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Induced Neutrosophic Topologies from Neutrosophic Graph Coloring: A Rigorous Approach to Approximation Under Indeterminacy

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ABSTRACT: The neutrosophic coloring of neutrosophic graphs is the source of a new class of topological structure that is introduced and studied in this paper: neutrosophic topologies. Neutosophic coloring, which builds on classical and intuitionistic graph colorings, gives each vertex a triple value: truth, indeterminacy, and falsity. This provides a more accurate representation of uncertainty. The authors use this enriched coloring model to construct a set of vertex subsets that use neutrosophic set operations to meet a topology's axioms. Neutronosophic color topological spaces are thus created. Two significant approximation operators—the neutrosophic lower and higher color approximations—are defined within this paradigm. Rough set theory serves as an inspiration for these, but neutrosophic logic enhances them. Important topological properties based on color-derived neighborhood systems, including monotonicity, idempotency, and boundary construction, are demonstrated for the operators. In addition, the relationship between the lower and upper approximations is discussed, and applications in fields where conventional binary or fuzzy techniques are inadequate, such as information systems and uncertain network structures, are presented. With implications for future studies in neutrosophic mathematics, granular computing, and uncertain data analysis, the study establishes a theoretical framework for unified reasoning in graph theory and topology under indeterminacy.

Key Words: Truth - membership, indeterminacy - membership, falsity - membership, neutrosophic graphs, neutrosophic coloring, neutrosophic lower approximation, neutrosophic upper approximation.

Contents

1. Introduction

Neutosophic sets, fuzzy sets, and rough sets are some of the frameworks that have been developed in response to the difficulty of handling ambiguity and uncertainty in complex systems [12,13,14]. These earlier models are generalized by neutrosophic sets, which were introduced by Smarandache [10]. They incorporate three independent membership functions: one for truth, one for indeterminacy, and one for falsehood. With applications ranging from image processing to decision-making, this triadic structure is especially helpful for modeling inconsistent or partial information. Researchers created neutrosophic graphs, [5,6,7], which use neutrosophic edges and vertices to depict relationship ambiguity, by fusing neutrosophic logic with graph theory. The features of these graphs, such as centrality, connectedness, and coloring, have been thoroughly examined. By giving vertex colors neutrosophic values, numerous writers [1,8] expand on classical graph coloring and enable more adaptable representations of confusing groupings. The specification of lower and higher approximations in a neutrosophic setting is made

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possible by this coloring technique, which is in line with rough set theory [9]. Neutosophic sets offer a more sophisticated method by explicitly representing indeterminacy, which is ubiquitous in real-world social data [2,3,11]. Rough sets and fuzzy sets have been used for approximate community discovery [4]. Compared to binary or fuzzy sets, neutrosophic sets provide a more sophisticated modeling tool in social network analysis, where linkages are unclear and data is frequently inadequate. They can help with tasks like anomaly detection and targeted engagement by identifying members of the core and periphery communities. Strong neutrosophic graphs, in which the vertices and edges exhibit different levels of indeterminacy, have even been introduced in recent work. By interpreting color-based clusters as generalized open sets, neutrosophic coloring yields a topology in such graphs. The lower and higher approximations, which were taken and modified from rough set theory, aid in capturing this uncertainty in an organized manner. These open sets represent the uncertainty present in the vertex attributes. Elements that unquestionably belong to a set are included in lower approximations, whilst elements that might belong to a set are included in upper approximations. More accurate classifications within ambiguous data are made possible by this method. Motivated by these concepts, the work analyzes social networks with indeterminate relationships by proposing a novel neutrosophic topology generated via graph coloring.

2. Preliminaries

This section reviews the key concepts needed to understand how neutrosophic topologies can be generated through neutrosophic coloring on the vertex sets of neutrosophic graphs. It includes the foundational ideas of the work.

Definition 2.1. *Let H be a universal set. A neutrosophic set N in H is defined by three membership functions: $T_N(y),\ I_N(y),\ F_N(y) : H \to [0,1]$, where for each element $y \in H$:*

- $T_N(y)$: *truth-membership degree,*
- $I_N(y)$: *indeterminacy-membership degree,*
- $F_N(y)$: *falsity-membership degree.*

These degrees are mutually independent and $0 \leq T_A(y) + I_A(y) + F_A(y) \leq 3$.

In the context of neutrosophic sets, a threshold is a boundary value used to filter, classify, or make decisions based on the truth (T), indeterminacy (I), and falsity (F) membership degrees of elements.

Definition 2.2. *In a neutrosophic set, a threshold refers to a predefined value (or set of values) applied to $T(x), I(x)$, and $F(x)$ to assist in inclusion, exclusion, or classification decisions. Thresholds help to filter elements based on conditions like:*

- *single values (as $T(x) \geq 0.7$) for filtering truth*
- *combined condition (as $T(x) \geq 0.6$ and $F(x) \leq 0.3$ and $I(x) \leq 0.2$)*
- *Context-specific dynamic thresholds*

Purpose of Thresholds in Neutrosophic Sets:

- Selecting elements that meet certain truth/falsity/indeterminacy conditions.
- Minimizing uncertainty by filtering highly indeterminate elements.
- Supporting decision-making and classification in practical applications like analysis of images and diagnostics of diseases.

Definition 2.3. *A neutrosophic graph is a graph where the edges are modeled as neutrosophic sets, that is, edges have three components: truth, falsehood, and indeterminacy. Each edge e between two vertices v_i and v_j is associated with a neutrosophic set $N(e) = \langle T(e), F(e), I(e)\rangle$, where:*

- $T(e)$ *gives the degree of edge existence,*
- $F(e)$ *shows the degree of edge non-existence,*
- $I(e)$ *captures the level of uncertainty about the edge.*

A neutrosophic topology generalizes classical topology by incorporating uncertainty and vagueness. Instead of having crisp open sets, the topology uses neutrosophic sets where each open set is characterized by its truth, falsity, and indeterminacy degrees.

Definition 2.4. *A collection $\mathcal{T}$ of neutrosophic sets forms a neutrosophic topology if:*

- $\emptyset$ *and the universal set X are in $\mathcal{T}$,*
- *The intersection of any finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$,*
- *The union of any number of sets in $\mathcal{T}$ is in $\mathcal{T}$.*

Each open set is now represented by a neutrosophic set, where the truth, falsehood, and indeterminacy values reflect the uncertainty about the openness of a set.

3. Topology Induced by Neutrosophic Coloring of Neutrosophic Graphs

Classical graph theory is extended to include uncertainty, indeterminacy, and inconsistency through the idea of neutrosophic coloring for the vertices of neutrosophic graphs. Adjacent vertices with high truth values should be given distinct colors in neutrosophic coloring. Color conflicts could be acceptable if there is a great deal of ambiguity in the relationship between vertices.
This model is particularly relevant to systems with uncertain or imprecise data, such as social or biological networks.

Definition 3.1. *Let $G = (V, E)$ be a neutrosophic graph, where each vertex $v \in V$ and edge $e \in E$ is associated with a neutrosophic value: $(T, I, F) \in [0, 1]^3$, representing truth-membership, indeterminacy-membership, and falsity-membership, respectively. A neutrosophic coloring is a mapping: $c : V \to \mathcal{C}$ where $\mathcal{C}$ is a set of colors, subject to certain neutrosophic conditions.*

Example 3.1. *Let us take a neutrosophic graph in which vertices are labelled a, b, c, d, e as in Fig.1.*

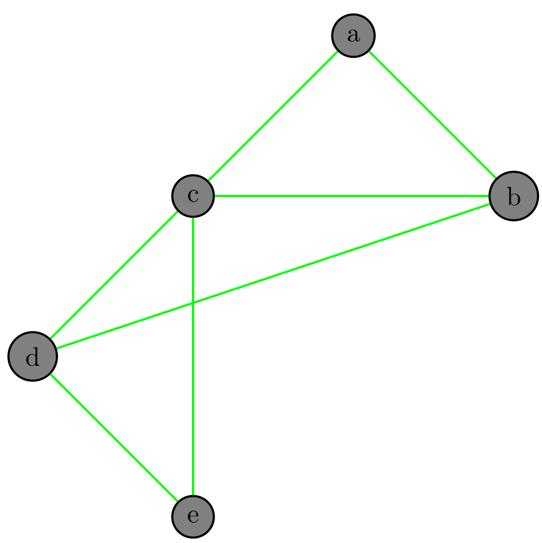

Figure 1:

The neutrosophic edge membership values (T, I, F) are given in the Table 1:

Edge	(T, I, F)
$a \leftrightarrow b$	(0.92, 0.12, 0.0)
$a \leftrightarrow c$	(0.72, 0.22, 0.12)
$b \leftrightarrow c$	(0.22, 0.62, 0.22)
$b \leftrightarrow d$	(0.86, 0.12, 0.05)
$c \leftrightarrow d$	(0.52, 0.52, 0.22)
$c \leftrightarrow e$	(0.12 0.81, 0.12)
$d \leftrightarrow e$	(0.96, 0.06, 0.0)

Table 1

Neutrosophic coloring Rule:
A threshold $T_{th} = 0.7$ is used:

- If $T \geq 0.7$: the edge is strong, and the connected vertices must have different colors
- If $T < 0.7$, and $I \geqslant 0.5$: color conflict is acceptable.

Step-by-Step Coloring:
Begin with vertex labeled a:
Color a = Red
The edge from a to b has $T = 0.92 \longrightarrow$ strong link
$\Rightarrow$ color b $\neq$ Red $\Longrightarrow$ color b = Blue
Vertex c:
$a \rightarrow c$ has $T = 0.72 \longrightarrow$ strong connection $\Rightarrow c \neq$ Red
$b \rightarrow c$ has $T = 0.22, I = 0.62 \longrightarrow$ weak and indeterminate $\Rightarrow c$ can be same as b.
Since $a \neq c$ is needed, and b = Blue, Color c = Blue (allowed through indeterminacy with b)
Vertex d:
$b \rightarrow d$ has $T = 0.86 \Rightarrow d \neq$ Blue
$c \rightarrow d$ has $T = 0.32, I = 0.52$. Hence conflict with Blue may be tolerated
So, Color d = Red
Vertex e:
$d \rightarrow e$ has $T = 0.96 \Rightarrow e \neq$ Red
$c \rightarrow e$ has $T = 0.12, I = 0.82$. Hence conflict with Blue is tolerable
So, Color e = Blue
Interpretation:

- Strong connections ($T \geqslant 0.7$) : vertices are colored differently.
- Weak connections and high indeterminacy: coloring conflicts are allowed.

This coloring uses 2 colors: Red and Blue, despite indeterminate links.

Definition 3.2. *A coloring is neutrosophically proper if high T implies different colors for adjacent vertices, but high I may allow same-color assignment.*

Definition 3.3. *The neutrosophic chromatic number, denoted $\chi_N(G)$ is the minimal number of colors wanted under neutrosophic coloring rules.*

Observation: 1. Neutrosophic coloring allows color conflicts between vertices that are adjacent if the connecting edge has: Low truth (T) value; High indeterminacy (I) value.
This rule makes the coloring suitable for modeling incomplete or uncertain systems.
2. Threshold values $T_{th} I_{th}$, and F_{th} can be defined based on application-specific needs. The coloring depends on these thresholds:

- Higher T_{th}: more constraints → more colors needed.

- Higher I_{th}: more flexibility → fewer colors may suffice.

Theorem 3.1. *Given a neutrosophic graph $G = (V, E)$, where each edge e_{ij} has a neutrosophic value (T_{ij}, I_{ij}, F_{ij}). If there is a threshold $T_{th} \in [0, 1]$ such that:*
Any two adjacent vertices (v_i, v_j) with $T_{ij} \geq T_{th}$ must be colored differently, and vertices connected by edges with $T_{ij} < T_{th}$ and $I_{ij} \geq I_{th}$ are allowed to share the same color, then the neutrosophic chromatic number $\chi_N(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of the underlying graph.

Proof: Let $G_c = (V, E_c)$ be the underlying graph of G, where $E_c = \{e_{ij} \in E \mid T_{ij} \geq T_{\text{th}}\}$. This edge set forms a subgraph where only edges with high certainty are retained, ignoring indeterminate or low-certainty connections. Now, consider a classical proper vertex coloring $c : V \to \mathcal{C}$ on G_c using $\chi(G_c) \leq \chi(G)$ colors, with the condition that:
If $(v_i, v_j) \in E_c$, then $c(v_i) \neq c(v_j)$.
This coloring fulfills the neutrosophic coloring constraints because:
• All strong edges $(T_{ij} \geq T_{\text{th}})$ are in E_c, and those vertices are assigned different colors.
• For edges not in E_c, color conflicts are allowed if $I_{ij} \geq I_{\text{th}}$, as per the neutrosophic coloring rule.
Hence, the coloring of G_c is also a valid neutrosophic coloring of G, and $\chi_N(G) \leq \chi(G_c) \leq \chi(G)$. Thus, the desired result is established. □

Corollary 3.1. *Let $G = (V, E)$ be a neutrosophic graph, and $\chi_N(G)$ its neutrosophic chromatic number with respect to a fixed truth threshold $T_{th} \in [0, 1]$ and indeterminacy threshold $I_{th} \in [0, 1]$. If the average indeterminacy across edges in G, defined as: $\bar{I} = \frac{1}{|E|} \sum_{e_{ij} \in E} I_{ij}$ increases, then the neutrosophic chromatic number $\chi_N(G)$ tends to decrease or remain the same.*

Proof: According to Theorem 3.1, vertices connected by edges with $T_{ij} < T_{\text{th}}$ and $I_{ij} \geq I_{\text{th}}$ can share the same color. Now, assume that the level of indeterminacy $\bar{I}$ increases, which means a greater number of edges are considered indeterminate. As a result, more edge pairs will be permitted to share the same color. This reduces the number of strict coloring restrictions those that require different colors. Consequently, the minimum number of distinct colors required under neutrosophic coloring constraints either decreases or remains unchanged. Hence as $\bar{I}$ rises, the neutrosophic chromatic number $\chi_N(G)$ decreases or stays constant. This completes the proof. □

Theorem 3.2. *Let $G = (V, E)$ be a neutrosophic graph. A neutrosophic coloring defines a relation $R \subseteq V \times V$ such that $(v_i, v_j) \in R \iff c(v_i) = c(v_j)$, meaning two vertices are related under R if and only if they are assigned the same color in a valid neutrosophic coloring. Then, R is an equivalence relation on V.*

Proof: Reflexivity: For every $v_i \in V$, $c(v_i) = c(v_i) \Rightarrow (v_i, v_i) \in R$ holds true. So R is reflexive.
Symmetry: If $(v_i, v_j) \in R$, then $c(v_i) = c(v_j)$. This implies $c(v_j) = c(v_i)$, so $(v_j, v_i) \in R$ holds true. So R is symmetric.
Transitivity: If $(v_i, v_j) \in R$ and $(v_j, v_k) \in R$, then $c(v_i) = c(v_j)$ and $c(v_j) = c(v_k)$. Hence $c(v_i) = c(v_k) \Rightarrow (v_i, v_k) \in R$ holds true. So R is transitive. □

Note 1. • *Each color class in a neutrosophic coloring corresponds to an equivalence class under R.*
• *These color classes partitions the vertex set V.*

Construction of Topology:
Let $G = (V, E)$ be a neutrosophic graph and $c : V \to C$, be a valid neutrosophic coloring assigning a color to each vertex. This coloring-induced an equivalence relation R where $v_i \sim v_j$ iff $c(v_i) = c(v_j)$.
Step 1: Define Basic Open Sets
Let each color class (i.e., pre-image of each color under c) be a basic open set: $U_k = \{v \in V \mid c(v) =$

$k\}, \quad \forall \quad k \in C$. These U_k are crisp open sets induced by the coloring.
Step 2: Define Neutrosophic Membership of Each Set
For each U_k, define a neutrosophic membership function $\mu_k(v) = (T_k(v), I_k(v), F_k(v))$, where:
• $T_k(v)$: degree to which vertex v certainly belongs to U_k
• $I_k(v)$: measure of indeterminacy about membership
• $F_k(v)$: degree to which v certainly does not belong to U_k
These values can be estimated based on:
• The indeterminacy of edges between v and other vertices in U_k
• The strength of the coloring constraint between v and U_k.
Step 3: Form the Topology $\mathcal{T}_N$
Define: $\mathcal{T}_N = \{U \subseteq V \mid \exists$ color classes $U_{k_1}, \dots, U_{k_r}$ such that $U = \bigcup_{i=1}^{r} U_{k_i}\}$. Each open set in $\mathcal{T}_N$ has a neutrosophic structure, i.e., each element $v \in U$ has a membership triple (T, I, F).
In summary, applying neutrosophic coloring to the vertices of a neutrosophic graph leads to a corresponding neutrosophic topology. This topology consists of open sets that encapsulate varying degrees of truth, indeterminacy, and falsity reflecting the inherent uncertainty present in the graph's structure.

Example 3.2. *We begin with a neutrosophic graph with vertex set* $V = \{1, 2, 3, 4, 5\}$ *as in Fig.2.*

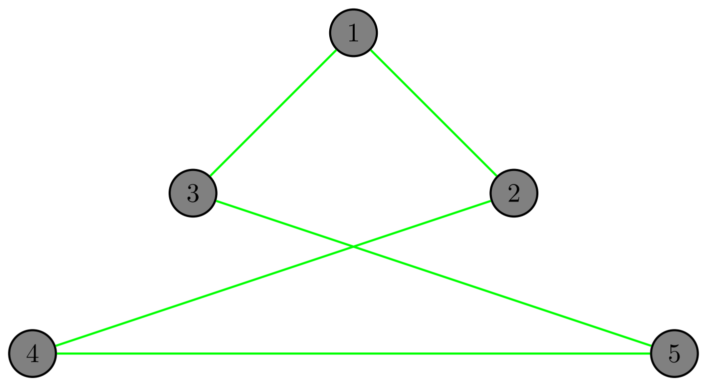

Figure 2:

The neutrosophic edge weights (T, I, F) *between connected vertices are given in the Table 2:*

Edge	*(T, I, F)*
$1 \leftrightarrow 2$	*(0.91, 0.11, 0.0)*
$1 \leftrightarrow 3$	*(0.61, 0.31, 0.11)*
$2 \leftrightarrow 4$	*(0.41, 0.51, 0.11)*
$3 \leftrightarrow 5$	*(0.21, 0.71, 0.11)*
$4 \leftrightarrow 5$	*(0.95, 0.05, 0.0)*

Table 2

Use a threshold: $T_{th} = 0.7$
Coloring based on constraints:

Vertex	*Color*
1	*Red*
2	*Blue*
3	*Red*
4	*Blue*
5	*Blue*

Table 3

Color Classes (Basic Open Sets):
• $U_{Red} = \{1, 3\}$ • $U_{Blue} = \{2, 4, 5\}$
Define Neutrosophic Membership for Each Open Set
We assign neutrosophic membership triples (T, I, F) based on how confidently a vertex belongs to the color class.
Red Open Set: U_{Red}

Vertex	T	I	F	Notes
1	1	0	0	Clear Member
2	1	0	0	Clear Member
3	0	0.11	0.91	Connected to 1 with high T
4	0	0.21	0.81	Indirect connection via 2
5	0	0.3	0.7	Weak link via 3

Table 4

Blue Open Set: U_{Blue}

Vertex	T	I	F	Notes
1	1	0	0	Clear member
2	1	0	0	Clear member
3	1	0	0	Clear member
4	0	0.11	0.91	Connected to 2 with high T
5	0	0.31	0.71	Weak link to 5 (low T)

Table 5

Neutrosophic Topology $\mathcal{T}_N$
Now define the neutrosophic topology: $\mathcal{T}_N = \{\emptyset, V, U_{Red}, U_{Blue}, U_{Red} \cup U_{Blue}, \dots\}$. Each set is now associated with a neutrosophic membership table as above.

Note 2. • *The color classes form the base of the topology.*
• *Each vertex has a graded membership (T, I, F) to each open set.*
• *The topology is soft and flexible, adapting to edge uncertainties.*

4. Lower and Upper Approximations in Neutrosophic Graph Topology

Definition 4.1. *Let $G = (V, E)$ be a neutrosophic graph. Let $\mathcal{T}_N$ be the neutrosophic topology on V induced by a neutrosophic coloring $c : V \to C$. Each neutrosophic open set $U \in \mathcal{T}_N$ has an associated neutrosophic membership function $\mu_U : V \to [0,1]^3$, $\mu_U(v) = (T_U(v), I_U(v), F_U(v))$, where $T_U(v), I_U(v), F_U(v)$ denote truth, indeterminacy, and falsity degrees of membership of v in U. Consider a subset $A \subseteq V$, represented as a neutrosophic set with membership $\mu_A(v) = (T_A(v), I_A(v), F_A(v))$. The neutrosophic lower approximation of A, $\underline{A}$, in the neutrosophic topology $\mathcal{T}_N$ is the largest neutrosophic open set contained in A. Formally, the membership values of $\underline{A}$ is: $\underline{T}_A(v) = \sup\{T_U(v) \mid U \in \mathcal{T}_N,\ U \subseteq A\}, \underline{I}_A(v) = \inf\{I_U(v) \mid U \in \mathcal{T}_N,\ U \subseteq A\}, \underline{F}_A(v) = \inf\{F_U(v) \mid U \in \mathcal{T}_N,\ U \subseteq A\}$. Intuitively, $\underline{A}$ identifies vertices that certainly belong to A (high truth, low indeterminacy and falsity) based on the neutrosophic open sets inside A.*
The neutrosophic upper approximation of $A, \overline{A}$, is the smallest neutrosophic closed set containing A, or equivalently, the neutrosophic closure of A. Its membership is defined by: $\overline{T}_A(v) = \inf\{T_F(v) \mid F \text{ closed},\ A \subseteq F\}, \overline{I}_A(v) = \sup\{I_F(v) \mid F \text{ closed},\ A \subseteq F\}, \overline{F}_A(v) = \sup\{F_F(v) \mid F \text{ closed},\ A \subseteq F\}$.

Remark 4.1. • *These approximations capture uncertainty and partial membership using neutrosophic values, extending traditional rough set theory into the neutrosophic context.*
• *Specifically, the lower approximation identifies vertices definitely in the set, while the upper approximation includes vertices that possibly belong to the set based on the topology.*

Example 4.1. *From the earlier example:*

Color	*Vertices*	*Membership triples* $\mu_{color}(v) = (T, I, F)$
Red	$\{1, 3\}$	$1 : (1, 0, 0)$, $3 : (1, 0, 0)$, $2 : (0, 0.11, 0.91)$, $4 : (0, 0.21, 0.81)$, $5 : (0, 0.31, 0.71)$
Blue	$\{2, 4, 5$	$2 : (1, 0, 0)$, $4 : (1, 0, 0)$, $5 : (1, 0, 0)$, $1 : (0, 0.11, 0.91)$, $3 : (0, 0.31, 0.71)$

Table 6

Let $A = \{1, 4\}$. *We look at vertices adjacent or linked by edges with indeterminacy or truth to vertices in* A *with their memberships extended.*
3 *connects to* 1 *with edge weight* $(0.61, 0.31, 0.11)$ *— moderate truth and indeterminacy.*
5 *connects to* 4 *with edge weight* $(0.95, 0.05, 0)$ *— high truth, low indeterminacy.*
Hence, extend memberships for $3, 5$*:*

Vertex	$T_{\overline{A}}$	$I_{\overline{A}}$	$F_{\overline{A}}$	*Reason*
1	*1*	*0*	*0*	$1 \in A$
2	*1*	*0*	*0*	$2 \in A$
3	*0.61*	*0.31*	*0.11*	*linked to* $1 \in A$
4	*1*	*0*	*0*	$4 \in A$
5	*0.95*	*0.05*	*0*	*linked to* $4 \in A$

Table 7

5. Properties and Structural Analysis

Theorem 5.1. *(Monotonicity) If* $A \subseteq B$, *then for every* $v \in V, T_{\underline{A}}(v) \leq T_{\underline{B}}(v), \quad I_{\underline{A}}(v) \geq I_{\underline{B}}(v), \quad F_{\underline{A}}(v) \geq F_{\underline{B}}(v)$, $T_{\overline{A}}(v) \leq T_{\overline{B}}(v), \quad I_{\overline{A}}(v) \geq I_{\overline{B}}(v), \quad F_{\overline{A}}(v) \geq F_{\overline{B}}(v)$.

Proof: As $A \subseteq B$, every neutrosophic open set U that is contained in A is also contained in B. Hence, the collection of open sets in A is a subset of those in B. Therefore, for each v, $\sup_{U \subseteq A} T_U(v) \leq \sup_{U \subseteq B} T_U(v), \inf_{U \subseteq A} I_U(v) \geq \inf_{U \subseteq B} I_U(v), \inf_{U \subseteq A} F_U(v) \geq \inf_{U \subseteq B} F_U(v)$. Similarly, for upper approximations, since every closed set containing B also contains A, the closed sets for B form a subset of those for A. Hence: $\inf_{F \supseteq A} T_F(v) \leq \inf_{F \supseteq B} T_F(v), \sup_{F \supseteq A} I_F(v) \geq \sup_{F \supseteq B} I_F(v), \sup_{F \supseteq A} F_F(v) \geq \sup_{F \supseteq B} F_F(v)$. □

Theorem 5.2. *(Idempotency)* $\underline{\underline{A}} = \underline{A}, \quad \overline{\overline{A}} = \overline{A}$.

Proof: The neutrosophic lower approximation $\underline{A}$ is the largest neutrosophic open set fully contained in A, and it is itself an open set within A. So, applying the neutrosophic lower approximation again does not change it. Formally, for all v, $T_{\underline{\underline{A}}}(v) = \sup_{U \subseteq \underline{A}} T_U(v) = T_{\underline{A}}(v)$, similarly for I and F. The same logic holds for the upper approximation: since $\overline{A}$ is already the smallest closed set containing A, applying the closure operator again leaves it unchanged. □

Theorem 5.3. *(Duality)* $\underline{A}^c = \overline{A^c}, \quad \overline{A}^c = \underline{A^c}$. *Here, the neutrosophic complement of* A *has membership* $T_{A^c}(v) = F_A(v), \quad I_{A^c}(v) = I_A(v), \quad F_{A^c}(v) = T_A(v)$.

Proof: In neutrosophic topology, open sets are complements of closed sets and vice versa. Using this complementarity, $\text{int}(A)^c = \text{cl}(A^c), \quad \text{cl}(A)^c = \text{int}(A^c)$.• Hence, the lower approximation of A^c equals the complement of the upper approximation of A, and similarly for the upper approximation. □

Observation: • For the empty set $\varnothing$, $T_{\underline{\varnothing}}(v) = 0, \quad I_{\underline{\varnothing}}(v) = 1, \quad F_{\underline{\varnothing}}(v) = 1$, and similarly for upper approximation, since no open sets lie inside $\varnothing$.
• For the whole set V, $T_{\underline{V}}(v) = 1, \quad I_{\underline{V}}(v) = 0, \quad F_{\underline{V}}(v) = 0$, and $T_{\overline{V}}(v) = 1, \quad I_{\overline{V}}(v) = 0, \quad F_{\overline{V}}(v) = 0$.

These properties show how the neutrosophic lower and upper approximations respect inclusion, repetition, and complementarity extending classical rough set theory into a neutrosophic framework. They also show how uncertainty affects the boundary behavior of sets in neutrosophic topology.

6. Applications

Graphs can be used to depict social networks, with vertices representing people and edges showing their connections. However, because of things like noisy data, privacy concerns, or contradicting information, these linkages and group memberships frequently involve uncertainty or partial truth in real-world circumstances.

Neutrosophic graphs help model this ambiguity by allotting a membership triple to each vertex and edge:

• T (truth): Degree of definite inclusion or acceptance

• I (indeterminacy): Degree of uncertainty

• F (falsity): Degree of exclusion or rejection

By applying neutrosophic coloring, the vertices are divided into groups (color classes) that reflect uncertain roles or affiliations. This grouping induces a neutrosophic topology over the vertex set.

The neutrosophic lower and upper approximations of a subset (like a social community) help distinguish:

• Core members (in the lower approximation): individuals clearly belonging to the group • Possible members (in the upper approximation): individuals who might belong, considering the uncertainties

Applications:

1. Community Detection under Uncertainty

• Neutrosophic Lower Approximation identifies definite members based on strong and reliable connections.

• Neutosophic Upper Approximation captures possible members who have partial or uncertain links.

Use Case:

Focus on the neutrosophic lower approximation for forming the stable "core" of a community, and analyze the upper approximation to explore boundaries and potential outreach.

2. Influence and Information Spread

• Information propagates more effectively within the lower approximation (the core).

• The neutrosophic upper approximation includes users who may potentially influence or be influenced under uncertain conditions.

Use Case:

Target core users for assured communication impact. Use upper approximation data to anticipate indirect or fringe influence pathways.

3. Role Assignment in Uncertain Networks

In networks with vague or evolving roles (like leader, influencer, connector):

• Neutrosophic Lower Approximation helps confidently assign a role.

• Neutrosophic Upper Approximation indicates individuals who may take on or shift into that role.

Use Case:

Supports flexible and adaptive modeling of organizational or social roles, accounting for ambiguity.

4. Noise and Anomaly Detection

Users who lie outside the upper approximation of a known group may be:

• Outliers

• Anomalies

• Fake accounts

Use Case:

This approach is useful in finding abnormal behavior or isolated nodes with minimal trusted connections.

Using upper and lower approximations derived from neutrosophic coloring of social network graphs enables:

• Handling uncertainty and indeterminacy in community membership.

• More nuanced community detection than classical crisp or fuzzy methods.

• Strategic decision-making for engagement, influence, and security.

Example: Neutrosophic Approximations in a Social Network Community

Setup

• Network: 6 users (vertices) $V = \{1, 2, 3, 4, 5, 6\}$. • Goal: Identify community $C \subseteq V$.

• Each user i has neutrosophic membership w.r.t community C: $\mu_C(v_i) = (T_i, I_i, F_i)$, where

• T_i = truth membership (degree user belongs),

• I_i = indeterminacy (uncertainty),

• F_i = falsity membership (degree user does NOT belong).
Given membership values:

User	T_i	I_i	F_i
1	0.91	0.05	0.05
2	0.71	0.21	0.11
3	0.41	0.41	0.21
4	0.61	0.31	0.11
5	0.21	0.61	0.21
6	0.11	0.11	0.81

Table 8

Step 1: Define neutrosophic topology induced by coloring
Suppose neutrosophic coloring groups users roughly by T_i:
• Color 1 (Strong members): $1, 2, 4$
• Color 2 (Uncertain members): $3, 5$
• Color 3 (Non-members): 6 The neutrosophic open sets (neutrosophic communities) induced by colors are:
$U_1 = \{1, 2, 4\}, U_2 = \{v_3, v_5\}, U_3 = \{v_6\}$
Step 2: Compute Neutrosophic Lower Approximation $\underline{C}$ Assuming $C = \{1, 2, 3, 4, 5\}$ (suspected community).
• $U_1 \subseteq C$
• $U_2 \subseteq C$
• $U_3 \nsubseteq C$
But 3 and 5 have low truth memberships and high indeterminacy, so considering a threshold for "definite" membership: $T_i \geq 0.6$ for core members. Thus only $1, 2, 4$ are in the neutrosophic lower approximation. So, $\underline{C} = \{1, 2, 4\}$.
Step 3: Compute Neutrosophic Upper Approximation $\overline{C}$
Upper approximation includes vertices possibly in C, i.e., vertices connected to or belonging to any closed set containing C. Practically, includes all with $T_i > 0.1.\overline{C} = \{1, 2, 3, 4, 5\}$
Step 4: Interpretations

User	Membership Status
1	Core member (high T_i)
2	Core member
4	Core member
3	Possible member (uncertain)
5	Possible member (more uncertain)
6	Outside community

Table 9

Step 5: Applications in social network
• Core outreach: Target $1, 2, 4$ for important messaging.
• Potential growth: Engage $3, 5$ to reduce indeterminacy, possibly converting them to core.
• Ignore or monitor 6 as likely non-member or outsider. Summary

User	Suggested Strategy	Reason
1,2,4	Core Outreach	High truth membership
3	Targeted engagement	Moderate truth, moderate uncertainty
5	Monitoring and light engagement	Low truth, high uncertainty
6	No engagement, monitor	High falsity membership

Table 10

Thus, the application of lower and upper approximations on the neutrosophic topology derived from neutrosophic vertex coloring of social network graphs facilitates the identification of influence zones, uncertain relationships, and community boundaries, thereby enhancing analysis and decision-making in complex, indeterminate social structures.

7. Conclusion and Future Work

The profound relationship between neutrosophic graph coloring and the neutrosophic topology it produced on a graph's vertex set was investigated in this work. We created a paradigm that can capture uncertainty, indeterminacy, and inconsistency in both the structural and topological aspects of graphs by expanding classical graph theory through the prism of neutrosophic logic. We showed how a graph's vertices can be given neutrosophic colors to create a neutrosophic topological space, with open sets that vary in degrees of truth, indeterminacy, and falsehood. When information is ambiguous or lacking, this generated topology is an effective tool for examining and grouping items inside a graph. This framework is further improved by the incorporation of rough set theory, which is done through the use of lower and higher approximations. By classifying vertices according to their definite or potential membership in subsets, these approximations enhance the accuracy of analysis in the neutrosophic setting.
Applying this approach to social networks in particular demonstrated how well it captured and explained community boundaries, unclear ties, and partial memberships. A more flexible and nuanced examination than traditional methods is provided by the upper approximation, which includes members with possible or borderline status, while the lower approximation finds members with strong supporting evidence.
A case study demonstrated how this approach supports:

- Targeted outreach
- Influence maximization
- Anomaly detection

Future Directions:
Further research could focus on:

- Designing efficient algorithms for computing neutrosophic topologies
- Studying dynamic updates in coloring-induced topologies
- Applying this model to real-world systems such as Bioinformatics and Multi-criteria decision-making environments

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