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The Solution of Fokker Planck Equation Using Two Different Methods: Aboodh Homotopy Perturbation Method and Aboodh Residuel Power Series Method

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ABSTRACT: This article's objective is to solve and analyse the time fractional form of a Fokker Planck equation using two techniques that are renowned for their effectiveness, precision, ease of use and computation, and adaptability: the Homotopy Perturbation Method and the Aboodh Residual Power Series Method. These techniques are well-known for their resilience when solving complicated issues, and they are especially well-suited for handling fractional differential equations. For the analysis of more complex nonlinear differential equations, the two suggested approaches—the Homotopy Perturbation Method and the Aboodh Residual Power Series Method—are very successful mathematical strategies. Beyond the particular equations under study, these techniques can yield accurate approximations of solutions to complex evolution equations. These techniques capture the memory and heredity characteristics present in many physical systems and offer a more realistic depiction of fractional order processes by employing the Caputo derivative.

Keywords: Fokker Planck Equation, Aboodh transform, Homotopy Perturbation Method, Aboodh residuel power series, fractional calculus.

Contents

1. Introduction

In recent years, the field of fractional calculus has experienced notable growth and increased interest, driven by both theoretical advancements and practical applications across diverse disciplines. Fractional calculus extends the capabilities of traditional PDEs by introducing fractional derivatives, which are particularly useful in modelling systems with memory, non-local interactions, and complex dynamics. This leads to more accurate and comprehensive descriptions of various physical, biological, and economic phenomena, thereby enhancing both theoretical understanding and practical applications [11], [18], [21],

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[26]. Researchers continue to develop new definitions and generalizations of fractional derivatives and integrals. For example, the introduction of the Caputo-Fabrizio derivative and the Atangana-Baleanu derivative offer new ways to model memory and hereditary properties in various systems. There has been progress in analytical methods for solving fractional differential equations, including novel techniques for handling boundary value problems and initial value problems. The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity (or position) of a particle under the influence of forces, particularly in the context of stochastic processes. It provides a way to model the diffusion and drift of particles within a system, especially under the influence of forces and random perturbations. It is widely used in various fields such as physics, finance, and biology. The Fokker-Planck [12]- [17] equation is related to stochastic differential equations (SDEs) through the correspondence between the drift and diffusion terms in the SDE and the terms in the Fokker-Planck equation. Example: Brownian Motion For a particle undergoing Brownian motion, the Fokker-Planck equation can be given by

$$\frac{\partial \vartheta\left(y,w\right)}{\partial w}=\left\{-\frac{\partial}{\partial x}\left(g_1\left(y,w\right)\right)+\frac{\partial^2}{\partial x^2}\left(g_2\left(y,w\right)\right)\right\}\vartheta\left(y,w\right), \tag{1.1}$$

$$\vartheta\left(y,0\right)=\phi(y) \tag{1.2}$$

Where $g_1\left(y,w\right)$ is the drift coefficient, representing the deterministic part of the evolution and $g_1\left(y,w\right)$ is the diffusion coefficient representing the random part of the evolution. The Fokker-Planck equation was independently derived by Adriaan Fokker and Max Planck. Adriaan Fokker, a Dutch physicist, worked on the equation in 1914. He used it to describe the behaviour of particles subjected to stochastic (random) forces, contributing to the field of statistical mechanics. Max Planck, a German physicist, also developed the equation around the same time in 1917. He applied it to explain the diffusion of particles, furthering its significance in the study of statistical physics and quantum mechanics. Hence, the equation bears the names of both scientists, recognizing their simultaneous and independent contributions to its formulation. The Fokker-Planck equation is crucial for understanding various physical systems, including diffusion processes, Brownian motion, and many other phenomena where random fluctuations play a significant role. In many real-world applications where random (stochastic) events affect systems, the Fokker-Planck equation is essential. It is significant because it can explain how the probability distribution of a system's state changes over time when both random and deterministic forces are at play. This equation has many applications in the fields of medical, physics, finance, biology, population dynamics, robotics, signal processing [29]- [33] etc. Knowing the Fokker-Planck equation [19]- [20] is essential for comprehending a wide range of physical systems, such as diffusion processes, Brownian motion, and numerous other phenomena where random fluctuations are important. In numerous real-world scenarios where systems are impacted by stochastic events, the Fokker-Planck equation is crucial. When both random and deterministic forces are present, it helps explain how the probability distribution of a system's state evolves over time, which makes it important and essential topic to discuss in research [27]- [28]. Mathematical modelling is being used as a highly useful tool in modern mathematical study. Since it converts real-world problems into mathematical terms, we put the results back into practical ones after using the essential techniques to forecast the goal. Every business plan to accomplish goals or make predictions about the future. Nonetheless, there are times when fractional calculus conclusions better capture the model than classical ones. Fractional calculus [1]- [2] is now used to model various technological, engineering, and physical scenarios. The history of fractional calculus is extensive, spanning around three centuries [4]- [10]. The main reason fractional models are used is because most systems exhibit memory and non-local effects, which make it challenging to model them with integer order operators. Numerous subjects make use of fractional calculus and its operators. The foundation of early study was the Riemann-Liouville fractional operator. However, as time went on, it was discovered that its kernel had a singularity; thus, several new definitions without singularity were introduced. The Atangana Baleanu operator is based on the Mittag-Leffler function, which produces the best results.

The structure of paper follows as: Section 1 deals with the introduction and background of the problem. Section 2 is about the pre-requisites. Segments 3 and 4 explains both the methods. Section 5 is having the applicational part of the methods while we conclude the article in section 6. At the end, we have attached the references.

2. Basic Definitions

In this segment, we are going to give brief information about the basic definitions of the operator and transformation used in the paper-

2.1. Definition 1

For function $\vartheta(y,w)$, the Riemann – Liouville fractional integral of order $\beta \in \Re^+$ is given in the following way:

$$I_w{}^\beta \vartheta(y,w) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^w (w-\upsilon)^{\beta-1} \vartheta(y,\upsilon) d\upsilon, & y \in I, \quad 0 < \upsilon, \\ \vartheta(y,w) & \beta = 0. \end{cases} \tag{2.1}$$

Now, we present the Caputo fractional derivative of order as below:

$$^cD_w{}^\beta \vartheta(y) = \begin{cases} \frac{1}{\Gamma\varepsilon - \beta} \int\limits_0^w \frac{\vartheta^{(\varepsilon)}\upsilon}{(y-\upsilon)^{\beta-\varepsilon+1}} d\upsilon, \quad w > 0 \quad \varepsilon - 1 < \beta < \varepsilon, \quad \forall \varepsilon \in N. \end{cases} \tag{2.2}$$

2.2. Definition 2

For $-1 < \beta < \varepsilon$, the Caputo time- fractional derivative of order $\beta \in \Re^+$ is given by

$$^cD_w{}^\beta \vartheta(y,w) = \begin{cases} \frac{1}{\Gamma\varepsilon - \beta} \int_0^w \frac{\partial^\varepsilon \vartheta(y,\upsilon)}{\partial \upsilon^\varepsilon} (w-\upsilon)^{\varepsilon-\beta-1} d\upsilon, \; w > 0 & \varepsilon - 1 < \beta < \varepsilon \\ \frac{\partial^\varepsilon \vartheta(y,\upsilon)}{\partial \upsilon^\varepsilon}, & \beta = \varepsilon \in N \end{cases} \tag{2.3}$$

For $-1 < \beta < \varepsilon$, the Caputo space- fractional derivative of order $\beta \in \Re^+$ is given by

$$^cD_w{}^\beta \vartheta(y) = \begin{cases} \frac{1}{\Gamma\varepsilon - \beta} \int\limits_0^w \frac{\vartheta^{(\varepsilon)}\upsilon}{(y-\upsilon)^{\beta-\varepsilon+1}} d\upsilon, \quad w > 0 \quad \varepsilon - 1 < \beta < \varepsilon, \quad \forall \varepsilon \in N. \end{cases} \tag{2.4}$$

2.3. Leema 1

If $I_w^\beta D_w^\beta \vartheta(y,w) = \vartheta(y,w) - \sum\limits_{i=0}^{\varepsilon} \frac{\partial^i \vartheta(y,0^+)}{\partial w^i} \frac{w^i}{i}$ where $y \in I, w > 0$ then we have D_w^β and I_w^β operators with some basic properties that can be found in I. Podlubny [3] and are given below:

$$D_w^\beta w^j = \frac{\Gamma(j+1)}{\Gamma(j-\beta+1)} t^{j-\beta} \tag{2.5}$$

$$I_w^\beta w^j = \frac{\Gamma(j+1)}{\Gamma(j+\beta+1)} w^{j+\beta} \tag{2.6}$$

2.4. Definition 3

Let the function $\vartheta(y,w)$ is of exponential order and piecewise continuous. The AT of $\vartheta(y,w)$ for $w \geq 0$, is defined as follows:

$$Ab\left[\vartheta\left(y,w\right)\right] = V\left(y,h\right) = \frac{1}{s}\int_0^\infty \vartheta\left(y,w\right) e^{-hw} dw, \qquad h \geq 0, \qquad f_1 \leq h \leq f_2, \tag{2.7}$$

and the description of the inverse of AT is given below:

$$Ab^{-1}\left[V\left(y,h\right)\right] = \vartheta\left(y,w\right) = \frac{1}{2\pi i}\int\limits_{q-i\infty}^{q+i\infty} he^{hw} V\left(y,h\right) dh. \tag{2.8}$$

2.5. Leema 2

Let we take two functions $\vartheta_1(y,w)$ and $\vartheta_2(y,w)$ are of exponential order and piecewise continuous on $[0,\infty\ ]$ then $Ab\left[\vartheta_1(y,w)\right]=V_1\left(y,h\right)$, $Ab\left[\vartheta_2(y,w)\ \right]=V_2\left(y,h\right)$ and Λ_1,Λ_2 are constants. Then following conditions are satisfied:
1. $Ab\left[\Lambda_1\vartheta_1(y,w)+\Lambda_2\vartheta_2(y,w)\ \right]=\Lambda_1V_1\left(y,h\right)+\Lambda_2V_2\left(y,h\right)$
2. $Ab^{-}\left[\Lambda_1V_1\left(y,h\right)+\Lambda_2V_2\left(y,h\right)\right]=\Lambda_1\vartheta_1(y,w)+\Lambda_2\vartheta_2(y,w)$
3. $\vartheta_0(y)=\lim\limits_{h\to\infty}s^2V\left(y,h\right)=\vartheta_2(y,0)$
4. $Ab\left[{}^{c}D_w^{\beta}\,\vartheta(y,w)\right]=s^{\beta}V\left(y,h\right)-\sum\limits_{m=0}^{l-1}\frac{D_w^{m}\,\vartheta(y,0)}{s^{m-\beta+2}},\ l-1<\beta\leq l,\ \ l\in N$
5. $Ab\left[{}^{c}D_w^{k\beta}\,\vartheta(y,w)\right]=s^{k\beta}V(y,w)-\sum\limits_{m=0}^{l-1}s^{\beta(l-m)-2c}D_w^{k\beta}\,\vartheta\left(y,0\right),\ 0<\beta\leq 1.$

2.6. Definition 4

The Caputo derivative of $\vartheta(y,w)$ with order β can be written as:

$$ {}^{c}{D_w}^{\beta}\vartheta(y,w)=I_{\beta}^{m-\beta}\vartheta^{(m)}(y,w),\qquad t\geq 0,\quad m-1<\beta\leq m, $$

where $I_{\beta}^{m-\beta}$ is the R-L integral of $\vartheta(y,w)$.

2.7. Definition 5

A fractional power series representation at $w=w_0$ has the following form

$$ \sum_{m=0}^{\infty}\vartheta_m(y)(w-w_0)^{m\beta}=\vartheta_0(y)+\vartheta_1(y)(w-w_0)^{\beta}+\vartheta_2(y)(w-w_0)^{2\beta}+........ \tag{2.9} $$

where $l-1<\beta<l,\ \ l\in N$, and $w=w_0$ is called multiple fractional power series (MFPS) about t_0.

2.8. Theorem 1

Let $V\left(y,h\right)=Ab\left[\vartheta\left(y,w\right)\right]$ has the following multiple fractional power series (MFPS) expansion at $w=w_0$:

$$ V\left(y,h\right)=\sum_{m=0}^{\infty}\frac{\vartheta_m(y)}{h^{m\beta+2}} $$

and we have

$$ \vartheta_m(y)=D_w^{m\beta}\vartheta(y,0) \tag{2.10} $$

where $D_w^{m\beta}=D_w^{\beta}.D_w^{\beta}......D_w^{\beta}$, (m times).

2.9. Remark 1

If $V\left(y,h\right)=\sum\limits_{m=0}^{\infty}\frac{\vartheta_m(y)}{h^{m\beta+2}}$ where $\vartheta_m(y)=D_w^{m\beta}\vartheta(y,0)$, then the inverse Aboodh transfer of $V\left(y,h\right)$ is given by

$$ \vartheta(y,w)=Ab\left\{V(y,h)\right\}=\frac{D_w^{m\beta}\,\vartheta(y,0)}{\Gamma\left(m\beta+1\right)}w^{^{m\beta}}. \tag{2.11} $$

2.10. Theorem 2

If $V\left(y,h\right)=Ab\left[\vartheta\left(y,w\right)\right]=\sum\limits_{m=0}^{\infty}\frac{\vartheta_m(y)}{h^{m\beta+2}}$ and $\left|h^2Ab\left[{}^{c}D_w^{(k+1)\beta}\,\vartheta\left(y,w\right)\right]\right|\leq M(y)$, then the remainder $R_k(y,h)$ of the fractional power series of $V\left(y,h\right)$ satisfies the following inequality:

$$ \left|R_k(y,h)\right|\leq\frac{M(y)}{h^{(k+1)\beta+2}}, \tag{2.12} $$

3. Aboodh Homotopy Perturbation Method

The Homotopy Perturbation Method is a semi-analytical technique designed to solve both linear and nonlinear differential equations. It effectively combines classical perturbation techniques with homotopy methods from topology, providing a powerful approach for addressing a wide range of mathematical problems. The Homotopy Perturbation Method was discovered by the Iranian mathematician Dr. Ji-Huan He. He introduced this method in the late 1990s as a new analytical technique to solve nonlinear problems in science and engineering. Dr. He has contributed significantly to the field of applied mathematics, particularly in the development and application of analytical methods for solving differential equations. Consider the following PDE with fractional order β:

$$^{c}D_{w}^{\beta}\,\vartheta(y,w) + L\vartheta(y,w) + N\vartheta(y,w) = g(y,w),$$

$$\vartheta\,(y,0) = f(y), \quad y \in \Re \tag{3.1}$$

Where L is the linear operator, $^{c}D_{w}^{\beta}$ is the Caputo fractional derivative, and N are the non- linear operators, y, w are the dependent variables. Now, by applying the Aboodh transform, we can write Eq. (3.1) as

$$Ab\left[{}^{c}D_{w}^{\beta}\,\vartheta(y,w)\right] + Ab\left[L\vartheta(y,w)\right] + Ab\left[N\vartheta(y,w)\right] = Ab\left[g(y,w)\right], \tag{3.2}$$

$$Ab\left[{}^{c}D_{w}^{\beta}\,\vartheta(y,w)\right] = Ab\left[g(y,w)\right] - Ab\left[\,L\vartheta(y,w) + N\vartheta(y,w)\right]$$

By applying the differentiation property of the Aboodh transform, we can write Eq. (3.2) as

$$h^{\beta}Ab\left[\vartheta(y,w)\right] - \sum_{m=0}^{l-1}\frac{D_{w}^{m}\,\vartheta(y,0)}{h^{m-\beta+2}} = Ab\left[g(y,w)\right] - Ab\left[\,L\vartheta(y,w) + N\vartheta(y,w)\right]$$

$$Ab\left[\vartheta(y,w)\right] = \frac{\vartheta(y,0)}{h^{2}} + \frac{1}{h^{\beta}}Ab\left[g(y,w)\right] - \frac{1}{h^{\beta}}Ab\left[\,L\vartheta(y,w) + N\vartheta(y,w)\right] \tag{3.3}$$

After applying the inverse transform, we can write the equation (3.3) as

$$\vartheta(y,w) = F(y,w) - Ab^{-1}\left[\frac{1}{h^{\beta}}Ab\left[\,L\vartheta(y,w) + N\vartheta(y,w)\right]\right] \tag{3.4}$$

where

$$F(y,w) = Ab^{-1}\left[\frac{1}{h^{2}}f(y) + \frac{1}{h^{\beta}}Ab\left[g(y,w)\right]\right]$$

The perturbation technique, for parameter $\wp$ is defined as

$$\vartheta(y,w) = \sum_{m=0}^{\infty}\wp^{m}\vartheta_{m}(y,w) \tag{3.5}$$

here is the perturbation parameter and $\wp \in$ [0,1]. The nonlinear term is defined as

$$N\vartheta(y,w) = \sum_{m=0}^{\infty}\wp^{m}H_{m}(\vartheta_{m}) \tag{3.6}$$

and He‘s polynomials are denoted by H_n with ϑ_0,ϑ_1,ϑ_2,ϑ_3 and are given as

$$H_{n}(\vartheta_{0},\vartheta_{1},\vartheta_{2},......\vartheta_{n}) = \frac{1}{\delta(n+1)}D_{\wp}^{m}\left[N\left(\sum_{m=0}^{\infty}\wp^{m}\vartheta_{m}\right)\right]_{\wp=0}. \tag{3.7}$$

By comparing both sides of the coefficient of $\wp$, we have

$$\wp^{0} : \vartheta_{0}\,(y,w) = Ab^{-1}\left[\frac{f(y)}{h^{2}}\right]$$

$$\wp^1 : \vartheta_1(y,w) = Ab^{-1}\left[\frac{1}{h^\beta}Ab\left[L\vartheta_0(y,w) + H_0(\vartheta)\right]\right]$$

$$\wp^2 : \vartheta_2(y,w) = Ab^{-1}\left[\frac{1}{h^\beta}Ab\left[L\vartheta_1(y,w) + H_1(\vartheta)\right]\right]$$

and so on, we get

$$\wp^l : \vartheta_l(y,w) = Ab^{-1}\left[\frac{1}{h^\beta}Ab\left[L\vartheta_{l-1}(y,w) + H_{l-1}(\vartheta)\right]\right] \tag{3.8}$$

The terms $\vartheta_l(y,w)$ are giving convergent series which is easily computable. On taking $\wp \to 1$

$$\vartheta(y,w) = \lim_{n\to\infty}\sum_{l=1}^{n}\vartheta_l(y,w). \tag{3.9}$$

4. Aboodh Residual Power Series Method

The Residual Power Series Method was introduced by Abdul-Majid Wazwaz, a mathematician renowned for his contributions to applied mathematics, particularly in the development of analytical and numerical methods for solving differential equations. Residual Power Series Method is known for its computational efficiency, often requiring less computational effort than other numerical methods because it directly constructs the solution as a power series. Wazwaz has published numerous papers and books on various methods for solving nonlinear equations, including the Adomian Decomposition Method and the Homotopy Perturbation Method [22]- [25]. The Algorithm for the solution of FPEs by Aboodh Residual Power Series Method is given in following steps:
The Fokker Planck equation is given by

$$\frac{\partial\vartheta(y,w)}{\partial w} = \left\{-\frac{\partial}{\partial y}(g_1(y,w)) + \frac{\partial^2}{\partial y^2}(g_2(y,w))\right\}\vartheta(y,w), \tag{4.1}$$

$$\vartheta(y,0) = \phi(y) \tag{4.2}$$

Applying the AT on Eq. (4.1), we get

$$V(y,h) - \frac{\vartheta(y,0)}{h^2} - \frac{1}{h^\beta}Ab\left\{-\frac{\partial}{\partial y}\left(g_1(y,w)Ab^{-1}V(y,w)\right) + \frac{\partial^2}{\partial y^2}\left(g_1(y,w)Ab^{-1}V(y,w)\right)\right\} = 0 \tag{4.3}$$

We assume that the solution of Eq. (4.1) and (4.2) is in the series form:

$$V(y,h) = \sum_{m=0}^{\infty}\frac{\vartheta_m(y)}{h^{m\beta+2}},\ y,w \geq 0, h > 0. \tag{4.4}$$

Using Lemma 2 (iii), we get following result,

$$\vartheta_0(y) = \lim_{h\to\infty}h^2V(y,h) = \vartheta(y,0) = 0$$

Now we define r^{th} truncated expansion of $V(y,h)$ as

$$V_t(y,h) = \sum_{m=0}^{t}\frac{\vartheta_m(y)}{h^{m\beta+2}},\ y,t \geq 0, h > 0. \tag{4.5}$$

The Aboodh residual function (ARF) of Eq. (4.3) is defined below:

$$\begin{aligned}&Ab\mathrm{Res}(y,h) = V(y,h) - \tfrac{\vartheta(y,0)}{h^2} - \\ &\tfrac{1}{h^\beta}Ab\left\{-\tfrac{\partial}{\partial y}\left(g_1(y,w)Ab^{-1}V(y,h)\right) + \tfrac{\partial^2}{\partial y^2}\left(g_2(y,w)Ab^{-1}V(y,h)\right)\right\}.\end{aligned} \tag{4.6}$$

r^{th} ARF is given by

$$\begin{aligned} &Ab\mathrm{Res}\left(y,h\right)=V_r\left(y,h\right)-\tfrac{\vartheta(y,0)}{h^2}-\\ &\tfrac{1}{h^\beta}Ab\left\{-\tfrac{\partial}{\partial y}\left(g_1\left(y,w\right)Ab^{-1}V_r\left(y,h\right)\right)+\tfrac{\partial^2}{\partial y^2}\left(g_2\left(y,w\right)Ab^{-1}V_r\left(y,h\right)\right)\right\}. \end{aligned} \tag{4.7}$$

Here we use some useful property of ARF which is given below:
1. $Ab\mathrm{Res}\left(y,h\right)=0 \qquad\qquad y\geq 0,\ \ h>0.$
2. $\lim\limits_{r\to\infty} Ab\mathrm{Res}_r\left(y,h\right)=Ab\mathrm{Res}\left(y,h\right), \quad y\geq 0,\ \ h>0.$
3. $\lim\limits_{h\to\infty} h\, Ab\mathrm{Res}\left(y,h\right)=0 \ \textit{then}\ \lim\limits_{h\to\infty} h\, Ab\mathrm{Res}_r\left(y,h\right)=0$
4. $\lim\limits_{h\to\infty} h^{r\beta+1}\, Ab\mathrm{Res}\left(y,h\right)=\lim\limits_{h\to\infty} h^{r\beta+1}\, Ab\mathrm{Res}_r\left(y,h\right)=0, \quad \forall\ r=1,\ 2,\ 3,\ .....$

We use the expansion form of $V_r\left(y,h\right)$ in $Ab\mathrm{Res}_r\left(y,h\right)$. After then we solve sequence of algebraic equations for $\vartheta_m(y)$ step by step, taking m = 1, 2, 3, ... To obtain the r^{th} approximate solution of the given problem, we apply Ab^{-1} on the final form of $V_r\left(y,h\right)$.

5. Applications

Here, we are going to demonstrate the applications of the above methods discussed.
Example 1 Let we take the time fractional Fokker Planck equation as

$$\frac{\partial^\beta}{\partial w^\beta}\vartheta\left(y,w\right)+\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta\left(y,w\right)\right)-\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta\left(y,w\right)\right)=0,\quad y,w>0,\quad \beta\in(0,1], \tag{5.1}$$

with initial condition

$$\vartheta\left(y,0\right)=y^2. \tag{5.2}$$

For $\beta=1,\ \ \vartheta\left(y,w\right)=y^2e^{\frac{w}{2}}$ which is exact solution of eq. (5.1) - (5.2).
Solution Taking the Aboodh transform of the equation (5.1), then we obtain,

$$Ab\left[\vartheta\left(y,w\right)\right]=\frac{y^2}{h^2}+\frac{1}{h^\beta}Ab\left[-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta\left(y,w\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta\left(y,w\right)\right)\right] \tag{5.3}$$

taking inverse Aboodh transform of Eq. (5.3)

$$\vartheta\left(y,w\right)=y^2+Ab^{-1}\left(\frac{1}{h^\beta}Ab\left[-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta\left(y,w\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta\left(y,w\right)\right)\right]\right) \tag{5.4}$$

By comparing both sides of the coefficient of $\wp$, we get

$$\wp^0:\vartheta_0\left(y,w\right)=y^2,$$

$$\wp^1:\vartheta_1\left(y,w\right)=Ab^{-1}\left[\frac{1}{h^\beta}Ab\left\{-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta_0\left(y,w\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta_0\left(y,w\right)\right)\right\}\right]=y^2\frac{w^\beta}{2\Gamma\left(\beta+1\right)},$$

$$\wp^2:\vartheta_2\left(y,w\right)=Ab^{-1}\left[\frac{1}{h^\beta}Ab\left\{-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta_1\left(y,w\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta_1\left(y,w\right)\right)\right\}\right]=y^2\frac{\left(w^\beta\right)^2}{4\Gamma\left(2\beta+1\right)},$$

$$\wp^3:\vartheta_3\left(y,w\right)=Ab^{-1}\left[\frac{1}{h^\beta}Ab\left\{-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta_2\left(y,w\right)\right)+\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta_2\left(y,w\right)\right)\right\}\right]=y^2\frac{\left(w^\beta\right)^3}{8\Gamma\left(3\beta+1\right)},$$

and so on.
Using HPM, we get the solution in series form

$$\vartheta(y,w)=\sum_{m=0}^{\infty}\wp^{m}\vartheta_m(y,w) \tag{5.5}$$

So we get,

$$\vartheta\left(y,w\right)=y^2\left\{1+\frac{w^{\beta}}{2\Gamma\left(\beta+1\right)}+\frac{\left(w^{\beta}\right)^2}{4\Gamma\left(2\beta+1\right)}+\frac{\left(w^{\beta}\right)^3}{8\Gamma\left(3\beta+1\right)}+........\right\} \tag{5.6}$$

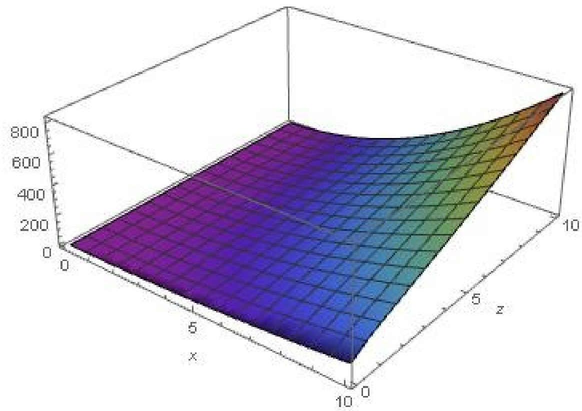

Figure 1: For β=0.5

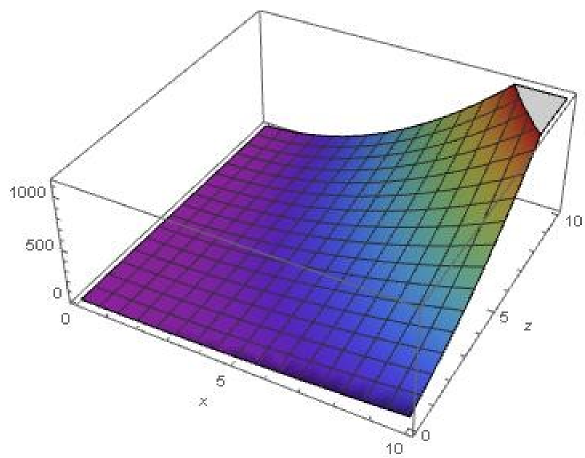

Figure 2: For β=0.7

Here, we solve this equation by ARPS method, First, we apply Aboodh Transform to both sides of Eq. (5.1). Then we get,

$$V\left(y,h\right)=\frac{y^2}{h^2}-\frac{1}{h^{\beta}}\frac{\partial}{\partial y}\left(\frac{y}{6}V\left(y,h\right)\right)+\frac{1}{h^{\beta}}\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}V\left(y,h\right)\right) \tag{5.7}$$

We get the solution of Eq. (5.1) - (5.2) are in the fractional power series.

$$V(y,h)=\sum_{m=0}^{\infty}\frac{\wp_m(y)}{h^{m\beta+2}} \tag{5.8}$$

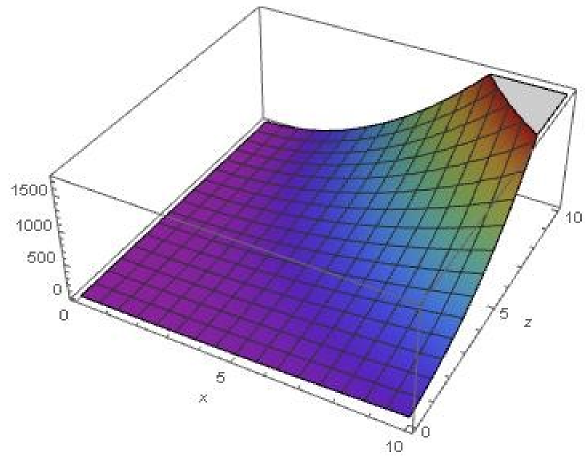

Figure 3: For β=0.9

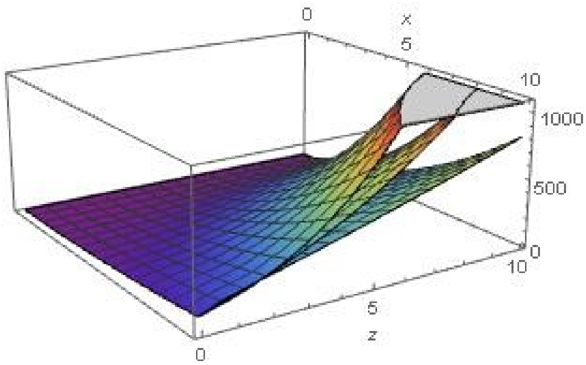

Figure 4: Comparison of results for β = 0.5, 0.7 and 0.9

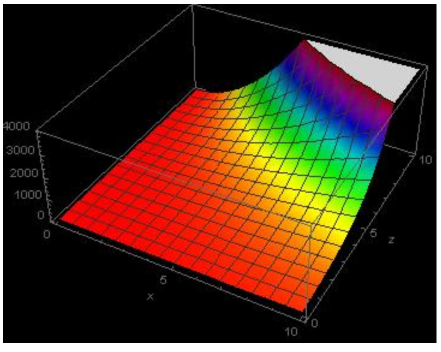

Figure 5: Exact solution for β = 1

We have $\wp_0(y) = \lim\limits_{h\to\infty} hV(y,h) = \vartheta(y,0) = y^2$. For r^{th} truncated series, we have

$$V_r(y,h) = \vartheta_0(y) + \sum_{m=1}^{l} \frac{\wp_m(y)}{h^{m\beta+2}} \tag{5.9}$$

Now, the Aboodh residual function $Ab\Re es(y,h)$ of Eq. (5.3) is determined as,

$$Ab\Re es(y,h) = V(y,h) - \frac{\vartheta(y,0)}{h^2} + \frac{1}{h^\beta} Ab\left[-\frac{\partial}{\partial y}\left(\frac{y}{6}\vartheta(y,w)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}\vartheta(y,w)\right)\right] \tag{5.10}$$

and, the r^{th} Aboodh residual function for Eq. (5.10) is given as

$$Ab\Re es_r(y,h) = V_r(y,h) - \frac{y^2}{h^2} + \frac{1}{h^\beta}\frac{\partial}{\partial y}\left(\frac{y}{6}V_r(y,h)\right) - \frac{1}{h^\beta}\frac{\partial^2}{\partial y^2}\left(\frac{x^2}{12}V_r(y,h)\right) \tag{5.11}$$

To determine the unknown $\wp_1(x)$, put r =1 in Eq. (5.11), then first residual function is obtained by

$$Ab\Re es_1(y,h) = V_1(y,h) - \frac{y^2}{h^2} + \frac{1}{h^\beta}\frac{\partial}{\partial y}\left(\frac{y}{6}V_1(y,h)\right) - \frac{1}{h^\beta}\frac{\partial^2}{\partial y^2}\left(\frac{y^2}{12}V_1(y,h)\right) \tag{5.12}$$

we put $V_1(y,h) = \vartheta_0(y) + \frac{\wp_1(y)}{h^{\beta+2}}$ in Eq. (5.12), we get

$$Ab\Re es_1(y,h) = \frac{\wp_1(y)}{h^{\beta+2}} + \frac{y^2}{2h^{\beta+2}} + \frac{y\wp^{`}(y)+\wp_1(y)}{6h^{2\beta+2}} - \frac{y^2}{h^{\beta+2}} - \frac{y^2\wp_1{}^{``}(y)+4y\wp^{`}{}_1(y)+2\wp_1(y)}{12h^{2\beta+2}}. \tag{5.13}$$

Multiplying the Eq. (5.13) by $h^{\beta+2}$ and applying the limit $h\to\infty$, we obtain

$$\lim_{h\to\infty} h^{\beta+2} Ab\Re es_1(y,h) = \wp_1(y) - \frac{y^2}{2} = 0$$

Hence, the value of $\wp_1(y) = \frac{y^2}{2}$ so,

$$Ab\Re es_2(y,h) = \frac{\wp_2(y)}{h^{2\beta+2}} - \frac{y^2}{4h^{2\beta+2}} + \frac{y\wp_2{}^{`}(y)+\wp_2(y)}{6h^{3\beta+2}} - \frac{\wp_2(y)+4x\wp^{`}{}_2(y)+2\wp_2(y)}{12h^{2\beta+2}}. \tag{5.14}$$

Multiplying the Eq. (5.14) by $h^{2\beta+2}$ and applying the limit $h\to\infty$, we obtain

$$\lim_{h\to\infty} h^{2\beta+2} Ab\Re es_1(y,h) = \wp_2(y) - \frac{y^2}{4} = 0$$

So, $\wp_2(y) = \frac{y^2}{4}$ Similarly, we can obtain the unknown coefficient by putting the r =3, 4, 5, ... in Equation (5.9) as follow:

$$\wp_3(y) = \frac{y^2}{8}$$

$$\wp_4(y) = \frac{y^2}{16}$$

$$\wp_5(y) = \frac{y^2}{32}$$

and

$$\wp_r(x) = \frac{y^2}{2^r}$$

hence

$$V(y,h) = \frac{y^2}{h^2} + \frac{y^2}{2h^{\beta+2}} + \frac{y^2}{4h^{2\beta+2}} + \frac{y^2}{8h^{3\beta+2}} + \ldots\ldots\ldots \tag{5.15}$$

By applying the inverse Aboodh transform to Eq. (5.15)

$$\vartheta(y,w) = y^2 + \frac{y^2 w^\beta}{2\Gamma(\beta+1)} + \frac{y^2 w^{2\beta}}{4\Gamma(2\beta+1)} + \frac{y^2 w^{3\beta}}{8\Gamma(3\beta+1)} + \ldots\ldots\ldots = y^2 E_\beta(\frac{w^\beta}{2}). \tag{5.16}$$

$$\vartheta(y,w) = y^2\left\{1 + \frac{w^\beta}{2\Gamma(\beta+1)} + \frac{(w^\beta)^2}{4\Gamma(2\beta+1)} + \frac{(w^\beta)^3}{8\Gamma(3\beta+1)} + \ldots\ldots\right\}$$

The graphical solutions of both the methods are given in the figures 1 to 11. We have shown the variation in solutions for β = 0.5, 0.7 and 0.9. We also compared the obtained results with the exact solution of the problem which reflects that our obtained solutions are almost very near to the exact solution of the problem.

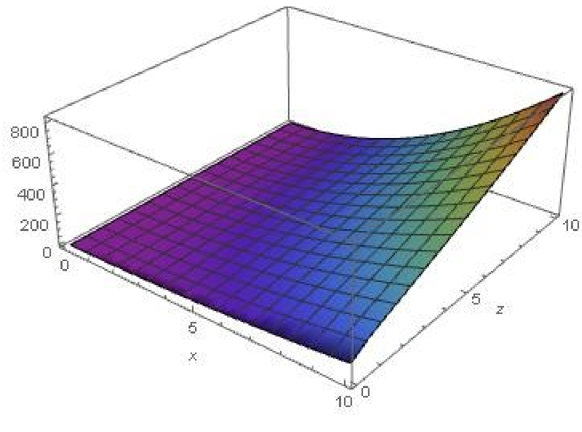

Figure 6: For β=0.5

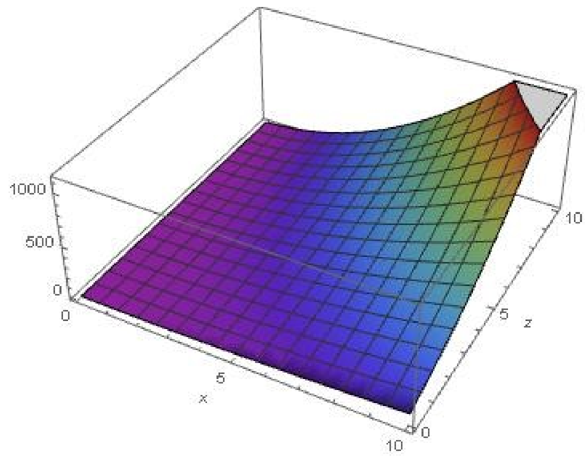

Figure 7: For β=0.7

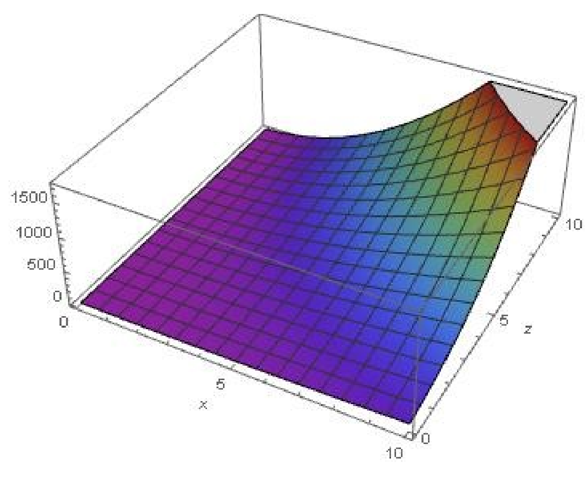

Figure 8: For β=0.9

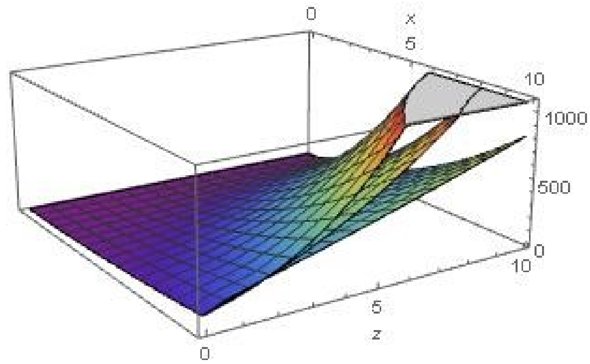

Figure 9: Comparison of results for $\beta = 0.5$, 0.7 and 0.9

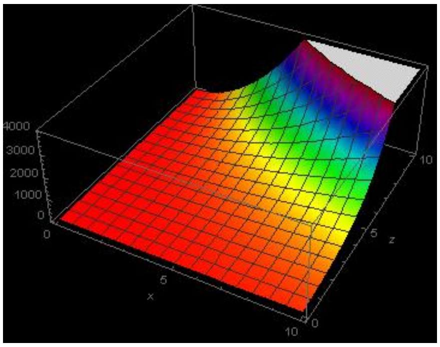

Figure 10: Exact solution for $\beta = 1$

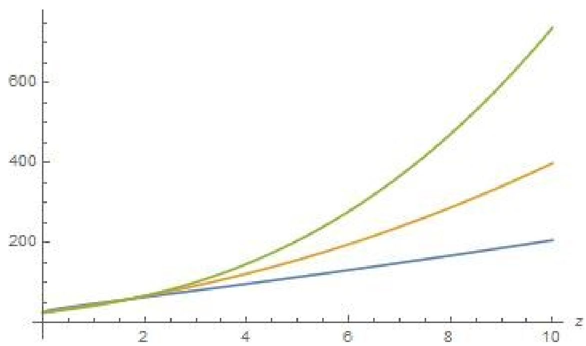

Figure 11: 2D representation of solution for x = 5 and $\beta = 0.5$, 0.7 and 0.9

6. Conclusion

Both Aboodh Residuel Power Series Method and Homotopy perturbation method, when combined with the Caputo fractional derivative, offer significant advantages in solving complex nonlinear fractional differential equations. The Caputo derivative's accurate portrayal of physical processes, along with the high efficiency, accuracy, and flexibility of Aboodh Residuel Power Series Method and Homotopy perturbation method, makes these methods powerful tools for analysing and solving a wide range of problems in science and engineering. Both methods are pretty easy to solve and they give best results if we compare the calculated results with the exact solutions.

Availability of statistics and materials: Availability of statistics is already cited in article.
Statement of disagreement: Researchers state that there aren't any conflicts of interest with the

article that is being discussed.

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