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A Study on Some Novel Fixed Point Results for Fuzzy Mapping via Θ-Contraction

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ABSTRACT: This study aims to analyze the presence of a fixed point for fuzzy mapping under θ-contraction. We intend to draw some conclusions in order to identify the condition under which these mappings have fixed points. The results of this investigation will be a useful addition to the existing body of research, providing support for the conclusions. The practical implication of our methodology show through example, demonstrating the strength of established outcomes. Our results extend and combine many results that exist in the significant area of research.

Key Words: Fuzzy sets, fuzzy mapping, fixed points, common fixed points, θ-contraction, multi-valued mappings.

Contents

1. Introduction

Fixed Point Theory is a central and celebrated field in mathematics, renowned for its vast applications across topology, functional analysis, computer science, game theory, and control theory.

Consider a non-empty set X, and let T be a self-map on X. The fundamental fixed-point equation is $T(x) = x$, which seeks to find a point that remains invariant under the map T. An alternative formulation of this problem is the equation $g(x) = 0, \quad \text{where} \quad g(x) = x - T(x)$, which elegantly recasts the search for a fixed point into the task of finding the root of a function.

Although the concept appears deceptively simple, solving the fixed-point equation can be extraordinarily challenging. In some cases, finding a solution may even be impossible. Nevertheless, fixed-point theorems are invaluable in establishing the existence and uniqueness of solutions to a wide variety of mathematical models.

Among the most celebrated results in this theory is the Banach Contraction Principle, which asserts that a contraction map on a complete metric space has a unique fixed point. This result is a cornerstone of the theory and has profound implications in numerous areas where the existence and uniqueness of solutions are critical [1].

The study of fixed points for multi-valued mappings was first introduced by Von Neumann. In 1969, Nadler [2] extended the Banach contraction theorem by incorporating the concept of multi-valued mappings and contractions. Among the most intriguing developments in this area is Zadeh's fuzzy set theory [3], a groundbreaking extension of classical set theory. Fuzzy sets allow elements to have varying degrees of membership in a set, represented by a value between 0 and 1, reflecting the strength of the association of the elements with the set. Heilpern [4] subsequently established a fixed point theorem within the framework of fuzzy sets.For further works employing fuzzy mappings or the fuzzy metric approach in the development of fixed point results, see Butnariu [5], Kanwal et al. [6], Weiss [7], Xu et al. [8], Moussaoui

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et al. [16,18], and Tassaddiq et al. [17].
The real world is rife with imprecision, ambiguity, and uncertainty. Numerous complicated problems exist in the fields of economics, engineering, environmental science, social science, medical science, biology and many more, that include data which isn't always crisp' Because of several kinds of uncertainties in these domains, we are unable to solve them with traditional approaches. There are several theories in the literature that deal with uncertainty in more effective way Zadeh fuzzy set theory [3], Atanassov intuitionistic fuzzy set theory [9] and Zdzislaw Pawlak rough set theory [10].
Klim and Wardowski [11], proposed the presence of a fixed point in complete metric space for set-valued contractions. Dhage [12], proved hybrid fixed point theorem and presented several implementations. Hong [13], demonstrated hybrid fixed point theorem using multi-valued mapping that satisfied weakly generalized contractive requirements in ordered complete metric space. It is both interesting and well-developed to investigate fixed point and common fixed point of hybrid pairs of mappings in metric spaces and Banach spaces. We refer to [14, 15] for a study of fixed point theory and multimap coincidences, their applications and associated conclusions.
Branciari presented generalized metric space, replacing the triangle inequality with the inequality $\rho(m,n) \leq \rho(m,s)+\rho(s,r)+\rho(r,n)$ for any pairwise distinct points $m,n,s,r \in \Pi$. Jleli and Sament [19], investigated fixed point theorem and presented a novel form of contractive mapping known as the θ-contraction in the framework of Branciari metric space.
The purpose of this study is to explore the fixed point of θ-contractions within the framework of fuzzy mappings. Our results offer notable enhancements to a number of well-established and conventional findings. To reinforce the robustness of these results, we present a carefully constructed non-trivial example. The key innovation of this work provides a solution approach for specific categories of non-linear integral equations, contributing valuable insights to their resolution.

2. Preliminaries

Definition 2.1 *[6] Let (Π,ρ) be a metric space. The function*

$$H : CB(\Pi) \times CB(\Pi) \to \mathbb{R}$$

is defined as,

$$H(W,M) = max\{\sup_{r\in W} \rho(r,M), \sup_{s\in M} \rho(W,s)\},$$

where

$$\rho(s,W) = \inf_{r\in W} \rho(s,r).$$

Such a mapp H is a Hausdorff metric, (Π,H) is Hausdorff metric space, and $CB(\Pi)$ denotes the family of all closed and bounded subsets of metric space (Π,ρ).

Definition 2.2 *[6] If Π is a universe of discourse and x be an arbitary element of Π. The fuzzy set K defined on Π is a collection of ordered pairs, $\{(q,K(q)) : q \in \Pi\}$, where $K : \Pi \to [0,1]$ is known as membership function. The $\mathcal{F}(\Pi)$ is the set of all Fuzzy sets.*

Example 2.1 *Let $\Pi = [1,10]$ and K, L are functions defined by,*

$$K(\mu) = \begin{cases} 1 - \frac{|\mu-3|}{2} & when\ 1 \leq \mu \leq 5, \\ 0 & otherwise. \end{cases}$$

$$L(\mu) = \begin{cases} 1 - \frac{|\mu-4|}{2} & when\ 2 \leq \mu \leq 6, \\ 0 & otherwise. \end{cases}$$

Where K and L are fuzzy sets on Π, the function values $K(\mu)$ and $L(\mu)$ are called the grade of memberships of μ in K and in L respectively.

Definition 2.3 *[6] The α-cut of fuzzy set K is denoted by $[K]_\alpha$,*

$$[K]_\alpha = \{\mu : K(\mu) \geq \alpha\} \quad where\ \alpha \in (0, 1].$$

Example 2.2 $\Pi = \{1, 2, 3\}$ *and* $K = \{(1, 0.5), (2, 0.8), (3, 1)\}$.
If $\alpha = 0.3$, *then* $[K]_{0.3} = \{1, 2, 3\}$.
If $\alpha = 0.7$, *then* $[K]_{0.7} = \{2, 3\}$.

Definition 2.4 *[6] Let P be a metric space and Π be a set. A fuzzy mapping $R : \Pi \to \digamma(\Pi)$ is a fuzzy set on $\Pi \times P$ with membership function $R(h)(j)$. $R(h)(j)$ is a membership value of j in $R(h)$. The α-cut of $R(x)$ is denoted by $[Rx]_\alpha$.*

Example 2.3 *Let $\Pi = [0, \infty)$ be complete metric space and $R : \Pi \to \digamma(\Pi)$, such that $Rx : \Pi \to [0, 1]$ is defiend as,*

$$Rx(t) = \begin{cases} \frac{1}{2} & if\ 0 \leq t \leq 2x + 2, \\ \frac{1}{5} & if\ 2x + 2 < t \leq 4x + 2, \\ \frac{1}{6} & if\ 4x + 2 < t < \infty. \end{cases}$$

$[Rx]_{\frac{1}{2}} = [0, 2x + 2]$, $[Rx]_{\frac{1}{5}} = [0, 4x + 2]$, $[Rx]_{\frac{1}{6}} = [0, \infty)$ *are alpha level sets of Rx.*

Definition 2.5 *Let Π be a metric space. A mapping $Z : \Pi \to \Pi$ is known as contraction on Π if real number exist $q \in [0, 1)$ such that $\forall\ s, r \in \Pi$*

$$\rho(Zs, Zr) \leq q\rho(s, r).$$

Theorem 2.1 *Let (Π, ρ) be a complete metric space, then a contraction $Z : \Pi \to \Pi$ on complete metric space has a unique fixed point.*

Theorem 2.2 *Let (Π, ρ) be a complete metric space. If $Z : \Pi \to CB(\Pi)$ is a multi-valued contraction on complete metric space, then Z has a fixed point.*

Theorem 2.3 *Let (Π, ρ) be a complete metric space and $Z : \Pi \to \digamma(\Pi)$ remunerate the requirment there exist $l \in (0, 1)$ such that*

$$D(Zs, Zr) \leq l\rho(s, r) \qquad for\ \ each\ \ s, r \in \Pi$$

then Z has fixed point.

Definition 2.6 *[6] Let $R : \Pi \to \digamma(\Pi)$ be fuzzy mapping. A point $z \in \Pi$ is called fuzzy fixed point of R if there exist $\alpha \in (0, 1]$ such that $z \in [Rz]_{\alpha_{Rz}}$.*

Definition 2.7 *[6] Let $Q, R : \Pi \to \digamma(\Pi)$ be two fuzzy mapping and for $u \in \Pi$, there exists $\alpha_{Q(u)}, \alpha_{R(u)} \in (0, 1]$. A point u is called fuzzy common fixed point of Q and R if $u \in [Qu]_{\alpha_{Qu}} \cap [Ru]_{\alpha_{Ru}}$.*

Definition 2.8 *[19] Let Π be a set and $\rho : \Pi \times \Pi \to [0, \infty)$, satisfy the conditions we have*

(1) $\rho(r, s) = 0 \Leftrightarrow r = s$,

(2) $\rho(r, s) = \rho(s, r)$,

(3) $\rho(r, s) \leq \rho(r, p) + \rho(p, t) + \rho(t, s)$.
Then (Π, ρ) is called Generalized metric space.

Example 2.4 *Let* $\Pi = \{1 - \frac{1}{w} : w = 1, 2, ...\} \bigcup \{1, 2\}$, $\rho : \Pi \times \Pi \to \mathbb{R}$ *as follow*

$$\rho(k,l) = \begin{cases} 0 & for \quad k = l \\ \frac{1}{w} \quad for \quad k \in \{1,2\} and l = 1 - \frac{1}{w} \ or \ l \in \{1,2\} and k = 1 - \frac{1}{w}, k \neq l \\ 1 & otherwise. \end{cases}$$

We can easily check (Π, ρ) *be a generalized metric space.*

Definition 2.9 *[19]*

Let Φ *is the collection of function,* $\Delta : (0, \infty) \to (1, \infty)$ *satisfying the following conditions.*

(Φ_1) *:* Δ *is increasing,*

(Φ_2) *: Each one sequence* $\{w_t\} \subset (0, \infty)$, $\lim_{t\to\infty} \Delta(w_t) = 1$ *iff* $\lim_{t\to\infty} w_t = 0$,

(Φ_3) *: there exist* $e \in (0, 1)$ *and* $h \in (0, \infty]$ *such that* $\lim_{w\to 0} \frac{\Delta(w)-1}{w^e} = h$,

(Φ_4) *:* $\Delta(\inf J) = \inf \Delta(J)$.

Definition 2.10 *[19] Let* (Π, ρ) *be a generalized metric space. A function* $Z : \Pi \to \Pi$ *is called* $\theta-$ *contraction, if there is a* $\Delta \in \Phi$ *and* $J \in (0, 1)$ *such that for all* $p, q \in \Pi$,

$$\rho(Zp, Zq) \neq 0 \Rightarrow \Delta(\rho(Zp, Zq)) \leq [\Delta(\rho(p, q))]^J.$$

Definition 2.11 *[19] Let* (Π, ρ) *be a generalized metric space. A function* $Z : \Pi \to CB(\Pi)$ *is called* $\theta-$ *contraction, if there is a* $\Delta \in \Phi$ *and* $P \in (0, 1)$ *such that for all* $p, q \in \Pi$,

$$H(Zp, Zq) \neq 0 \Rightarrow \Delta(H(Zp, Zq)) \leq [\Delta(\rho(p, q))]^P. \tag{2.1}$$

Definition 2.12 *Let* (Π, ρ) *be a generalized metric space. A function* $Z : \Pi \to \digamma(\Pi)$ *is fuzzy mapping. Suppose for each* $u \in \Pi$ *there exist* $\alpha_{Zu} \in (0, 1]$ *such that* $[Zu]_{\alpha_{Zu}} \in CB(\Pi)$. *A mapping* Z *is called* $\theta-$ *contraction, if there is a* $\Delta \in \Phi$ *and* $P \in (0, 1)$ *such that for all* $p, q \in \Pi$,

$$H([Zp]_{\alpha_{Zp}}, [Zq]_{\alpha_{Zq}}) \neq 0 \Rightarrow \Delta(H([Zp]_{\alpha_{Zp}}, [Zq]_{\alpha_{Zq}})) \leq [\Delta(\rho(x, y))]^P. \tag{2.2}$$

Lemma 2.1 *[6] Let* (Π, ρ) *be a metric space. Let* $K, L \in CB(\Pi)$ *and* $\epsilon > 0$
$\rho(a, L) \leq H(K, L)$ *where* $a \in K$.

Lemma 2.2 *[6] For* $\epsilon > 0$ *and* $p \in K$*, then there exists* $q \in L$ *such that*
$\rho(p, q) \leq H(K, L) + \epsilon$.

.

3. Main Results

This section contains our research work regarding the existence of fixed point of fuzzy mapping. Moreover, the obtained result are supported with example.

Theorem 3.1 *Let* (Π, ρ) *be a generalized metric space and* $Z : \Pi \to \digamma(\Pi)$ *be a fuzzy mapping. Suppose for each* $\omega \in \Pi$ *there exist* $\alpha_{Z\omega} \in (0, 1]$ *such that* $[Z\omega]_{\alpha_{Z\omega}} \in CB(\Pi)$ *and assume that* $\Delta \in \Phi$ *and* $P \in (0, 1)$ *such that* (2.2) *holds. Then* Z *has fixed point.*

Proof: Let $\nu_0 \in \Pi$ there exist $\alpha_{Z\nu_o} \in (0, 1]$ such that $[Z\nu_0]_{\alpha_{Z\nu_0}} \in CB(\Pi)$. Let $\nu_1 \in [Z\nu_0]_{\alpha_{Z\nu_0}}$. Let $\nu_1 \in \Pi$ there exist $\alpha_{Z\nu_1} \in (0, 1]$ such that $[Z\nu_1]_{\alpha_{Z\nu_1}} \in CB(\Pi)$. Let $\nu_2 \in [Z\nu_1]_{\alpha_{Z\nu_1}}$.
Assume $H([Z\nu_0]_{\alpha_{Z\nu_0}}, [Z\nu_1]_{\alpha_{Z\nu_1}}) > 0$, by using Lemma 2.1

$$\Delta(\rho(\nu_1, [Z\nu_1]_{\alpha_{Z\nu_1}})) \leq \Delta(H([Z\nu_0]_{\alpha_{Z\nu_0}}, [Z\nu_1]_{\alpha_{Z\nu_1}})) \leq [\Delta(\rho(\nu_0, \nu_1))]^P, \tag{3.1}$$

from Φ_4, we have

$$\Delta(\rho(\nu_1,[Z\nu_1]_{\alpha_{Z\nu_1}})) = \inf_{\nu_2\in[Z\nu_1]_{\alpha_{Z\nu_1}}} \Delta(\rho(\nu_1,\nu_2)), \tag{3.2}$$

from (3.1), (3.2)

$$\inf_{\nu_2\in[Z\nu_1]_{\alpha_{Z\nu_1}}} \Delta(\rho(\nu_1,\nu_2)) \leq [\Delta(\rho(\nu_0,\nu_1))]^P$$

$$\Delta(\rho(\nu_1,\nu_2)) \leq [\Delta(\rho(\nu_0,\nu_1))]^P. \tag{3.3}$$

Assume $H([Z\nu_1]_{\alpha_{Z\nu_1}},[Z\nu_2]_{\alpha_{Z\nu_2}}) > 0$, then by using Lemma 2.1

$$\Delta(\rho(\nu_2,[Z\nu_2]_{\alpha_{Z\nu_2}})) \leq (H([Z\nu_1]_{\alpha_{Z\nu_1}},[Z\nu_2]_{\alpha_{Z\nu_2}})) \leq [\Delta(\rho(\nu_1,\nu_2))]^P, \tag{3.4}$$

from Φ_4, we have

$$\Delta(\rho(\nu_2,[Z\nu_2]_{\alpha_{Z\nu_2}})) = inf_{\nu_3\in[Z\nu_2]_{\alpha_{Z\nu_2}}} \rho(\nu_2,\nu_3), \tag{3.5}$$

from (3.4), (3.5)

$$\inf_{\nu_3\in[Z\nu_2]_{\alpha_{Z\nu_2}}} \Delta(\rho(\nu_2,\nu_3)) \leq [\Delta(\rho(\nu_1,\nu_2))]^P$$

$$\Delta(\rho(\nu_2,\nu_3)) \leq [\Delta(\rho(\nu_1,\nu_2))]^P.$$

Similarly, we generate a sequence $\{\nu_n\}$ in Π such that for $\nu_n \in \Pi$ there exist $\alpha_{Z\nu_n} \in (0,1]$ such that $[Z\nu_n]_{\alpha_{Z\nu_n}} \in CB(\Pi)$.
Let $\nu_{n+1} \in [Z\nu_n]_{\alpha_{Z\nu_n}}, \quad \nu_{n+2} \in [Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}}$,
Assume $H([Z\nu_n]_{\alpha_{Z\nu_n}},[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}}) > 0$, then by using Lemma 2.1

$$\Delta(\rho(\nu_{n+1},[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}})) \leq \Delta(H([Z\nu_n]_{\alpha_{Z\nu_n}},[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}})) \leq [\Delta(\rho(\nu_n,\nu_{n+1}))]^P, \tag{3.6}$$

from Φ_4, we have

$$\Delta(\rho(\nu_{n+1},[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}})) = \inf_{\nu_{n+2}\in[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}}} \Delta(\rho(\nu_{n+1},\nu_{n+2})), \tag{3.7}$$

from (3.6), (3.7)

$$\inf_{\nu_{n+2}\in[Z\nu_{n+1}]_{\alpha_{Z\nu_{n+1}}}} \Delta(\rho(\nu_{n+1},\nu_{n+2})) \leq [\Delta(\rho(\nu_n,\nu_{n+1}))]^P$$

$$\begin{aligned}\Delta(\rho(\nu_{n+1},\nu_{n+2})) &\leq [\Delta(\rho(\nu_n,\nu_{n+1}))]^P\\ &\leq [\Delta(\rho(\nu_{n-1},\nu_n))]^{P^2}\\ &\leq [\Delta(\rho(\nu_{n-2},\nu_{n-1}))]^{P^3}\\ &\leq [\Delta(\rho(\nu_0,\nu_1))]^{P^n}.\end{aligned}$$

Since $\Delta \in \Phi$, we have at the limit $n \to \infty$.

$$\lim_{n\to\infty} \Delta(\rho(\nu_n,\nu_{n+1})) = 1.$$

Thus,

$$\lim_{n\to\infty} \rho(\nu_n,\nu_{n+1}) = 0 \ \ by \ \ \Phi_2.$$

In view of Φ_3, there are $f \in (0,1)$ and $g \in (0,\infty]$, so that,

$$\lim_{n\to\infty} \frac{\Delta(\rho(\nu_n,\nu_{n+1})) - 1}{(\rho(\nu_n,\nu_{n+1}))^f} = g.$$

Case 1
Let $g < \infty$ and $\frac{g}{2} = C > 0$, hence, there is $n_0 \in \mathbb{N}$ so that for all $n > n_0$

$$|\frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{(\rho(\nu_n, \nu_{n+1}))^f} - g| \leq C.$$

$$\frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{[\rho(\nu_n, \nu_{n+1})]^f} \geq g - C = C.$$

$$\Delta(\rho(\nu_n, \nu_{n+1})) - 1 \geq C[\rho(\nu_n, \nu_{n+1})]^f.$$

$$\frac{1}{C}[\Delta(\rho(\nu_n, \nu_{n+1})) - 1] \geq [\rho(\nu_n, \nu_{n+1})]^f.$$

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[\Delta(\rho(\nu_n, \nu_{n+1})) - 1].$$

where $D = \frac{1}{C}$.
Case 2
Suppose $g = \infty$ and $C > 0$ be a real, there is $n_0 \in \mathbb{N}$ so that,

$$C \leq \frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{[\rho(\nu_n, \nu_{n+1})]^f},$$

for all $n > n_0$. This implies that,

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[\Delta(\rho(\nu_n, \nu_{n+1})) - 1].$$

Now, we have,

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[(\Delta(\rho(\nu_0, \nu_1)))^{P^n} - 1].$$

As $n \to \infty$, the above inequality yields that,

$$\lim_{n\to\infty} n[\rho(\nu_n, \nu_{n+1})]^f = 0.$$

Hence, there is an integers n_1 so that for all $n > n_1$,

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq 1,$$

$$\rho(\nu_n, \nu_{n+1}) \leq \frac{1}{n^{\frac{1}{f}}},$$

for all $n > n_1$. Now to show $\{\nu_n\}$ is a cauchy sequence. Suppose $m, n \in \mathbb{N}$ such that $m > n > n_1$. we have,

$$\begin{aligned}\rho(\nu_n, \nu_m) &\leq \rho(\nu_n, \nu_{n+1}) + \rho(\nu_{n+1}, \nu_{n+2}) + ... + \rho(\nu_{m-1}, \nu_m)\\ &\leq \frac{1}{n^{\frac{1}{f}}} + \frac{1}{(n+1)^{\frac{1}{f}}} + \frac{1}{(n+2)^{\frac{1}{f}}} + ... + \frac{1}{(m-1)^{\frac{1}{f}}}\\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{f}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{f}}}.\end{aligned}$$

Since $0 < f < 1$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{f}}}$ is converage. When $n, m \to \infty$, we obtain $\rho(\nu_n, \nu_m) \to 0$. Hence $\{\nu_n\}$ is a cauchy sequence.
Since Π is complete then there exist an element say $z \in \Pi$ such that $\nu_n \to z$ as $n \to \infty$.

$$\lim_{n\to\infty} \nu_n = z.$$

Now,

$$\Delta(\rho(\nu_{n+1}, [Zz]_{\alpha_{Zz}})) \leq \Delta(H([Z\nu_n]_{\alpha_{Z\nu_n}}, [Zz]_{\alpha_{Zz}})) \leq [\Delta(\rho(\nu_n, z))]^P,$$

in limiting case, $\rho(z, [Zz]_{\alpha_{Zz}}) \to 0$, this implies that $z \in [Zz]_{\alpha_{Zz}}$. □

Corollary 3.1 *Let (Π, ρ) be complete metric space and $Z : \Pi \to F(\Pi)$ be fuzzy mapping. Suppose that for each $s \in \Pi$ there exist $\alpha_{Zs} \in (0,1]$ such that $[Zs]_{\alpha_{Zs}} \in CB(\Pi)$ and assume $\Delta \in \Phi$ and $K \in (0,1)$ such that $H([Zs]_{\alpha_{Zs}}, [Zr]_{\alpha_{Zr}}) \neq 0$ this implies that*

$$\Delta(\rho(s, [Zr]_{\alpha_{Zr}})) \leq \Delta(H([Zs]_{\alpha_{Zs}}, [Zr]_{\alpha_{Zr}})) \leq [\Delta(\rho(s,r))]^K.$$

Then Z has a fixed point.

Proof: *Observe that the Heilpern fixed point theorem follow from corollary 3.1. If Z is a Heilpern contraction there exist $\beta \in (0,1)$ such that*

$$H([Zs]_{\alpha_{Zs}}, [Zr]_{\alpha_{Zr}}) \leq \rho(s,r) \quad for \;\; all \;\; s,r \in \Pi$$

then we have

$$e^{H(Zs,Zr)} \leq [e^{\rho(s,r)}]^K \quad for \;\; all \;\; s,r \in \Pi$$

Clearly the function $\Delta : (0,\infty) \to (1,\infty)$ defined by $\Delta(t) = e^{\sqrt{t}}$ belongs to Φ. So the existence of fixed point follow from corollary 3.1. □

4. Application

Here, we make a new contribution by developing a θ contraction-based fixed point theorem that applies to limited and closed subsets in a Hausdorff metric spaces. In this context, we have generalized the concept of θ-contractions to set-valued mappings, which broadens the theoretical foundation and extends the applicability of the result to more complex mathematical settings.

Theorem 4.1 *Let (Π, ρ) be a generalized metric space and $Z : \Pi \to CB(\Pi)$ be a multi-valued mapping. Suppose that there exist $\Delta \in \Phi$ and $P \in (0,1)$ such that for all $s, r \in \Pi$,*

$$H(Zs, Zr) \neq 0 \Rightarrow \Delta(H(Zs,Zr)) \leq [\Delta(\rho(s,r))]^P.$$

Then Z has fixed point.

Proof: Let $\nu_0 \in \Pi$, then $Z\nu_0 \in CB(\Pi)$. Since $Z\nu_0 \neq \phi$, so there is $\nu_1 \in Z\nu_0$. Let $\nu_1 \in \Pi$, then $Z\nu_1 \in CB(\Pi)$. Since $Z\nu_1 \neq \phi$, so there is $\nu_2 \in Z\nu_1$.
Assume $H(Z\nu_0, Z\nu_1) > 0$, by using Lemma 2.1

$$\Delta(\rho(\nu_1, Z\nu_1)) \leq \Delta(H(Z\nu_0, Z\nu_1)) \leq [\Delta(\rho(\nu_0,\nu_1))]^P, \tag{4.1}$$

from Φ_4, we have

$$\Delta(\rho(\nu_1, Z\nu_1)) = inf_{\nu_2 \in Z\nu_1} \Delta(\rho(\nu_1,\nu_2)), \tag{4.2}$$

by (4.1), (4.2)

$$\inf_{\nu_2 \in Z\nu_1} \Delta(\rho(\nu_1,\nu_2)) \leq [\Delta(\rho(\nu_0,\nu_1))]^P$$

$$\Delta(\rho(\nu_1,\nu_2)) \leq [\Delta(\rho(\nu_0,\nu_1))]^P.$$

Suppose $H(Z\nu_1, Z\nu_2) > 0$, then by using Lemma 2.1

$$\Delta(\rho(\nu_2, Z\nu_2)) \leq (H(Z\nu_1, Z\nu_2)) \leq [\Delta(\rho(\nu_1,\nu_2))]^P, \tag{4.3}$$

from Φ_4, we have

$$\Delta(\rho(\nu_2, Z\nu_2)) = inf_{\nu_3 \in Z\nu_2} \rho(\nu_2,\nu_3). \tag{4.4}$$

By (4.3), (4.4)

$$\inf_{\nu_3 \in Z\nu_2} \Delta(\rho(\nu_2,\nu_3)) \leq [\Delta(\rho(\nu_1,\nu_2))]^P$$

$$\Delta(\rho(\nu_2,\nu_3)) \leq [\Delta(\rho(\nu_1,\nu_2))]^P.$$

Similarly, we generate a sequence $\{\nu_n\}$ in Π such that
$\nu_{n+1} \in Z\nu_n, \quad \nu_{n+2} \in Z\nu_{n+1}$.
Suppose $H(Z\nu_n, Z\nu_{n+1}) > 0$, by using Lemma 2.1

$$\Delta(\rho(\nu_{n+1}, Z\nu_{n+1})) \leq \Delta(H(Z\nu_n, Z\nu_{n+1})) \leq [\Delta(\rho(\nu_n, \nu_{n+1}))]^P, \tag{4.5}$$

from Φ_4, we have

$$\Delta(\rho(\nu_{n+1}, Z\nu_{n+1})) = \inf_{\nu_{n+2} \in Z\nu_{n+1}} \Delta(\rho(\nu_{n+1}, \nu_{n+2})). \tag{4.6}$$

By (4.5), (4.6)

$$\begin{aligned}
\inf_{\nu_{n+2} \in Z\nu_{n+1}} \Delta(\rho(\nu_{n+1}, \nu_{n+2})) &\leq [\Delta(\rho(\nu_n, \nu_{n+1}))]^P \\
\Delta(\rho(\nu_{n+1}, \nu_{n+2})) &\leq [\Delta(\rho(\nu_n, \nu_{n+1}))]^P \\
&\leq [\Delta(\rho(\nu_{n-1}, \nu_n))]^{P^2} \\
&\leq [\Delta(\rho(\nu_{n-2}, \nu_{n-1}))]^{P^3} \\
&\leq [\Delta(\rho(\nu_0, \nu_1))]^{P^n}.
\end{aligned}$$

Since $\Delta \in \Phi$, we have at the limit $n \to \infty$.

$$\lim_{n \to \infty} \Delta(\rho(\nu_n, \nu_{n+1})) = 1.$$

Thus,

$$\lim_{n \to \infty} \rho(\nu_n, \nu_{n+1}) = 0 \ \ by \ \ \Phi_2.$$

In view of Φ_3, there are $f \in (0,1)$ and $g \in (0, \infty]$, so that,

$$\lim_{n \to \infty} \frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{(\rho(\nu_n, \nu_{n+1}))^f} = g.$$

Case 1
Let $g < \infty$ and $\frac{g}{2} = C > 0$, hence there is $n_0 \in \mathbb{N}$ so that for all $n > n_0$,

$$\left|\frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{(\rho(\nu_n, \nu_{n+1}))^f} - g\right| \leq C.$$

$$\frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{[\rho(\nu_n, \nu_{n+1})]^f} \geq g - C = C.$$

$$\Delta(\rho(\nu_n, \nu_{n+1})) - 1 \geq C[\rho(\nu_n, \nu_{n+1})]^f.$$

$$\frac{1}{C}[\Delta(\rho(\nu_n, \nu_{n+1})) - 1] \geq [\rho(\nu_n, \nu_{n+1})]^f.$$

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[\Delta(\rho(\nu_n, \nu_{n+1})) - 1].$$

where $D = \frac{1}{C}$.
Case 2
Suppose $g = \infty$ and $C > 0$ be a real number, there is $n_0 \in \mathbb{N}$ so that,

$$C \leq \frac{\Delta(\rho(\nu_n, \nu_{n+1})) - 1}{[\rho(\nu_n, \nu_{n+1})]^f}, \ \ for \ \ all \ \ n > n_0$$

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[\Delta(\rho(\nu_n, \nu_{n+1})) - 1].$$

Now, we have,

$$n[\rho(\nu_n, \nu_{n+1})]^f \leq nD[(\Delta(\rho(\nu_0, \nu_1)))^{P^n} - 1].$$

As $n \to \infty$, the above inequality yields that,

$$\lim_{n\to\infty} n[\rho(\nu_n, \nu_{n+1})]^f = 0.$$

Hence, there is an integer n_1 so that for all $n > n_1$.

$$n[\rho(\nu_n, \nu_{n+1})]^f \le 1,$$
$$\rho(\nu_n, \nu_{n+1}) \le \frac{1}{n^{\frac{1}{f}}},$$

for all $n > n_1$. Now to show $\{\nu_n\}$ is a cauchy sequence. Suppose $m, n \in \mathbb{N}$ such that $m > n > n_1$, we have,

$$\begin{aligned}\rho(\nu_n, \nu_m) &\le \rho(\nu_n, \nu_{n+1}) + \rho(\nu_{n+1}, \nu_{n+2}) + ... + \rho(\nu_{m-1}, \nu_m)\\ &\le \frac{1}{n^{\frac{1}{f}}} + \frac{1}{(n+1)^{\frac{1}{f}}} + \frac{1}{(n+2)^{\frac{1}{f}}} + ... + \frac{1}{(m-1)^{\frac{1}{f}}}\\ &\le \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{f}}} \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{f}}}.\end{aligned}$$

Since $0 < f < 1$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{f}}}$ is converage. When $n, m \to \infty$, we obtain $\rho(\nu_n, \nu_m) \to 0$. Hence $\{\nu_n\}$ is a cauchy sequence.
Since Π is complete then there exist an element say $v \in \Pi$ such that $\nu_n \to v$ as $n \to \infty$.

$$\lim_{n\to\infty} \nu_n = v.$$

Now,

$$\Delta(\rho(\nu_{n+1}, Zv)) \le \Delta(H(Z\nu_n, Zv)) \le [\Delta(\rho(\nu_n, v))]^P,$$

in limiting case, $\rho(v, Zv) \to 0$ this implies that $v \in Zv$. □

Example 4.1 *Let $\Pi = \eta_w, w \in \mathbb{N}$ where $\eta_w = \frac{w(w+1)}{2}$ for all $w \in \mathbb{N}$. we endow Π with metric ρ given by $\rho(s, r) = |s - r|$ for all $s, r \in \Pi$. Easily to show that (Π, ρ) is complete metric space. Let $Z : \Pi \to CB(\Pi)$ be multi-valued mappin defined as*

$$Z\eta_w = [0, \frac{w(w+1)}{2}] \ \ for \ \ all \ \ w \ge 2.$$

we can easily check that

$$\lim_{w\to\infty} \frac{H(Z\eta_w, Z\eta_1)}{\rho(\eta_w, \eta_1)} = \lim_{w\to\infty} \frac{H([0,3],[0,1])}{\rho(3,1)} = 1.$$

Now consider a mapping $\Phi : (0, \infty) \to (1, \infty)$ defined by

$$\Delta(t) = e^{\sqrt{te^t}}.$$

it easily to show $\Delta \in \Phi$. Now to prove Z satisfy the condition 2.1, such that

$$H(Z\eta_w, Z\eta_h) \ne 0 \Rightarrow e^{\sqrt{H(Z\eta_w, Z\eta_h)e^{H(Z\eta_w, Z\eta_h)}}} \le e^{y\sqrt{\rho(\eta_w,\eta_h)e^{\rho(\eta_w,\eta_h)}}}$$

for some $y \in (0, 1)$. The above condition is equivalent to

$$H(Z\eta_w, Z\eta_h) \ne 0 \Rightarrow H(Z\eta_w, Z\eta_h)e^{H(Z\eta_w, Z\eta_h)} \le y^2 \rho(\eta_w, \eta_h)e^{\rho(\eta_w,\eta_h)} \tag{4.7}$$

for some $y \in (0, 1)$. Consider two Possibilities
Case 1:
$w = 1$, $h > 2$ we have

$$\frac{H(Z\eta_w, Z\eta_h)e^{H(Z\eta_w, Z\eta_h)}}{\rho(\eta_w, \eta_h)e^{\rho(\eta_w,\eta_h)}} = \frac{H([0,1],[0,\frac{h^2+h}{2}])e^{H([0,1],[0,\frac{h^2+h}{2}]) - \rho(1,\frac{h^2+h}{2})}}{\rho(1, \frac{h^2+h}{2})} \le e^{-1}.$$

Case 2:
$h > w > 1$ we have

$$\frac{H(Z\eta_w, Z\eta_h)e^{H(Z\eta_w,Z\eta_h)}}{\rho(\eta_w,\eta_h)e^{\rho(\eta_w,\eta_h)}} = \frac{H([0,\frac{w^2+w}{2}],[0,\frac{h^2+h}{2}])e^{H([0,\frac{w^2+w}{2}],[0,\frac{h^2+h}{2}])-\rho(\frac{w^2+w}{2},\frac{h^2+h}{2})}}{\rho(\frac{w^2+w}{2},\frac{h^2+h}{2})} \leq e^{-1}.$$

Thus, the equation 4.7 is satisfied with $k = e^{\frac{-1}{2}}$. Then theorem 4.1 implies Z has fixed point.

5. Conclusion

This paper explores a fixed point results for fuzzy mapping defined on generalized metric spaces, using a θ-contraction approach. The theorem offers a broader perspective on classical concepts and is particularly relevant in settings where uncertainty and vagueness are part of the problem structure.Fixed point techniques have shown to be very successful and appealing, with applications ranging from functional inclusions to optimization theory, fractal design, discrete dynamics involving set-valued operators, and a variety of other fields in nonlinear functional analysis. We validate these theorems through examples, adding to the existing knowledge in this area. Additionally, we highlight the practical applications of these concepts, thereby enhancing the reliability and usefulness of our research findings.The existence results presented in this study offer a robust framework for approximating a wider range of operator equations, thereby contributing meaningfully to the evolving landscape of applied scientific research. These findings are poised to inform and guide future scholarly efforts.

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