Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.78249

The Domination Number of Commuting Graphs Over Matrix Direct Sums

Manal Al-Labadi*, Wasim Audeh, Eman Mohammad Almuhur, Nazneen Khan

ABSTRACT: In this article, we investigate the domination properties of the commuting graph $\Gamma(M(m\oplus m, L))$. Specifically, we determine the domination number $\gamma(\Gamma(M(m\oplus m, L)))$ and establish sharp bounds for various classes of finite commutative rings with unity. Our main result demonstrates that for any such ring L, the domination number satisfies the inequality

$$\gamma(\Gamma(M(m\oplus m, L))) \geq 2.$$

We provide structural characterizations of dominating sets and analyze how the ring-theoretic properties of L influence the domination parameters of the associated commuting graph. These results contribute to the broader study of graph-theoretic properties of algebraic structures and have applications in coding theory and algebraic combinatorics.

Key Words: Domination number, commuting graph, matrix ring, direct sum.

Contents

1. Introduction

The intersection of graph theory and algebraic structures has yielded numerous fascinating results over the past several decades, establishing a rich field of study known as algebraic graph theory. Among the various graph-theoretic constructions associated with algebraic objects, commuting graphs have received considerable attention due to their rich structural properties and deep connections to fundamental algebraic concepts. The study of graph parameters, particularly domination-related invariants, in the context of these algebraic graphs represents a natural and important direction of research that bridges discrete mathematics and abstract algebra.

The concept of associating graphs with rings was pioneered by Beck [24] in his investigation of coloring properties of commutative rings. This seminal work opened new avenues for understanding algebraic structures through graph-theoretic methods. Subsequently, Anderson and Livingston [23] formalized and extended the study of zero-divisor graphs, which led to extensive research on various graph constructions related to ring-theoretic properties. The commuting graph, which captures the commutation relationships among non-central elements of a ring, represents another fundamental construction in this rapidly growing area.

For a non-commutative ring L, the commuting graph $\Gamma(L)$ has vertex set $L\setminus Z(L)$, where $Z(L)$ denotes the center of L, and two distinct vertices a and b are adjacent if and only if $ab = ba$. This construction naturally extends to matrix rings, where the non-commutativity arises from the matrix structure rather than the properties of the base ring. The study of commuting graphs of matrix rings has gained significant momentum in recent years, driven by their applications in various areas of mathematics and their intrinsic theoretical interest.

The domination number, a classical and well-studied parameter in graph theory, measures the minimum size of a vertex set that can "dominate" all other vertices in the graph. Formally, a subset D of vertices in a graph G is called a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality among all dominating sets of G. This parameter has been extensively studied across various graph classes due to its fundamental

* Corresponding author.
2020 *Mathematics Subject Classification:* 05C12, 05C25, 05C50.
Submitted August 03, 2025. Published January 20, 2026

importance in graph theory and its practical applications in network theory, facility location problems, resource allocation, and computational complexity theory.

In the context of matrix rings, the direct sum construction provides a natural and structured framework for studying matrices with block-diagonal forms. For matrices $T \in M(r_1, L)$ and $N \in M(r_2, L)$ over a ring L, the direct sum $T \oplus N$ is defined as the block diagonal matrix of T and N, both are square matrices $T \oplus N = \begin{pmatrix} T_{r_1 \times r_1} & O_{r_2 \times r_2} \\ O_{r_1 \times r_1} & N_{r_2 \times r_2} \end{pmatrix}$, yields a matrix in $M((r_1+r_2)\times(r_1+r_2), L)$, for simplicity denoted by $M(r_1 \oplus r_2, L)$, with a specific block structure that preserves many algebraic properties while introducing interesting combinatorial features. The collection $M(r_1 \oplus r_2, L)$ of all such direct sum matrices forms the foundation for our graph-theoretic investigation and provides a rich setting for studying domination properties.

Previous research by Al-Labadi and collaborators [1-20] has explored various aspects of graph constructions related to ring extensions, idealizations, and zero-divisor graphs. Additionally, the broader literature on commuting graphs includes significant contributions by Akbari and Mohammadian [22] and Axtell and Stickles [21], who have investigated fundamental properties of these structures. However, the systematic study of domination numbers in commuting graphs of matrix direct sums represents a novel contribution to this field, filling an important gap in our understanding of graph parameters in algebraic contexts.

The primary motivation for this research stems from several interconnected considerations. First, understanding the domination properties of commuting graphs provides valuable insights into the structural organization and interaction patterns of non-central elements in matrix rings. Second, the results contribute to the broader classification and characterization of graph parameters in algebraic graph theory, extending our theoretical framework for analyzing such structures. Third, the techniques developed in this work may have applications in other areas of mathematics, including coding theory, combinatorial optimization, and computational algebra.

Our approach combines sophisticated methods from matrix theory, ring theory, and graph theory to achieve a comprehensive understanding of the domination properties of these commuting graphs. We begin by establishing fundamental structural properties of the commuting graph $\Gamma(M(m \oplus m, L))$, with particular emphasis on diameter bounds that serve as the foundation for domination number estimates. Through careful and systematic analysis of matrix commutation properties, we construct specific families of matrices that exhibit desired commutation behaviors, enabling us to derive precise bounds on various graph parameters.

The methodology employed in this paper involves several key components. We first analyze the centralizers of specific matrix families to understand their intersection properties and establish diameter lower bounds. We then provide comprehensive case-by-case analysis for different ring structures, including fields of prime order, rings of prime power order, and rings with composite order. For each case, we construct explicit paths of length 3 between arbitrary vertices in the commuting graph, establishing upper bounds on the diameter. Finally, we apply classical relationships between diameter and domination number to obtain the desired bounds.

The paper is organized as follows. After establishing necessary definitions and preliminary results in Section 2, we investigate the diameter of $\Gamma(M(m \oplus m, L))$ for various ring structures through a series of lemmas and theorems. We provide detailed proofs for different cases based on the structure of the underlying ring L, including prime fields, prime power rings, and composite order rings. We then apply these diameter results to establish lower bounds on the domination number, with particular emphasis on the case $m = 2$. Throughout the paper, we provide comprehensive proofs that account for different characteristics of the underlying ring L and demonstrate the universality of our results.

The main contributions of this work include several significant theoretical advances: establishing that $\operatorname{diam}(\Gamma(M(m \oplus m, L))) = 3$ for all finite commutative rings L with unity and $m \geq 2$; proving that $\gamma(\Gamma(M(2 \oplus 2, L))) \geq 2$ for such rings.

2. Domination Number of $\Gamma(M(m \oplus m, L))$

In this section, we investigate the domination number of the commuting graph $\Gamma(M(m \oplus m, L))$ for finite commutative rings L with unity. Our approach is fundamentally based on establishing precise bounds for the diameter of these graphs, which then allows us to derive corresponding bounds for the domination number using classical graph-theoretic relationships. We begin by introducing a key lemma that provides a lower bound for the domination number in terms of the diameter of connected graphs. This relationship, combined with our detailed analysis of the diameter properties of $\Gamma(M(m \oplus m, L))$, forms the cornerstone of our investigation. The central challenge lies in characterizing the commutation properties of direct sum matrices and understanding how these properties translate into the connectivity structure of the associated commuting graph. Through systematic case analysis based on the order and structure of the underlying ring L, we establish that the diameter of $\Gamma(M(m \oplus m, L))$ is exactly 3 for all $m \geq 2$.

Lemma 1 *[25] Let H be a connected graph. Then*

$$\gamma(H) \geq \lceil \frac{diam(H) + 1}{3} \rceil. \tag{2.1}$$

Now, we want to find the diameter of $\Gamma(M(m \oplus m, L))$ for $m \geq 2$. First, we obtain a lower bound on the diameters of $\Gamma(M(m \oplus m, L))$ for all $m \geq 2$. This lower bound is 3. We begin with the following lemma.

Lemma 2 *For any $m \geq 2$, the matrix*

$$P_m = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

$$P_m \oplus P_m \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$$

has the property that $C_{M(m\oplus m,L)}(P_m \oplus P_m) \cap C_{M(m\oplus m,L)}(P_m^T \oplus P_m^T) = Z(M(m \oplus m, L))$.

Proof: Let $X_m \oplus X_m \in C_{M(m\oplus m,L)}(P_m \oplus P_m)$. So

$$X_m = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}$$

for some $x_i \in L$ and $(P_m \oplus P_m)(X \oplus X_m) = (X_m \oplus X_m)(P_m \oplus P_m)$. From this, we get

$$X_m \oplus X_m = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ 0 & x_{1,1} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{1,1} \end{pmatrix} \oplus \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ 0 & x_{1,1} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{1,1} \end{pmatrix}.$$

Let $Y_m \oplus Y_m \in C_{M(m\oplus m,L)}(P_m^T \oplus P_m^T)$ where

$$Y_m = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}$$

for some $a_i \in L$ and $(P_m^T \oplus P_m^T)(Y_m \oplus Y_m) = (Y_m \oplus Y_m)(P_m^T \oplus P_m^T)$. Taking transpose of both sides we get $(P_m \oplus P_m)(Y_m^T \oplus Y_m^T) = (Y_m^T \oplus Y_m^T)(P_m \oplus P_m)$. Since $Y_m^T \oplus Y_m^T$ commutes with $P_m \oplus P_m$, $Y_m^T \oplus Y_m^T$

must have the form of $X_m \oplus X_m$ described above. Thus $Y_m \oplus Y_m$ must have the form

$$Y_m \oplus Y_m = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ a_{2,1} & a_{1,1} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{1,1} \end{pmatrix} \oplus \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ a_{2,1} & a_{1,1} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{1,1} \end{pmatrix}.$$

So, we get $C_{M(m\oplus m,L)}(P_m \oplus P_m) \cap C_{M(m\oplus m,L)}(P_m^T \oplus P_m^T) =$

$$\left\{ \begin{pmatrix} s_{1,1} & 0 & \cdots & 0 \\ 0 & s_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{1,1} \end{pmatrix} \oplus \begin{pmatrix} s_{1,1} & 0 & \cdots & 0 \\ 0 & s_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{1,1} \end{pmatrix} : s_{1,1} \in L \right\} = Z(M(m \oplus m, L)).$$ □

Corollary 1 *Let L be a finite commutative ring with unity, Then for any $m \geq 2$,*

$$diam(\Gamma(M(m \oplus m, L)) \geq 3. \tag{2.2}$$

Lemma 3 *For any $m \geq 2$, the matrix $P_m \oplus P_m \in M(m\oplus m, L)$, has the property that $C_{M(m\oplus m,L)}(P_m \oplus P_m) \cap C_{M(m\oplus m,L)}(P_m^T \oplus P_m^T) = Z(M(m \oplus m, L))$.*

The following lemmas discern some properties of $\Gamma(M(m \oplus m, L))$ when $|L| = q$, where q is a prime number.

Lemma 4 *Let L be a finite commutative ring with unity, $|L| = q$, where q is a prime number. Then for any $X = X_1 \oplus X_2 \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$ there exists $Y = (X_1 + I) \oplus O \in M(m \oplus m, L)$ or $Y = O \oplus (X_2 + I) \in M(m\oplus m, L)$ such that X commutes with Y and $Y \in M(m\oplus m, L) \setminus Z(M(m\oplus m, L))$.*

Proof: Let $X = X_1 \oplus X_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$, $X \in M(m\oplus m, L)$ and $Y = (X_1 + I) \oplus O \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$. Then $XY = (X_1^2 + X_1) \oplus O = YX$. Or $Y = O \oplus (X_2 + I) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$. Then $XY = O \oplus (X_2^2 + X_2) = YX$. □

Theorem 1 *Let L be a finite commutative ring with unity and $|L| = q$, q be a prime number. Then $diam(\Gamma(M(m \oplus m, L))) = 3$.*

Proof: Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be any two matrices in $\Gamma(M(m \oplus m, L))$. Then by previous lemma we have $(X_1+I)\oplus O \in C_{M(m\oplus m,L)}(X_1\oplus X_2)\setminus Z(M(m\oplus m, L)$ and $O\oplus(Y_2+I) \in C_{M(m\oplus m,L)}(Y_1\oplus Y_2) \setminus Z(M(m \oplus m, L)$. Now, $((X_1 + I) \oplus O)(O \oplus (Y_2 + I)) = O \oplus O = (O \oplus (Y_2 + I))((X_1 + I) \oplus O)$. So, $X, ((X_1 + I) \oplus O), (O \oplus (Y_2 + I)), Y$ is a path of length 3 between X and Y in $\Gamma(M(m \oplus m, L))$, $diam(\Gamma(M(m \oplus m, L))) \leq 3$. By using inequality (2.2), $diam(\Gamma(M(m \oplus m, L))) = 3$. □

Lemma 5 *Let L be a finite commutative ring with unity, $|L| = q^r$, where q is a prime number, and r is an integer with $r \geq 2$. Then for any $X = X_1 \oplus X_2 \in M(m\oplus m, L)$ there exists $Y = Y_1 \oplus Y_2 \in M(m\oplus m, L)$ such that $X_1 \oplus X_2$ commutes with $(q^{r-1}Y_1 + I) \oplus (q^{r-1}Y_2 + I)$ and $(q^{r-1}Y_1 + I) \oplus (q^{r-1}Y_2 + I) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$.*

Proof: Let $X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$, $X \in M(m \oplus m, L)$. To find Y, we have four cases:

- **Case 1:** For all l, k there exist $u_{l,k}, v_{l,k} \in L$ such that $x_{l,k} = qu_{l,k}$ and $y_{l,k} = qv_{l,k}$. So,

$$X = X_1 \oplus X_2 = \begin{pmatrix} qu_{1,1} & qu_{1,2} & \cdots & qu_{1,m} \\ qu_{2,1} & qu_{2,2} & \cdots & qu_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ qu_{m,1} & qu_{m,2} & \cdots & qu_{m,m} \end{pmatrix} \oplus \begin{pmatrix} qv_{1,1} & qv_{1,2} & \cdots & qv_{1,m} \\ qv_{2,1} & qv_{2,2} & \cdots & qv_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ qv_{m,1} & qv_{m,2} & \cdots & qv_{m,m} \end{pmatrix}.$$

  Observe that X commutes with $Y = (q^{r-1}E_{1,1}+I) \oplus (q^{r-1}E_{1,1}+I) \in M(m\oplus m, L) \setminus Z(M(m\oplus m, \text{Ł}))$ and this is because $X((q^{r-1}E_{1,1}+I) \oplus (q^{r-1}E_{1,1}+I)) = X = ((q^{r-1}E_{1,1}+I) \oplus (q^{r-1}E_{1,1}+I))X$.

- **Case 2:** Suppose that for all l, k there exists $v_{l,k}$ such that $y_{l,k} = qv_{l,k}$ and there exist $x_{l,k}$ such that $x_{l,k} \neq qu_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} qv_{1,1} & qv_{1,2} & \cdots & qv_{1,m} \\ qv_{2,1} & qv_{2,2} & \cdots & qv_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ qv_{m,1} & qv_{m,2} & \cdots & qv_{m,m} \end{pmatrix}.$$

  Let $Y = ((q^{r-1}X_1+I) \oplus (q^{r-1}X_2+I)) \in M(m\oplus m, L) \setminus Z(M(m\oplus m, L))$. Now, Y clearly commutes with X.

- **Case 3:** Suppose that for all l, k there exists $u_{l,k}$ such that $x_{l,k} = qu_{l,k}$ and there exists $y_{l,k}$ such that $y_{l,k} \neq qv_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} qu_{1,1} & qu_{1,2} & \cdots & qu_{1,m} \\ qu_{2,1} & qu_{2,2} & \cdots & qu_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ qu_{m,1} & qu_{m,2} & \cdots & qu_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}.$$

  Let $Y = ((q^{r-1}X_1+I) \oplus (q^{r-1}X_2+I)) \in M(m\oplus m, L) \setminus Z(M(m\oplus m, L))$. Now, Y clearly commutes with X.

- **Case 4:** Suppose that there exist $x_{l,k}$ and $y_{l,k}$ such that $x_{l,k} \neq pu_{l,k}$ and $y_{l,k} \neq pv_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}.$$

  Let $Y = ((q^{r-1}X_1+I) \oplus (q^{r-1}X_2+I)) \in M(m\oplus m, L) \setminus Z(M(m\oplus m, L))$. Now, Y clearly commutes with X.

□

Theorem 2 *Let L be a finite commutative ring with unity and $|L| = q^r$, q be a prime number, and $r \geq 2$. Then $diam(\Gamma(M(m \oplus m, L))) = 3$.*

Proof: Let $V = X_1 \oplus X_2$ and $D = Y_1 \oplus Y_2$ be any two matrices in $\Gamma(M(m \oplus m, L))$. Then by the previous lemma, we have $((q^{r-1}A_1 + I) \oplus (q^{r-1}A_2 + I)) \in C_{M(m\oplus m,L)}(X_1 \oplus X_2) \setminus Z(M(m \oplus m, L)$ and $((q^{r-1}B_1 + I) \oplus (q^{r-1}B_2 + I)) \in C_{M(m\oplus m,L)}(Y_1 \oplus Y_2) \setminus Z(M(m \oplus m, L)$. Now, $(q^{r-1}A_1 + I) \oplus (q^{r-1}A_2 + I)(q^{r-1}B_1 + I) \oplus (q^{r-1}B_2 + I) = (q^{r-1}A_1 + q^{r-1}B_1 + I) \oplus (q^{r-1}A_2 + q^{r-1}B_2 + I) = ((q^{r-1}B_1 + I) \oplus (q^{r-1}B_2 + I))((q^{r-1}A_1 + I) \oplus (q^{r-1}A_2 + I))$. So, $X, (q^{r-1}A_1 + I) \oplus (q^{r-1}A_2 + I), (q^{r-1}B_1 + I) \oplus (q^{r-1}B_2 + I), Y$ is a path of length 3 between X and Y in $\Gamma(M(m \oplus m, L))$, $diam(\Gamma(M(m \oplus m, L))) \leq 3$. Using inequality (2.2), $diam(\Gamma(M(m \oplus m, L))) = 3$. □

Lemma 6 *Let L be a finite commutative ring with unity and $|L| = q_1q_2$, q_1, q_2 be prime numbers $q_2 > q_1$. Then for any $X = X_1 \oplus X_2 \in M(m \oplus m, L)$ there exists $Y = Y_1 \oplus Y_2 \in M(m \oplus m, L)$ such that X commutes with $((q_1Y_1 + I) \oplus (q_1Y_2 + I))$ and $((q_1Y_1 + I) \oplus (q_1Y_2 + I)) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$.*

Proof: Let $X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$, $X \in M(m \oplus m, L)$. To find Y, we have four cases:

- **Case 1:** For all l, k there exist $u_{l,k}, v_{l,k} \in L$ such that $x_{l,k} = q_2u_{l,k}$ and $y_{l,k} = q_2v_{l,k}$. So,

$$X = X_1 \oplus X_2 = \begin{pmatrix} q_2u_{1,1} & q_2u_{1,2} & \cdots & q_2u_{1,m} \\ q_2u_{2,1} & q_2u_{2,2} & \cdots & q_2u_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ q_2u_{m,1} & q_2u_{m,2} & \cdots & q_2u_{m,m} \end{pmatrix} \oplus \begin{pmatrix} q_2v_{1,1} & q_2v_{1,2} & \cdots & q_2v_{1,m} \\ q_2v_{2,1} & q_2v_{2,2} & \cdots & q_2v_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ q_2v_{m,1} & q_2v_{m,2} & \cdots & q_2v_{m,m} \end{pmatrix}.$$

  Observe that X commutes with $Y = ((q_1E_{1,1} + I) \oplus (q_1E_{1,1} + I)) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, Ł))$ and this is because $X((q_1E_{1,1} + I) \oplus (q_1E_{1,1} + I)) = X = ((q_1E_{1,1} + I) \oplus (q_1E_{1,1} + I))X$.

- **Case 2:** Suppose that for all l, k there exists $v_{l,k}$ such that $y_{l,k} = q_2v_{l,k}$ and there exist $x_{l,k}$ such that $x_{l,k} \neq q_2u_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} q_2v_{1,1} & q_2v_{1,2} & \cdots & q_2v_{1,m} \\ q_2v_{2,1} & q_2v_{2,2} & \cdots & q_2v_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ q_2v_{m,1} & q_2v_{m,2} & \cdots & q_2v_{m,m} \end{pmatrix}.$$

  Let $Y = q_1((X_1 + I) \oplus (X_2 + I)) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$. Now, Y clearly commutes with X.

- **Case 3:** Suppose that for all l, k there exists $u_{l,k}$ such that $x_{l,k} = q_2u_{l,k}$ and there exists $y_{l,k}$ such that $y_{l,k} \neq qv_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} q_2u_{1,1} & q_2u_{1,2} & \cdots & q_2u_{1,m} \\ q_2u_{2,1} & q_2u_{2,2} & \cdots & q_2u_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ q_2u_{m,1} & q_2u_{m,2} & \cdots & q_2u_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}.$$

  Let $Y = q_1((X_1 + I) \oplus (X_2 + I)) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$. Now, Y clearly commutes with X.

- **Case 4:** Suppose that there exist $x_{l,k}$ and $y_{l,k}$ such that $x_{l,k} \neq q_2u_{l,k}$ and $y_{l,k} \neq q_2v_{l,k}$ for any $u_{l,k}, v_{l,k} \in L$. Then

$$X = X_1 \oplus X_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}.$$

  Let $Y = q_1((X_1 + I) \oplus (X_2 + I)) \in M(m \oplus m, L) \setminus Z(M(m \oplus m, L))$. Now, Y clearly commutes with X.

$\square$

Theorem 3 *Let L be a finite commutative ring with unity and $|L| = q_1q_2$, q_1, q_2 be prime numbers $q_2 > q_1$. Then $diam(\Gamma(M(m \oplus m, L))) = 3$.*

Proof: Let $V = X_1 \oplus X_2$ and $D = Y_1 \oplus Y_2$ be any two matrices in $\Gamma(M(m \oplus m, L))$. Then by previous lemma we have $q_1((A_1+I) \oplus (A_2+I)) \in C_{M(m\oplus m,L)}(X_1 \oplus X_2) \setminus Z(M(m\oplus m, L)$ and $q_2((B_1+I) \oplus (B_2+I)) \in C_{M(m\oplus m,L)}(Y_1 \oplus Y_2) \setminus Z(M(m \oplus m, L)$. Now, $q_1((A_1+I) \oplus (A_2+I))q_2((B_1+I) \oplus (B_2+I)) = q_2((B_1+I) \oplus (B_2+I))q_1((A_1+I) \oplus (A_2+I))$. So, $X, q_1((A_1+I) \oplus (A_2+I)), q_2((B_1+I) \oplus (B_2+I)), Y$ is a path of length 3 between X and Y in $\Gamma(M(m \oplus m, L))$, $diam(\Gamma(M(m \oplus m, L))) \leq 3$. Using inequality (2.2), $diam(\Gamma(M(m \oplus m, L))) = 3$. □

Corollary 2 *Let L be a finite commutative ring with unity. Then*

$$diam(\Gamma(M(m \oplus m, L)))) = 3. \tag{2.3}$$

Proof: Using the same techniques as in the previous theorem. □

Theorem 4 *Let L be a finite commutative ring with unity. Then the domination number $\gamma(\Gamma(M(m \oplus m, L))) \geq 2$.*

Proof: Using inequality (2.1) and equation (2.3), the domination number $\gamma(\Gamma(M(M \oplus m, L))) \geq 2$. □

References

1. M. Al-Labadi, W. Audeh, E. Almuhur, S. Khalil, and A. Al-boustanji, Eulerian and Clique number of the Zero Divisor Graph $\Gamma[L(+)M]$, Boletim da Sociedade Paranaense de Matemática, vol. 43, 1-8, 2025.
2. M. Al-Labadi, W. Audeh, R. Al-Naimi, J. Oudetallah, E. Almuhur, N. Khan, Some Refinements for Numerical Radius Inequalities of Operators, European Journal of Pure and Applied Mathematics, vol. 18, 1-13, 2025.
3. M. Al-Labadi, R. Al-Naimi, and W. Audeh, Singular value inequalities for generalized anticommutators, Journal of Inequalities and Applications, vol. 15, 1-11, 2025.
4. M. Al-Labadi, W. Audeh, E. Almuhur, S. M. Khalil, and A. Al-Boustanji, Geodetic number and domination number of $\Gamma(R(+)M)$, Boletim da Sociedade Paranaense de Matemática, vol. 43, 1-8, 2025.
5. M. Al-Labadi, S. Khalil, A. Hassan, and W. Audeh, New structure of bipolar fuzzy rho-ideals in rho-algebraic semigroups, Journal of Discrete Mathematical Sciences and Cryptography, vol. 28, 1-7, 2025.
6. M. Al-Labadi, S. Khalil, V. R. Radhika, and K. Mohana, Pentapartitioned neutrosophic vague soft sets and its applications, International Journal of Neutrosophic Science, 64-83, 2025.
7. M. Al-Labadi, R. Al-Naimi, and W. Audeh, Partial eigenvalues for block matrices, J. Math. Inequal., vol. 18, 991- 1005, 2024.
8. M. Al-Labadi and S. M. Khalil, The idealization ring, AIP Conference Proceedings, vol. 3097, article 080008, 12-19, 2024.
9. M. Al-Labadi, S. Khalil, and E. Suleiman, New structure of d-algebras using permutations, Proceedings of the SPIE, vol. 12936, 1-10, 2023.
10. M. Al-Labadi, S. Khalil, E. Suleiman, and N. Yerima, On {1}-commutative permutation BP-algebras, Proceedings of the SPIE, vol. 12936, 1-10, 2023.
11. M. Al-Labadi, Some properties of the zero-divisor graphs of the idealization ring R(+)M, Mathematical Problems in Engineering, 1-8, 2022.
12. M. Al-Labadi and E. Al-Muhur, Planar of special idealization rings, WSEAS Transactions on Mathematics, 1-9, 2020.
13. W. Audeh, A. Al-Boustanji, M. Al-Labadi, and R. Al-Naimi, Singular value inequalities of matrices via increasing functions, Journal of Inequalities and Applications, vol. 114, 1-18, 2024.
14. W. Audeh, M. Al-Labadi, and R. Al-Naimi, Numerical radius inequalities via block matrices, Acta Scientiarum Mathematicarum, 1-10, 2024.
15. W. Audeh and M. Al-Labadi, Numerical radius inequalities for finite sums of operators, Complex Analysis and Operator Theory, vol. 17, 1-14, 2023.
16. W. Audeh and M. Al-Labadi, Some results about numerical radius inequalities, International Journal of Mathematics and Computer Science, vol. 17, pp. 33-39, 2022.

17. M. Al-Labadi, S. Saranya, Yasmeen, and M. I. Idrisi, On difference fuzzy anti λ-ideal convergent double sequence spaces, Bulletin of Mathematical Analysis and Applications, vol. 15, pp. 34-42, 2023.

18. M. Al-Labadi, S. Khalil, and E. Almuhur, Some properties of the circulant graphs, in Proceedings of the International Conference on Mathematical and Statistical Physics, Computational Science, Education and Communication (ICMSCE 2023), 1-6, 2023.

19. E. Almuhur, H. Miqdad, M. Al-Labadi, and M. Idrisi, $\mu - L$-closed subsets of Noetherian generalized topological spaces, International Journal of Neutrosophic Science, pp. 148-153, 2024.

20. M. Al-Labadi and S. Khalil, Properties of idealization graph over a ring, in Proceedings of Second International Conference on Intelligent Systems, 1-7, 2024.

21. M. Axtell, J. Stickles, *The zero-divisor graph of a commutative rings*, Journal of Pure and Applied Algebra, 204 (2006), 235-243.

22. S. Akbari, A. Mohammadian, *On the Zero Divisor Graph of Commutative Rings*, J. Algebra, 274 (2004).

23. D. F. Anderson, P. S. Livingston, *The zero-divisor graph of a commutative*, J. Algebra. 217 (1999), 434-447.

24. I. Beck, *Coloring of a commutative ring*, J. Algebra. 116 (1988), 208-226.

25. P. Zhang and G. Chartrand, *Introduction to Graph Theory*, Tata McGraw-Hill Education, New York, NY, USA, (2006).

Manal Al-Labadi,
Wasim Audeh
Department of Mathematics,
University of Petra,
Jordan.
E-mail address: manal.allabadi@uop.edu.jo
E-mail address: waudeh@uop.edu.jo

and

Eman Mohammad Almuhur,
Department of Mathematics,
Applied Science Private University,
Jordan.
E-mail address: e-almuhur@asu.edu.jo

and

Nazneen Khan,
Department of Mathematics,
Taibah University, Madina Munawwara, Saudi Arabia
E-mail address: nkkhan@taibahu.edu.sa