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Infinite Families of Congruences Modulo Powers of 5 for 2-Color Partition

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ABSTRACT: In this work, we investigate the arithmetic properties of $p^t_{1,\ell}(n)$, which counts 2-color partitions of n where one color appears only in parts that are not multiples of t, and the other color appears only in parts that are multiples of ℓ. By constructing generating functions for $p^t_{1,\ell}$ across specific arithmetic progressions, we establish Ramanujan-type infinite families of congruences modulo powers of 5 for $p^t_{1,\ell}(n)$.

Key Words: Partitions, generating functions, congruences.

Contents

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sums is n. Let $p(n)$ denote the number of partitions of n, with the generating function given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1}.$$

Here and throughout the paper, we set

$$f_r := (q^r;q^r)_\infty = \prod_{m=1}^{\infty}(1-q^{rm}).$$

Ramanujan [5] conjectured, and Watson [9] proved that

$$p(5^k n+\delta_k)\equiv 0 \pmod{5^k}, \tag{1.1}$$

where $k\geq 1$ and δ_k is the reciprocal modulo 5^k of 24.
Hirschhorn and Hunt [3] proved (1.1) by establishing generating functions

$$\sum_{n\geq 0} p\left(5^{2k-1}n+\delta_{2k-1}\right)q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1}\frac{f_5^{6j-1}}{f_1^{6j}} \tag{1.2}$$

and

$$\sum_{n\geq 0} p\left(5^{2k}n+\delta_{2k}\right)q^n = \sum_{j\geq 1} x_{2k,j}\, q^{j-1}\frac{f_5^{6j}}{f_1^{6j+1}}, \tag{1.3}$$

where the coefficient vectors $\mathbf{x}_k=(x_{k,1},x_{k,2},\dots)$ are given by

$$\mathbf{x}_1=(x_{1,1},x_{1,2},\dots)=(5,0,0,\dots),$$

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and for $k \geq 1$,

$$x_{k+1,i} = \begin{cases} \sum_{j\geq 1} x_{k,j}\, m_{6j,j+i}, & \text{if } k \text{ is odd}, \\ \sum_{j\geq 1} x_{k,j}\, m_{6j+1,j+i}, & \text{if } k \text{ is even}, \end{cases}$$

where the first five rows of $M = (m_{i,j})_{i,j\geq 1}$ are

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2\times 5 & 5^3 & 0 & 0 & 0 & 0 & \dots \\ 9 & 3\times 5^3 & 5^5 & 0 & 0 & 0 & \dots \\ 4 & 22\times 5^2 & 4\times 5^5 & 5^7 & 0 & 0 & \dots \\ 1 & 4\times 5^3 & 8\times 5^5 & 5^8 & 5^9 & 0 & \dots \end{bmatrix}$$

and for $i \geq 6$, $m_{i,1} = 0$, and for $j \geq 2$,

$$m_{i,j} = 25m_{i-1,j-1} + 25m_{i-2,j-1} + 15m_{i-3,j-1} + 5m_{i-4,j-1} + m_{i-5,j-1}.$$

Let $p_{1,\ell}(n)$ be the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of ℓ; its generating function is given by

$$\sum_{n\geq 0} p_{1,\ell}(n)q^n = \frac{1}{f_1 f_\ell}. \tag{1.4}$$

Ahmed, Baruah, and Dastidar [1] found several new congruences modulo 5:

$$p_{1,\ell}(25n + t) \equiv 0 \pmod{5}, \tag{1.5}$$

where $\ell \in \{0, 1, 2, 3, 4, 5, 10, 15, 20\}$ and $\ell + t = 24$. They also conjectured that the congruence also holds for $\ell = 7, 8, 17$. This conjecture was confirmed by Chern [2], moreover he proved that

$$p_{1,4}(49n + t) \equiv 0 \pmod{7}, \tag{1.6}$$

where $t \in \{11, 25, 32, 39\}$.

In [8], Wang derived congruences for $p_{1,5}(n)$ analogous to (1.1). He proved that

$$p_{1,5}\left(5^{\beta+1}n + \frac{3\cdot 5^{\beta+1}+1}{4}\right) \equiv 0 \pmod{5^{\beta+1}}, \tag{1.7}$$

$$p_{1,5}\left(5^{\beta+2}n + \frac{11\cdot 5^{\beta+1}+1}{4}\right) \equiv 0 \pmod{5^{\beta+2}}, \tag{1.8}$$

$$p_{1,5}\left(5^{\beta+2}n + \frac{19\cdot 5^{\beta+1}+1}{4}\right) \equiv 0 \pmod{5^{\beta+2}}, \tag{1.9}$$

for each $n, \beta \geq 0$. Note that, (1.7) is stronger result than (1.5) for $\ell = 5$.

Ranganatha [6] extended these results to $p_{1,25}(n)$. For each $n, \beta \geq 0$, he proved that

$$p_{1,25}\left(5^{2\beta+1}n + \frac{7\cdot 5^{2\beta+1}+13}{12}\right) \equiv 0 \pmod{5^{\beta+1}} \tag{1.10}$$

and

$$p_{1,25}\left(5^{2\beta+2}n + \frac{11\cdot 5^{2\beta+2}+13}{12}\right) \equiv 0 \pmod{5^{\beta+2}}. \tag{1.11}$$

Recently Shivashankar et.al [7] proved the congruences for $p_{1,5^k}$ for all positive integers k, which are generalizations of the above results derived by Wang and Ranganatha. The results are as follows:

For each $n, \beta \geq 0$, and $k \geq 1$, we have

$$p_{1,5^{2k-1}}\left(5^{2k+\beta-1}n+\frac{18\cdot 5^{2k+\beta-1}+5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta-1}}, \tag{1.12}$$

$$p_{1,5^{2k}}\left(5^{2k+2\beta-1}n+\frac{14\cdot 5^{2k+2\beta-1}+5^{2k}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta-1}}, \tag{1.13}$$

$$p_{1,5^{2k}}\left(5^{2k+2\beta}n+\frac{22\cdot 5^{2k+2\beta}+5^{2k}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta}}, \tag{1.14}$$

$$p_{1,5^{2k-1}}\left(5^{2k+\beta}n+\frac{(24r+18)\cdot 5^{2k+\beta-1}+5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta}} \tag{1.15}$$

and

$$p_{1,5^{2k}}\left(\frac{5^{2k+2\beta+2}n+22\cdot 5^{2k+2\beta+2}+5^{2k}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta+1}} \tag{1.16}$$

where $r\in\{2,3,4\}$.

Let $p^t_{1,\ell}(n)$, be the 2-color partitions of n where one of the colors appears only in parts that are not multiples of t, and another color appears only in parts which are multiples of ℓ, its generating function is given by

$$\sum_{n\geq 0}{p^t}_{1,\ell}(n)q^n=\frac{f_t}{f_1f_\ell}. \tag{1.17}$$

In this paper, we establish congruences for $p^t_{1,\ell}$ with $\ell\in\{5^{2k},5^{2k-1}\}$ and $t\in\{5^{2k-1},5^{2k}\}$. The main results are as follows:

Theorem 1.1 *For each $\beta\geq 0$ and $k\geq 1$, we have*

$$p^{5^{2k-1}}_{1,5^{2k}}\left(5^{2k+2\beta-1}n+\frac{3\cdot 5^{2k+2\beta}+4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta-1}}, \tag{1.18}$$

$$p^{5^{2k-1}}_{1,5^{2k}}\left(5^{2k+2\beta}n+\frac{3\cdot 5^{2k+2\beta}+4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta-1}}, \tag{1.19}$$

$$p^{5^{2k-1}}_{1,5^{2k}}\left(5^{2k+2\beta}n+\frac{r\cdot 5^{2k+2\beta-1}+4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta}} \tag{1.20}$$

and

$$p^{5^{2k-1}}_{1,5^{2k}}\left(5^{2k+2\beta+1}n+\frac{s\cdot 5^{2k+2\beta}+4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+\beta}} \tag{1.21}$$

where $r\in\{39,63,87,111\}$ and $s\in\{51,75,99\}$.

Theorem 1.2 *For each $\beta\geq 0$ and $k\geq 1$, we have*

$$p^{5^{2k}}_{1,5^{2k-1}}\left(5^{2k+2\beta-1}n+\frac{23\cdot 5^{2k+2\beta}-4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+2\beta-1}}, \tag{1.22}$$

$$p^{5^{2k}}_{1,5^{2k-1}}\left(5^{2k+2\beta}n+\frac{19\cdot 5^{2k+2\beta}-4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+2\beta}} \tag{1.23}$$

and

$$p^{5^{2k}}_{1,5^{2k-1}}\left(5^{2k+2\beta}n+\frac{s\cdot 5^{2k+2\beta-1}-4\cdot 5^{2k-1}+1}{24}\right)\equiv 0 \pmod{5^{2k+2\beta}} \tag{1.24}$$

where $s\in\{71,95,119\}$.

2. Preliminary Results

In this section, we state some lemmas which play a vital role in proving our results. Let H be the "huffing" operator modulo 5, that is,

$$H\left(\sum a_n q^n\right) = \sum a_{5n} q^{5n}.$$

If $G = \frac{f_5^6}{q^4 f_1 f_{25}^5}$ and $u = \frac{f_5^6}{q^5 f_{25}^6}$, then

$$H\left(G^i\right) = \sum_{j=1}^{i} m_{i,j} u^{i-j}. \tag{2.1}$$

Lemma 2.1 *For all $i \geq 1$, we have*

$$H\left(q^{i-1} \frac{f_5^{6i-2}}{f_1^{6i-1}}\right) = \sum_{j=1}^{5i} m_{6i-1,i+j-1}\, q^{5j-5} \frac{f_{25}^{6j-5}}{f_5^{6j-4}}, \tag{2.2}$$

$$H\left(q^{i-4} \frac{f_5^{6i-5}}{f_1^{6i-4}}\right) = \sum_{j=1}^{5i-3} m_{6i-4,i+j-1}\, q^{5j-5} \frac{f_{25}^{6j-2}}{f_5^{6j-1}}, \tag{2.3}$$

$$H\left(q^{i+1} \frac{f_5^{6i+1}}{f_1^{6i+1}}\right) = \sum_{j=1}^{5i+1} m_{6i+1,i+j}\, q^{5j} \frac{f_{25}^{6j-1}}{f_5^{6j-1}} \tag{2.4}$$

and

$$H\left(q^{i} \frac{f_5^{6i}}{f_1^{6i}}\right) = \sum_{j=1}^{5i} m_{6i,i+j}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}}. \tag{2.5}$$

Proof: We can rewrite (2.1) as

$$H\left(q^{i} \frac{f_{25}^{i}}{f_1^{i}}\right) = \sum_{j=1}^{i} m_{i,j}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}}. \tag{2.6}$$

From (2.6) and the fact that $m_{6i-1,j} = 0$ for $1 \leq j < i$, we have

$$\begin{aligned} H\left(\left(q\frac{f_{25}}{f_1}\right)^{6i-1}\right) &= \sum_{j=i}^{6i-1} m_{6i-1,j}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}} \\ &= \sum_{j=1}^{5i} m_{6i-1,j+i-1}\, q^{5j+5i-5} \frac{f_{25}^{6j+6i-6}}{f_5^{6j+6i-6}}, \end{aligned}$$

which yields (2.2). Similarly, we can prove (2.3)–(2.5). □

3. Generating Functions

In this section, we establish generating functions for $p_{1,\ell}^{t}(n)$ within specific arithmetic progressions.

Theorem 3.1 *For each $\beta \geq 0$ and $k \geq 1$, we have*

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta-1} n + \frac{3\cdot 5^{2k+2\beta} + 4\cdot 5^{2k-1} + 1}{24}\right) q^n = \sum_{i\geq 1} z_{2\beta+1,i}^{(2k-1)}\, q^{i-1} \frac{f_5^{6i-2}}{f_1^{6i-1}} \tag{3.1}$$

and

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta} n + \frac{3\cdot 5^{2k+2\beta} + 4\cdot 5^{2k-1} + 1}{24}\right) q^n = \sum_{i\geq 1} z_{2\beta+2,i}^{(2k-1)}\, q^{i-1} \frac{f_5^{6i-5}}{f_1^{6i-4}} \tag{3.2}$$

where the coefficient vectors are defined as follows:

$$z_{1,j}^{(2k-1)} = x_{2k-1,j}$$

and

$$z_{\beta+1,j}^{(2k-1)} = \begin{cases} \sum_{i\geq 1} z_{\beta,i}^{(2k-1)} m_{6i-1,j+i-1} & \text{if } \beta \text{ is odd,} \\ \sum_{i\geq 1} z_{\beta,i}^{(2k-1)} m_{6i-4,j+i-1} & \text{if } \beta \text{ is even,} \end{cases}$$

for all $\beta, j \geq 1$.

Proof: By setting $\ell = 5^{2k}$ and $t = 5^{2k-1}$ in 1.17,we have

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}(n)q^n = \frac{f_{5^{2k-1}}}{f_1 f_{5^{2k}}} \tag{3.3}$$

$$= \frac{f_{5^{2k-1}}}{f_{5^{2k-1+1}}} \sum_{n\geq 0} p(n)q^n. \tag{3.4}$$

Extracting the terms involving $q^{5^{2k-1}n+\delta_{2k-1}}$ on both sides of (3.4) and dividing throughout by $q^{\delta_{2k-1}}$, we obtain

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}(5^{2k-1}n + \delta_{2k-1})q^{5^{2k-1}n} = \frac{f_{5^{2k-1}}}{f_{5^{2k-1+1}}} \sum_{n\geq 0} p(5^{2k-1}n + \delta_{2k-1})q^{5^{2k-1}n}.$$

If we replace $q^{5^{2k-1}}$ by q and use (1.2), we get

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}(5^{2k-1}n + \delta_{2k-1})q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1} \frac{f_5^{6j-2}}{f_1^{6j-1}},$$

which is the case $\beta = 0$ of (3.1) We now assume that (3.1) is true for some integer $\beta \geq 0$. Applying the operator H to both sides, by (2.2), we have

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}} \left(5^{2k+2\beta}n + \frac{3\cdot 5^{2k+2\beta} + 4\cdot 5^{2k-1} + 1}{24}\right) q^{5n} \\
&= \sum_{i\geq 1} z_{2\beta+1,i}^{(2k-1)}\, H\left(q^{i-1}\frac{f_5^{6i-2}}{f_1^{6i-1}}\right) \\
&= \sum_{i\geq 1} z_{2\beta+1,i}^{(2k-1)} \sum_{j\geq 1} m_{6i-1,i+j-1}\, q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-4}} \\
&= \sum_{j\geq 1}\left(\sum_{i\geq 1} z_{2\beta+1,i}^{(2k-1)}\, m_{6i-1,i+j-1}\right) q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-4}} \\
&= \sum_{j\geq 1} z_{2\beta+2,j}^{(2k-1)}\, q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-4}}
\end{aligned}$$

that is

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}} \left(5^{2k+2\beta}n + \frac{3\cdot 5^{2k+2\beta} + 4\cdot 5^{2k-1} + 1}{24}\right) q^{n} = \sum_{j\geq 1} z_{2\beta+2,j}^{(2k-1)}\, q^{j-1}\frac{f_5^{6j-5}}{f_1^{6j-4}}.$$

Hence, if (3.1) is true for some integer $\beta \geq 0$, then (3.2) is true for β. Suppose that (3.2) is true for some integer $\beta \geq 0$. Applying the operator H to both sides, by (2.3), we obtain

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta+1}n+\frac{3\cdot 5^{2k+2\beta+2}+4\cdot 5^{2k-1}+1}{24}\right)q^{5n}$$
$$=\sum_{i\geq 1} z_{2\beta+2,i}^{(2k-1)}\sum_{j\geq 1} m_{6i-4,i+j-1}q^{5j-5}\frac{f_{25}^{6j-2}}{f_5^{6j-1}}$$
$$=\sum_{j\geq 1}\left(\sum_{i\geq 1} z_{2\beta+2,i}^{(2k-1)}\, m_{6i-4,i+j-1}\right)q^{5j-5}\frac{f_{25}^{6j-2}}{f_5^{6j-1}}$$
$$=\sum_{j\geq 1} z_{2\beta+3,j}^{(2k-1)}\, q^{5j-5}\frac{f_{25}^{6j-2}}{f_5^{6j-1}}$$

which yields

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta+1}n+\frac{3\cdot 5^{2k+2\beta+2}+4\cdot 5^{2k-1}+1}{24}\right)q^{n}=\sum_{j\geq 1} z_{2\beta+3,j}^{(2k-1)}\, q^{j-1}\frac{f_5^{6j-2}}{f_1^{6j-1}}.$$

This is (3.1) with β replaced by $\beta+1$. This completes the proof. □

Theorem 3.2 *For each $\beta \geq 0$ and $k \geq 1$, we have*

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta-1}n+\frac{23\cdot 5^{2k+2\beta-1}-4\cdot 5^{2k-1}+1}{24}\right)q^{n}=\sum_{i\geq 1} w_{2\beta+1,i}^{(2k-1)}\, q^{i-1}\frac{f_5^{6i}}{f_1^{6i+1}} \tag{3.5}$$

and

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta}n+\frac{19\cdot 5^{2k+2\beta}-4\cdot 5^{2k-1}+1}{24}\right)q^{n}=\sum_{i\geq 1} w_{2\beta+2,i}^{(2k-1)}\, q^{i-1}\frac{f_5^{6i-1}}{f_1^{6i}} \tag{3.6}$$

where the coefficient vectors are defined as follows:

$$w_{1,j}^{(2k-1)}=x_{2k-1,j}$$

and

$$w_{\beta+1,j}^{(2k-1)}=\begin{cases}\sum_{i\geq 1} w_{\beta,i}^{(2k-1)} m_{6i+1,j+i} & \text{if } \beta \text{ is odd,}\\ \sum_{i\geq 1} w_{\beta,i}^{(2k-1)} m_{6i,j+i} & \text{if } \beta \text{ is even,}\end{cases}$$

for all $\beta, j \geq 1$.

Proof: From 1.17 by setting $\ell = 5^{2k-1}$, $t = 5^{2k}$, we have

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}(n)q^n=\frac{f_{5^{2k}}}{f_1 f_{5^{2k-1}}}$$

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}(n)q^n=\frac{f_{5^{2k-1+1}}}{f_{5^{2k-1}}}\sum_{n\geq 0} p(n)q^n. \tag{3.7}$$

Extracting the terms involving $q^{5^{2k-1}n+\delta_{2k-1}}$ on both sides of (3.7) and dividing throughout by $q^{\delta_{2k-1}}$, we obtain

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}(5^{2k-1}n+\delta_{2k-1})q^{5^{2k-1}n}=\frac{f_{5^{2k-1+1}}}{f_{5^{2k-1}}}\sum_{n\geq 0} p(5^{2k-1}n+\delta_{2k-1})q^{5^{2k-1}n}.$$

If we replace $q^{5^{2k-1}}$ by q and use (1.2), we get

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}(5^{2k-1}n+\delta_{2k-1})q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1}\frac{f_5^{6j}}{f_1^{6j+1}}$$

which is the case $\beta = 0$ of (3.5) We now assume that (3.5) is true for some integer $\beta \geq 0$. Applying the operator H to both sides, by (2.4), we have

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta}n + \frac{19\cdot 5^{2k+2\beta} - 4\cdot 5^{2k-1}+1}{24}\right)q^{5n+5} \\
&= \sum_{i\geq 1} w_{2\beta+1,i}^{(2k-1)}\, H\left(q^{i+1}\frac{f_5^{6i}}{f_1^{6i+1}}\right) \\
&= \sum_{i\geq 1} w_{2\beta+1,i}^{(2k-1)} \sum_{j\geq 1} m_{6i+1,i+j}\, q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j}} \\
&= \sum_{j\geq 1}\left(\sum_{i\geq 1} w_{2\beta+1,i}^{(2k-1)}\, m_{6i+1,i+j}\right) q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j}} \\
&= \sum_{j\geq 1} w_{2\beta+2,j}^{(2k-1)}\, q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j}}
\end{aligned}$$

that is

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta}n + \frac{19\cdot 5^{2k+2\beta} - 4\cdot 5^{2k-1}+1}{24}\right)q^{n} = \sum_{j\geq 1} w_{2\beta+2,j}^{(2k-1)}\, q^{j-1}\frac{f_5^{6j-1}}{f_1^{6j}}.$$

Hence, if (3.5) is true for some integer $\beta \geq 0$, then (3.6) is true for β. Suppose that (3.6) is true for some integer $\beta \geq 0$. Applying the operator H to both sides, by (2.5), we obtain

$$\begin{aligned}
&\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta+1}n + \frac{23\cdot 5^{2k+2\beta+1} - 4\cdot 5^{2k-1}+1}{24}\right)q^{5n+5} \\
&= \sum_{i\geq 1} w_{2\beta+2,i}^{(2k-1)} \sum_{j\geq 1} m_{6i,i+j} q^{5j}\frac{f_{25}^{6j}}{f_5^{6j+1}} \\
&= \sum_{j\geq 1}\left(\sum_{i\geq 1} w_{2\beta+2,i}^{(2k-1)}\, m_{6i,i+j}\right) q^{5j}\frac{f_{25}^{6j}}{f_5^{6j+1}} \\
&= \sum_{j\geq 1} w_{2\beta+3,j}^{(2k-1)}\, q^{5j}\frac{f_{25}^{6j}}{f_5^{6j+1}}
\end{aligned}$$

which yields

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta+1}n + \frac{23\cdot 5^{2k+2\beta+1} - 4\cdot 5^{2k-1}+1}{24}\right)q^{n} = \sum_{j\geq 1} w_{2\beta+3,j}^{(2k-1)}\, q^{j-1}\frac{f_5^{6j}}{f_1^{6j+1}}.$$

This is (3.5) with β replaced by $\beta+1$. This completes the proof. □

4. Proof of Congruences

For a positive integer n, let $\pi(n)$ be the highest power of 5 that divides n, and define $\pi(0) = +\infty$.

Lemma 4.1 **([3], Lemma 4.1)** *For each $i, j \geq 1$, we have*

$$\pi\left(m_{i,j}\right) \geq \frac{5j-i-1}{2}. \tag{4.1}$$

Lemma 4.2 **([3], Lemma 4.3)** *For each $k, j \geq 1$, we have*

$$\pi\left(x_{2k-1,j}\right) \geq 2k-1+\left[\frac{5j-5}{2}\right]. \tag{4.2}$$

For each $j, k \geq 1$ and $\beta \geq 0$, we have

$$\pi\left(z^{(2k-1)}_{2\beta+1,j}\right) \geq 2k+\beta-1+\left[\frac{5j-5}{2}\right] \tag{4.3}$$

and

$$\pi\left(z^{(2k-1)}_{2\beta+2,j}\right) \geq 2k+\beta-1+\delta_{1,j}+\left[\frac{5j-6}{2}\right] \tag{4.4}$$

where

$$\delta_{1,j}=\begin{cases}1 & \text{if } j=1,\\ 0 & \text{if } j\neq 1.\end{cases}$$

Proof: In view of Theorem 3.1, we have

$$z^{(2k-1)}_{1,j}=x_{2k-1,j}.$$

From (4.2), we can see that the inequality (4.3) holds for $\beta=0$.

We now assume that (4.3) is true for some $\beta \geq 0$, then

$$\begin{aligned}\pi\left(z^{(2k-1)}_{2\beta+2,j}\right) &\geq \min_{i\geq 1}\left\{\pi\left(z^{(2k-1)}_{2\beta+1,i}\right)+\pi\left(m_{6i-1,i+j-1}\right)\right\}\\ &\geq \min_{i\geq 1}\left\{2k+\beta-1+\left[\frac{5i-5}{2}\right]+\left[\frac{5j-i-5}{2}\right]\right\}\\ &\geq 2k+\beta+1+\delta_{1,j}+\left[\frac{5j-6}{2}\right]\end{aligned}$$

which is (4.4). Now, suppose (4.4) holds for some $\beta \geq 0$. Then,

$$\begin{aligned}\pi\left(z^{(2k-1)}_{2\beta+3,j}\right) &\geq \min_{i\geq 1}\left\{\pi\left(z^{(2k-1)}_{2\beta+2,i}\right)+\pi\left(m_{6i-4,i+j-1}\right)\right\}\\ &\geq \min_{i\geq 1}\left\{2k+\beta-1+\delta_{1,i}+\left[\frac{5i-6}{2}\right]+\left[\frac{5j-i-2}{2}\right]\right\}\\ &\geq 2k+\beta-1+\left[\frac{5j-3}{2}\right]\\ &\geq 2k+\beta+\left[\frac{5j-5}{2}\right]\end{aligned}$$

which is (4.3) with $\beta+1$ for β. This completes the proof. □

Lemma 4.3 *For each $j, k \geq 1$ and $\beta \geq 0$, we have*

$$\pi\left(w^{(2k-1)}_{2\beta+1,j}\right) \geq 2k-1+2\beta+\left[\frac{5j-5}{2}\right] \tag{4.5}$$

and

$$\pi\left(w^{(2k-1)}_{2\beta+2,j}\right) \geq 2k+2\beta+\left[\frac{5j-5}{2}\right]. \tag{4.6}$$

Proof: In view of Theorem 3.2, we have

$$w_{1,j}^{(2k-1)} = x_{2k-1,j}.$$

From (4.2), we can see that the inequality (4.5) holds for $\beta = 0$.

We now assume that (4.5) is true for some $\beta \geq 0$, then

$$\begin{aligned}
\pi\left(w_{2\beta+2,j}^{(2k-1)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(w_{2\beta+1,i}^{(2k-1)}\right) + \pi\left(m_{6i+1,i+j}\right)\right\} \\
&\geq \min_{i\geq 1}\left\{2k-1+2\beta+\left[\frac{5i-5}{2}\right]+\left[\frac{5j-i-2}{2}\right]\right\} \\
&\geq 2k+2\beta+\left[\frac{5j-5}{2}\right]
\end{aligned}$$

which is (4.6). Now suppose (4.6) holds for some $\beta \geq 0$. Then,

$$\begin{aligned}
\pi\left(w_{2\beta+3,j}^{(2k-1)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(w_{2\beta+2,i}^{(2k-1)}\right) + \pi\left(m_{6i,i+j}\right)\right\} \\
&\geq \min_{i\geq 1}\left\{2k+2\beta+\left[\frac{5i-5}{2}\right]+\left[\frac{5j-i-1}{2}\right]\right\} \\
&\geq 2k+2\beta+\left[\frac{5j-2}{2}\right] \\
&\geq 2k+2\beta+1+\left[\frac{5j-5}{2}\right]
\end{aligned}$$

which is (4.5) with $\beta+1$ for β. This completes the proof. □

Proof of Theorem 1.1 Congruence (1.18) follows from (4.3) together with (3.1) and congruence (1.19) follows from (4.4) and (3.2).

The 2-color partition $p_{-2}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{f_1^2}. \tag{4.7}$$

It has been shown by Ramanathan [4] that for $n \geq 0$ and $s \in \{2,3,4\}$.

$$p_{-2}(5n+s) \equiv 0 \pmod{5}. \tag{4.8}$$

In view of (4.3) and using binomial theorem, we can express(3.1) as

$$\begin{aligned}
\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta-1}n+\frac{3\cdot 5^{2k+2\beta}+4\cdot 5^{2k-1}+1}{24}\right)q^n &\equiv z_{2\beta+1,1}^{(2k-1)}\frac{f_5^4}{f_1^5} \pmod{5^{2k+\beta}} \\
&\equiv z_{2\beta+1,1}^{(2k-1)} f_5^3 \pmod{5^{2k+\beta}}
\end{aligned} \tag{4.9}$$

Equating the coefficients of q^{5n+r}, $r \in \{1,2,3,4\}$, we arrive at (1.20).

Extracting the terms q^{5n} in (4.9) and replace q^5 by q, and using (4.7) we get

$$\sum_{n\geq 0} p_{1,5^{2k}}^{5^{2k-1}}\left(5^{2k+2\beta}n+\frac{3\cdot 5^{2k+2\beta}+4\cdot 5^{2k-1}+1}{24}\right)q^n \equiv z_{2\beta+1,1}^{(2k-1)} f_5 \sum_{n=0}^{\infty} p_{-2}(n)q^n \pmod{5^{2k+\beta}},$$

using (4.8), we arrive at (1.21).

Proof of Theorem 1.2 Congruence (1.22) follows from (4.5) together with (3.5), and congruence (1.23) follows from (4.6) and (3.6).
In view of (4.5) and using binomial theorem, we can express(3.5) as

$$\sum_{n\geq 0} p_{1,5^{2k-1}}^{5^{2k}}\left(5^{2k+2\beta-1}n+\frac{23\cdot 5^{2k+2\beta-1}-4\cdot 5^{2k-1}+1}{24}\right)q^n \equiv w_{2\beta+1,1}^{(2k-1)}\frac{f_5^6}{f_1^7} \pmod{5^{2k+2\beta}}$$
$$\equiv w_{2\beta+1,1}^{(2k-1)}\frac{f_{25}}{f_1^2} \pmod{5^{2k+2\beta}}. \tag{4.10}$$

Using (4.7) and (4.8), Equating the coefficients of q^{5n+s}, $s\in\{2,3,4\}$, we arrive at (1.24).

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