assertions are equivalent:*

(2) *$K \bowtie I$ is an $(S \bowtie I)$-weakly quasi n-absorbing ideal of $R_1 \bowtie I$;*

(2) *$K \bowtie I$ is an S'-weakly quasi n-absorbing ideal of $R_1 \bowtie I$;*

(1) *K is an S-weakly quasi n-absorbing ideal of R_1, and if there exist $a, b \in R_1$ with $a^n b = 0$ but $sa^n \notin I$ and $sa^{n-1}b \notin I$, then*

$$\sum_{k=0}^{n-1} \binom{n}{k} a^k b i^{n-k} + j \sum_{k=0}^{n} \binom{n}{k} a^k i^{n-k} = 0 \textit{ for all } i, j \in J.$$

Let S be a multiplicative set of R_1 and K an ideal of $f(R_1)+J$. Then clearly, the set

$$\overline{T}^f := \{(s, f(s)+j) \mid s \in R_1, j \in J\ , f(s)+j \in T\}$$

is multiplicative subset of $R_1 \bowtie^f J$, and the set

$$\overline{K}^f := \{(a, f(a)+j) \mid a \in R_1, j \in J\ , f(a)+j \in K\}$$

is an ideal of $R_1 \bowtie^f J$.

Theorem 3.3 *Consider the amalgamation of rings R_1, R_2 along the ideal J of R_2 with respect to an epimorphism f. Let K be an ideal of R_2 and T be a multiplicative subset R_2 disjoint with K. Then the following statements are equivalent:*

1. *$\overline{K}^f$ is a $\overline{T}^f$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$;*
2. *K is a T-weakly quasi n-absorbing ideal of R_2 and for $f(a), f(b) \in R_2$ and a T-weakly element $f(s)$ of K if $(f(a)+i)^n(f(b)+j) = 0$ for every $i, j \in J$, $f(s)(f(a)+i)^n \notin K$ and $f(s)(f(a)+i)^{n-1}(f(b)+j) \notin K$, then $a^n b = 0$.*

Proof:

One can easily verify that $T \cap S = \emptyset$ if and only if $\overline{T}^f \cap \overline{K}^f = \emptyset$.

(1) ⇒ (2) Suppose that $\overline{K}^f$ is a $\overline{T}^f$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$ and a choose a $\overline{T}^f$-weakly element $(s, f(s)+j)$ of $\overline{K}^f$. Let $a_1^n, b_1 \in R_2$ such that $0 \neq a_1^n b_1 \in K$, say, $a_1^n = f(a^n)$ and $b_1 = f(b)$ for $a, b \in R_1$. Hence, $(a, f(a)), (b, f(b)) \in R_1 \bowtie^f J$ with $(0,0) \neq (a, f(a))^n (b, f(b)) = (a^n b, f(a^n b)) \in \overline{K}^f$. So, we have either

$$(s, f(s)+j)(a, f(a))^n = (sa^n, (f(s)+j)f(a^n)) \in \overline{K}^f$$

or

$$(s, f(s)+j)(a, f(a))^{n-1}(b, f(b)) = (sa^{n-1}b, (f(s)+j)f(a^{n-1}b)) \in \overline{K}^f.$$

Thus, $f(s)+j \in T$ and $(f(s)+j)f(a)^n \in K$ or $(f(s)+j)f(a)^{n-1}f(b) \in K$. It follows that K is a T-weakly quasi n-absorbing ideal of R_2. Now, let $f(s)$ be an S-weakly element of K, and $f(a), f(b) \in R_2$ such that $(f(a)+i)^n(f(b)+j) = 0$, $f(s)(f(a)+i)^n \notin K$ and $f(s)(f(a)+i)^{n-1}(f(b)+j) \notin K$. Since $\overline{K}^f$ is $\overline{T}^f$-weakly quasi n-absorbing, for every $i, j \in J$, $(a, f(a)+i)^n(b, f(b)+j) \in \overline{K}^f$, then we must have $a^n b = 0$ and we are done.

(2) ⇒ (1) Choose a T-weakly element $f(s)$ of K. Let $(0,0) \neq (a, f(a)+i)^n(b, f(b)+j) = (a^n b, (f(a)+i)^n(f(b)+j)) \in \overline{K}^f$ for $(a, f(a)+i), (b, f(b)+j) \in R_1 \bowtie^f J$. Then $(f(a)+i)^n(f(b)+j) \in K$. If $(f(a)+i)^n(f(b)+j) \neq 0$. Thus,

$$(s, f(s))(a, f(a)+i)^n = (sa^n, f(s)(f(a)+i)^n) \in \overline{K}^f$$

or

$$(s, f(s))(a, f(a)+i)^{n-1}(b, f(b)+j) = (sa^{n-1}b, f(s)(f(a)+i)^{n-1}(f(b)+j)) \in \overline{K}^f.$$

The result follows. Suppose $(f(a)+i)^n(f(b)+j) = 0$. Then $a^n b \neq 0$, and so by our assumption, we conclude either $f(s)(f(a)+i)^n \in K$ or $f(s)(f(a)+i)^{n-1}(f(b)+j) \in K$. Hence, again

$$(s, f(s))(a, f(a)+i)^n \in \overline{K}^f$$

or

$$(s, f(s))(a, f(a)+i)^{n-1}(b, f(b)+j) \in \overline{K}^f$$

and so $\overline{K}^f$ is a $\overline{T}^f$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$.

□

In particular, if S is a multiplicative subset of R_1, $S \times f(S)$ is also a multiplicative subset of $R_1 \bowtie^f J$. Thus, we get the following corollary of Theorem 3.3.

Corollary 3.3 *Let R_1, R_2, J, S and f be as in Theorem 4.6. Let K be an ideal of R_2 and $T = f(S)$. Then the following assertions are equivalent.*

(1) $\overline{K}^f$ *is $(S \times T)$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$;*

(2) $\overline{K}^f$ *is a $\overline{T}^f$-weakly quasi n-absorbing ideal of $R_1 \bowtie^f J$;*

(3) *K is a T-weakly quasi n-absorbing ideal of R_2 and for $f(a), f(b) \in R_2$ and a T-weakly element $f(s)$ of K, if $(f(a)+i)^n(f(b)+j) = 0$ for every $i, j \in J$, $f(s)(f(a)+i)^n \notin K$ and $f(s)(f(a)+i)^{n-1}(f(b)+j) \notin K$, then $a^n b = 0$.*

Remark 3.1 Let $f : R \to T$ be a ring homomorphism and J be an ideal of T. Consider I (resp., H) be an ideal of R (resp., $f(R) + J$) such that $f(I)J \subseteq H \subseteq J$. Observe that

$$I \bowtie^f H = \{(i, f(i) + h) : i \in I, h \in H\}$$

is an ideal of $R \bowtie^f J$.

The preceding remark offers an alternative path to the following result.

Proposition 3.2 *Under the above notation, if $I \bowtie^f H$ is an S'-weakly quasi n-absorbing ideal of $R \bowtie^f J$ if and only if I is an S-weakly quasi n-absorbing ideal of R and for each $i, j \in R$ such that $i^n j = 0$, $si^n \notin I$ and $si^{n-1}j \notin I$, we have:*

$$\sum_{\ell=0}^{n-1} \binom{n}{\ell} f(i^\ell j) h^{n-\ell} + j \sum_{\ell=0}^{n} \binom{n}{\ell} f(i^\ell) h^{n-\ell} = 0 \text{ for each } h, k \in H.$$

Proof: The argument follows in the same manner as in Theorem 3.2.

□

The next corollary is a direct result of Proposition 3.2, which addresses duplications.

Corollary 3.4 *Let R be a ring and I, H, J be ideals of R with $IJ \subseteq H \subset J$. If $I \bowtie H$ is an S'-weakly quasi n-absorbing ideal of $R \bowtie J$ if and only if I is an S-weakly quasi n-absorbing ideal of R and for each $i, j \in R$ such that $i^n j = 0$, $si^n \notin I$ and $si^{n-1}j \notin I$, we have:*

$$\sum_{\ell=0}^{n-1} \binom{n}{\ell} i^\ell j h^{n-\ell} + j \sum_{\ell=0}^{n} \binom{n}{\ell} i^\ell h^{n-\ell} = 0 \text{ for each } h, k \in H.$$

Let R be a ring, I an ideal of R, and M an R-module. Consider the set

$$M \bowtie I := \{(m_1, m_2) \in M \times M : m_1 - m_2 \in IM\}$$

which becomes an $R \bowtie I$-module with scalar multiplication defined by

$$(a, a + i)(m_1, m_2) = (am_1, (a + i)m_2),$$

for all $a \in R$, $i \in I$ and $(m_1, m_2) \in M \bowtie I$. The set $M \bowtie I$ is called the duplication of the R-module M along the ideal I. Let N be a submodule of the R-module M. Consider the set:

$$N \bowtie I := \{(n, m) \in N \times M : n - m \in IM\}.$$

Clearly, $N \bowtie I$ is a submodule of $M \bowtie I$. If S is a multiplicative set of R, then the sets $S \bowtie I := \{(s, s + i) \mid i \in I, s \in S\}$ is a multiplicative subset of $R \bowtie I$.

Lemma 3.2 *Let R be a ring, and consider N a submodule of the R-module M. We then observe the following relation:*

$$(N \bowtie I :_{R \bowtie I} M \bowtie I) = (N :_R M) \bowtie I$$

Proof: Let $(a, a+i) \in (N \bowtie I :_{R\bowtie I} M \bowtie I)$. Then by definition $(a, a+i)(M \bowtie I) \subseteq N \bowtie I$. In particular, this implies that $aM \subseteq N$, and hence $(a, a+i) \in (N :_R M) \bowtie I$. Conversely, assume that $(a, a+i) \in (N :_R M) \bowtie I$ and consider any $(m, m') \in M \bowtie I$. Then $(a, a+i)(m, m') = (am, (a+i)m') \in M \times M$. Since $m - m' \in IM$, it follows that $a(m - m') - im' \in IM$, which implies that $(a, a+i) \in (N \bowtie I :_{R\bowtie I} M \bowtie I)$. We thus conclude the equality: $(N \bowtie I :_{R\bowtie I} M \bowtie I) = (N :_R M) \bowtie I$. □

The lemma established earlier serves as a cornerstone for the proof of the upcoming theorem.

Proposition 3.3 *Let N be a submodule of M such that $(N :_R M) \cap S = \emptyset$. If $N \bowtie I$ is an $(S \bowtie I)$-weakly quasi n-absorbing submodule of $M \bowtie I$, then N is an S-weakly quasi n-absorbing submodule of M.*

Proof: Note that $(N \bowtie I :_{R\bowtie I} M \bowtie I) \cap (S \cap I) = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$. Assume that $N \bowtie I$ is an $(S \bowtie I)$-weakly quasi n-absorbing submodule of $M \bowtie I$. Let $(s, s+i)$ be an $S \bowtie I$-weakly element of $N \bowtie I$, and suppose that $0 \neq a^n m \in N$ for some elements $a \in R$ and $m \in M$. Then $(0, 0) \neq (a, a)^n (m, m) \in N \bowtie I$. By hypothesis, since $N \bowtie I$ is an $(S \bowtie I)$-weakly quasi n-absorbing submodule $M \bowtie I$, we have either $(s, s+i)(a, a)^n \in (N \bowtie I :_{R\bowtie I} M \bowtie I)$ or $(s, s+i)(a, a)^{n-1}(m, m) \in N \bowtie I$. In the first case, by Lemma 3.2, it follows that $sa^n \in (N :_R M)$. In the second case, we have $(sa^{n-1}m, (s+i)a^{n-1}m) \in N \bowtie I$, which implies that $sa^{n-1}m \in N$. Therefore, in both cases, N is an S-weakly quasi n-absorbing submodule of M.

□

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