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The Characteristic Polynomial of a Uniform Hypercycle

Shashwath S. Shetty and K. Arathi Bhat*

ABSTRACT: It is (and was) an increasingly significant field of study to analyze the graph properties using the spectra of the various matrices associated with the graph. The computation of the characteristic polynomial (either explicitly or recursively) of the hypermatrices associated with the uniform hypergraphs is really a challenging and interesting task. In this article, we obtain all the coefficients of the characteristic polynomial of the degree-based extended adjacency hypermatrices associated with an r-uniform hypercycle of arbitrary length.

Keywords: Hypercycle, extended adjacency, hypermatrix, generalized trace, characteristic polynomial, quality education.

Contents

1. Introduction

Motivated by the study of positive definiteness of (the homogeneous polynomial associated with) the hypermatrix discussed in [24], Qi [32] proposed the definition of the eigenvalues of a real symmetric hypermatrix and studied some of its fundamental properties, including its trace, (hyper) determinant, characteristic polynomial, and the distribution of the eigenvalues. Later, Lim [25,26] has given the definition for the eigenvalues of the non-symmetric hypermatrices. Though these types of definitions for the eigenvalues of the hypermatrix are found way back in [28], the systematic study has begun in [32]. The generalization of the Perron-Frobenius theorems to the multi-linear forms and some of its variations can be found in [4,5,20,40,41]. The adjacency [12], Laplacian and signless-Laplacian [23,33] hypermatrices are defined in later years. The study of the spectral symmetry of hypergraphs has been carried out in various capacities [12,18,31]. The generalization of the Harary-Sachs theorem for hypergraphs can be found in [10] and its applications in [11]. Determining the spectrum of power hypergraphs using the signed subgraphs of the original graph was proposed in [43] and well studied in [8,9,19].

Let $g_1, g_2, \dots, g_n$ be a system of n homogeneous polynomials of (total) degrees $r_1, \dots, r_n$, respectively, in n variables. Then, this system has a non-zero solution if it satisfies $R_{r_1 \dots r_n}\{g_1, \dots, g_n\} = 0$, where R is the polynomial in the coefficients of $g_1, \dots, g_n$ called the resultant. For the detailed study of resultants, one can refer to [14] and [21]. It has been shown [32] that the eigenvalues of the hypermatrix $\mathcal{M}$ are exactly the roots of the polynomial $det(\mathcal{M} - \lambda\mathcal{I})$, which is defined to be the characteristic polynomial of the hypermatrix. There are mainly four approaches for computing the characteristic polynomial of a hypermatrix. The first one is through the Poisson product formula, where one can get the expression for the characteristic polynomial in a recursive way. The second one is through the generalized traces of a hypermatrix, where the t-th Schur polynomial in these generalized traces will be the coefficient of the co-degree t term of the characteristic polynomial. The third one is through the determinant of the Koszul complex [6,1], where for a given non-linear map one can construct a Koszul complex, and the determinant of the Koszul complex [3] will be exactly equal to the resultant of the original map. The fourth one is

* Corresponding author.
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through the series of contour integrals [30]. It is very well observed that the first two techniques are very much used in studying the special classes of hypermatrices. Cooper and Dutle [12] in the year 2012 have given the expression for the characteristic polynomial of a (single) hyperedge containing r vertices and some coefficients of the characteristic polynomial of some special classes of hypergraphs. In the year 2015, the same authors [13] have given the explicit expression for the characteristic polynomial of the 3-uniform hyperstars. In the same year, Shao and others [34] have given the graph theoretical formula for generalized traces of the hypermatrices. In the year 2020, authors [2] have provided the combinatorial approach for the computation of the characteristic polynomial of the star-like hypergraphs. A reduction formula for the characteristic polynomial of the uniform hypergraphs with pendant hyperedges is found in [7]. Zheng [42] in the year 2021 has given the expression for the characteristic polynomial of the complete 3-uniform hypergraphs. The explicit expression for the characteristic polynomial of an r-uniform hypercycle of lengths 3 and 4 is given in [16] and [17], respectively.

Let $\mathcal{M} = (\mathcal{M}_{j_1,\dots,j_r}), j_i \in [n] := \{1,\dots,n\}, i \in [r]$ be an order $r (r \geq 2)$, dimension n hypermatrix, and y be a (complex) vector of length n (order 1, dimension n hypermatrix). Then $\mathcal{M}\mathbf{y}^{r-1}$ and $\lambda\mathbf{y}^{r-1}$ are the vectors of length n each of whose j-th entry is given by

$$(\mathcal{M}\mathbf{y}^{r-1})_j = \sum_{j_2,\dots,j_r \in [n]} \mathcal{M}_{j,j_2,\dots,j_r} y_{j_2} \cdots y_{j_r} \text{ and } (\lambda\mathbf{y}^{r-1})_j = \lambda(y_j^{r-1}).$$

Despite several attempts to generalize the idea of the eigenvalues of the matrix to the eigenvalues of the higher-order form, the following Definition 1.1 by Qi and Lim became well-known because it allows many of the properties of matrices to be generalized to hypermatrices.

Definition 1.1 *A complex number λ is called an eigenvalue of the order r, dimension n hypermatrix $\mathcal{M}$ corresponding to an eigenvector $\mathbf{y} \in \mathbb{C}^n$ if it satisfies*

$$\mathcal{M}\mathbf{y}^{r-1} = \lambda\mathbf{y}^{r-1}.$$

A hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(H))$ (or simply $(\mathcal{V}, \mathcal{E})$, if there is no ambiguity in the hypergraph considered), where the elements of the set $\mathcal{V}$ are called the vertices of $\mathcal{H}$, and $\mathcal{E}$ contains a collection of non-empty subsets of $\mathcal{V}$ that are called the hyperedges of $\mathcal{H}$. By a hypergraph $\mathcal{H}$, we mean a simple (no hyperedge is contained inside the other), undirected (elements of $\mathcal{E}$ are non-empty subsets of $\mathcal{V}$) hypergraph with no self-loops (elements of $\mathcal{E}$ are sets of size greater than one). Degree of a vertex u in a hypergraph $\mathcal{H}$ is the number of hyperedges containing the vertex u, and is denoted by $d_{\mathcal{H}}(u)$ or simply d_u. An r power hypergraph of a graph $G = (V, E)$ on n vertices is an r-uniform hypergraph $G^{(r)} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = V \cup \{v_1^{(e)}, \dots, v_{r-2}^{(e)}|$ for each $e \in E\}$, and the hyperedge set is given by $\mathcal{E} = \{e \cup \{v_1^{(e)}, \dots, v_{r-2}^{(e)}\} | e \in E\}$. An r-uniform hypercycle with m hyperedges, denoted by $C_m^{(r)}$ is the r power hypergraph of the cycle C_m of length m.

Definition 1.2 *[12] Let $\mathcal{H}$ be an r-uniform hypergraph on the vertex set $[n]$. Then the adjacency hypermatrix of $\mathcal{H}$ is an order r, dimension n hypermatrix denoted by $\mathcal{A}_{\mathcal{H}} = (\mathcal{A}_{j_1,\dots,j_r})$ is defined as*

$$\mathcal{A}_{j_1,\dots,j_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{j_1,\dots,j_r\} \in \mathcal{E}(\mathcal{H}) \\ 0, & \text{otherwise} \end{cases}.$$

A hypermatrix $\mathcal{M} = (\mathcal{M}_{j_1 \dots j_r})$ is said to be a symmetric hypermatrix [24] if $\mathcal{M}_{j_1 \dots j_r} = \mathcal{M}_{\sigma(j_1)\dots\sigma(j_r)}$ for any permutation $\sigma \in S_r$, where S_r is the set of all permutation on r elements. For an undirected hypergraph $\mathcal{H}$, it is easy to see that $\mathcal{A}_{\mathcal{H}}$ is always symmetric. Similarly, a function f on r variables $x_1, \dots, x_r$ is said to be symmetric if $f(x_1, \dots, x_r) = f(\sigma(x_1), \dots, \sigma(x_r))$ for any permutation $\sigma \in S_r$.

The generalization of the vertex-degree-based topological indices from graphs to hypergraphs has been proposed in [37,35,39]. One of the degree-based topological indices is the atom-bond connectivity (ABC) index, and the study on the spectral radius of the degree-based extended adjacency hypermatrix corresponding to the ABC index has been carried out in [27]. The degree-based extended adjacency hypermatrix corresponding to a symmetric vertex-degree-based topological index can be defined as follows.

Definition 1.3 *Let $\mathcal{H}$ be an r-uniform hypergraph on the vertex set $[n]$. Then, the degree-based extended adjacency hypermatrix of $\mathcal{H}$ corresponding to a symmetric function f of the vertex degrees of a hyperedge is an order r, dimension n hypermatrix denoted by $\mathcal{F}_{\mathcal{H}} = (\mathcal{F}_{j_1,\ldots,j_r})$ is defined as*

$$\mathcal{F}_{j_1,\ldots,j_r} = \begin{cases} \frac{f(d_{j_1},\ldots,d_{j_r})}{(r-1)!}, & \text{if } \{j_1,\ldots,j_r\} \in \mathcal{E}(\mathcal{H}) \\ 0, & \text{otherwise} \end{cases}.$$

Recently, the notion of adjacency tensor has been generalized to degree-based extended adjacency tensors, and the bounds for the spectral radius of uniform hypergraphs are discussed in [36].

The rest of the paper is organized as follows. In Section 2, we state some of the important definitions and results from the literature and we obtain the characteristic polynomial of the degree-based extended adjacency hypermatrix of an r-uniform hyperedge. The main result of the paper is the characteristic polynomial of the degree based extended adjacency hypermatrix of an r-uniform hypercycle, whose coefficients are the polynomial in the generalized traces of the corresponding hypermatrix, and the expression for which has been discussed in Section 3.

2. Preliminary Results

Following a summary of some of the most relevant concepts and findings from the literature, we derive the characteristic polynomial of the degree-based extended adjacency tensor of an r-uniform hyperedge in this section.
For the order r, dimension n hypermatrices, Morozov and Shakirov [29] have given the formula for $det(\mathcal{I} - \mathcal{M})$ using Schur polynomials in the generalized traces.

Definition 2.1 *Let A be an $n \times n$ matrix with variable entries A_{ij}. Define t^{th} order trace (for some positive integer t) of an order r, dimension n hypermatrix $\mathcal{M}$ as*

$$Tr_t(\mathcal{M}) = (r-1)^{n-1} \sum_{p_1+\cdots+p_n=t} \left(\prod_{i=1}^{n} \frac{\hat{g_i}^{p_i}}{(p_i(r-1))!} \right) tr(A^{t(r-1)}),$$

where $g_i(y_1,\ldots,y_r) := \left(\mathcal{M}\mathbf{y}^{r-1}\right)_i$ and $\hat{g}_i = g_i\left(\frac{\partial}{\partial A_{i1}}, \frac{\partial}{\partial A_{i2}}, \ldots, \frac{\partial}{\partial A_{in}}\right)$.

Definition 2.2 *Let $P_0 = 1$ and for $t > 0$, t^{th} Schur polynomial $P_t \in \mathbb{Z}[x_1,\ldots,x_r]$ is defined as*

$$P_t(x_1,\ldots,x_r) = \sum_{h=1}^{t} \sum_{\substack{t_1+\cdots+t_h=t \\ t_j>0,\ \forall j}} \frac{x_{t_1}\ldots x_{t_h}}{h!}.$$

In other words,

$$exp\left(\sum_{r=1}^{\infty} x_r z^r\right) = \sum_{r=1}^{\infty} P_t(x_1,\ldots,x_r) z^r.$$

Theorem 2.1 *[29] Let $\mathcal{M}$ be an order r, dimension n hypermatrix and $\mathcal{I}$ be the identity hypermatrix (same order and dimension). Then,*

$$det(\mathcal{I} - \mathcal{A}) = \sum_{t=1}^{\infty} P_t\left(\frac{Tr_1(\mathcal{M})}{1}, \ldots, \frac{Tr_t(\mathcal{M})}{t}\right) = exp\left(\sum_{t=1}^{\infty} -\frac{Tr_t}{t}\right).$$

For an integer $t > 0$, define

$$\mathcal{S}_t = \{((j_1,\beta_1),\ldots,(j_t,\beta_t)) | j_1 \le \cdots \le j_t, \beta_i \in [n]^{r-1}, 1 \le i \le t\}.$$

Given an element $S = ((j_1,\beta_1),\ldots,(j_t,\beta_t)) \in \mathcal{S}_t$, we call each $(j_i,\beta_i), 1 \le i \le t$, a component of S. Also, let $j_i, 1 \le i \le t$ be the primary element of S, and the components of β_i be the secondary elements of S.

An element $S \in S_t$ is said to be r-valent, if every $i \in [n]$ occurs exactly $0 \mod r$ times as some element of S. Also, let

$$S'_t = \{S \in S_t | S \text{ is } r-\text{valent}\}.$$

Hereafter, for the sake of simplicity, we denote an element of $\mathcal{S}_t$ by $(j_1\beta_1, \ldots, j_t\beta_t)$ instead of $((j_1, \beta_1), \ldots, (j_t, \beta_t))$.

Definition 2.3 *[34] Let $\vec{G}(M)$ be the weighted digraph associated with an $n\times n$ square matrix $M = (m_{ij})$, that is defined as, $V(\vec{G}) := [n]$ and there will be an arc (i, j) with weight m_{ij} in $\vec{G}$ if and only if $m_{ij} \neq 0$.*

Definition 2.4 *[34] Let $S = (j_1\beta_1, \ldots, j_t\beta_t) \in \mathcal{S}_t$, where $j_i \in [n]$, $\beta_i \in [n]^{r-1}$ for $1 \le i \le t$. Then*

- $E(S) := \bigcup_{k=1}^{n} E_k(S)$, *where* $E_k(S) := \{(j_k, u_2), \ldots, (j_k, u_r)\}$ *and* $\beta_k = u_2 \ldots u_r$.
- $b(S)$ *denotes the product of the factorials of the multiplicities of all the arcs of* $E(S)$.
- $c(S)$ *denotes the product of the factorials of the out-degrees of all the vertices that are incident with some arcs of* $E(S)$.
- $\mathbf{W}(S)$ *denotes the set of all Eulerian cycles* W *with arc multi-set* $E(W) = E(S)$.

Theorem 2.2 *[34] Let $\mathcal{M} = (\mathcal{M}_{i_1\ldots i_r})$ be an order r, dimension n hypermatrix, and $Tr_t(\mathcal{M})$ be the t^{th} order trace of $\mathcal{M}$. Then*

$$Tr_t(\mathcal{M}) = (r-1)^{n-1} \sum_{S\in\mathcal{S}'_t} \frac{b(S)}{c(S)} \pi_S(\mathcal{M}) |\mathbf{W}(S)|, \tag{2.1}$$

where $\pi_S(\mathcal{M}) = \prod_{i=1}^{n} \mathcal{M}_{j_i\beta_i}$ and $S = (j_1\beta_1, \ldots, j_t\beta_t)$.

The following two results for the adjacency spectrum of an r-uniform hypergraph are stated in [12], and the proof is omitted as it is direct for the extended case.

Lemma 2.1 *The extended adjacency spectrum of an r-uniform r-partite hypergraph is invariant under the multiplication of any r^{th} root of unity.*

Lemma 2.2 *Let $\mathcal{H}$ be an r-uniform hypergraph, then for $i = 0, 1, \ldots r-1$ the codegree i coefficient of $\Phi_{\mathcal{F}_\mathcal{H}}(\lambda)$ is zero.*

Lemma 2.3 *Let $\mathcal{H}$ be an r-uniform d-regular hypergraph on n vertices with m hyperedges. Then, codegree r coefficient of $\Phi_{\mathcal{F}_\mathcal{H}}(\lambda)$ is*

$$-f(d, \ldots, d)^r m r^{r-2} (r-1)^{n-r}.$$

Proof: Let $f(d_e) = f(d_{j_1}, \ldots, d_{j_r})$ be a symmetric (positive valued) function of the vertex degrees of a hyperedge $e = \{j_1 \ldots, j_r\}$, and if A is an $n \times n$ auxiliary matrix with variable entries A_{ij}, then

$$\left(\sum_{e\in\mathcal{E}} \prod_{i\in e} \hat{g}_i\right) tr(A^{r(r-1)}) = \left(\sum_{e\in\mathcal{E}} f(\mathbf{d}_e)^r \prod_{\substack{i,j\in e\\ i\neq j}} \frac{\partial}{\partial A_{ij}}\right) tr(A^{r(r-1)}).$$

Furthermore, if $\mathcal{H}$ is d-regular, then the product of the operators is independent of the hyperedge under consideration. That is,

$$\left(\sum_{e\in\mathcal{E}} \prod_{i\in e} \hat{g}_i\right) tr(A^{r(r-1)}) = m f(\mathbf{d}_e)^r \left(\prod_{\substack{1\le i,j\le r\\ i\neq j}} \frac{\partial}{\partial \bar{A}_{ij}}\right) tr(\bar{A}^{r(r-1)}),$$

where $\bar{A}$ is an $r \times r$ auxiliary matrix with variable entries $\bar{A}_{ij}$. The rest of the proof is similar to the case of the adjacency hypermatrix of $\mathcal{H}$. □

Theorem 2.3 *[22] Suppose $Tr_t(\mathcal{M})$ denote the t^{th} order trace of an order r, dimension n hypermatrix $\mathcal{M}$. Then $Tr_t(\mathcal{M}) = \sum_{\lambda \in \sigma(\mathcal{M})} \lambda^t$, $\sigma(\mathcal{M})$ is the multiset of the roots of the characteristic polynomial of $\mathcal{M}$.*

Theorem 2.4 *Let $\mathcal{H}$ be an r-uniform hypergraph containing single hyperedge and no isolated vertices. If $\mathcal{F}_\mathcal{H}$ denotes the degree-based extended adjacency hypermatrix $\mathcal{H}$ with corresponding degree function f (positive valued), then the characteristic polynomial of $\mathcal{F}_\mathcal{H}$ is given by*

$$\Phi_{\mathcal{F}_\mathcal{H}}(\lambda) = \lambda^{r(r-1)^{r-1} - r^{r-1}} (\lambda^r - f(1, \dots, 1))^{r^{r-2}}.$$

Proof: Since f is a function from positive octant(in an r-dimensional space) to a positive reals, it is simple to show that all the components of the eigenvector corresponding to a non-zero eigenvalue are non-zero. That is, suppose λ is a non-zero eigenvalue of $\mathcal{F}_\mathcal{H}$ with corresponding eigenvector $\mathbf{y}$. If $y_s = 0$ for some $1 \leq s \leq r$, then for any $y_j \neq 0$ we have

$$\lambda {y_j}^{r-1} = (\mathcal{F}_\mathcal{H} \mathbf{y}^{r-1})_j = f(1, \dots, 1) \prod_{i \neq j} y_i = 0,$$

a contradiction since $\lambda \neq 0$. Now,

$$\lambda^r \prod_{j=1}^r y_j^{r-1} = \prod_{j=1}^r \lambda y_j^{r-1} = \prod_{j=1}^r (\mathcal{F}_\mathcal{H} \mathbf{y}^{r-1})_j = \prod_{j=1}^r \prod_{i \neq j} y_i^{r-1} = f(1, \dots, 1)^r \prod_{j=1}^r y_j^{r-1}.$$

This implies that for any $\lambda \neq 0$, $\lambda^r - f(1, \dots, 1)^r = 0$. By using lemma 2.1, we have the characteristic polynomial of $\mathcal{F}_\mathcal{H}$ is of the form $\Phi_{\mathcal{F}_\mathcal{H}}(\lambda) = \lambda^p (\lambda^r - f(1, \dots, 1)^r)^q$. □

Example 2.1 *Let $\mathcal{H}$ be an r-uniform hypergraph containing single hyperedge and no isolated vertices. The Sombor index of the hypergraph is defined [35] as $\mathcal{SO}(H) = \sum_{e \in \mathcal{E}} \sqrt{\sum_{u \in e} d_u^2}$. If $SO_\mathcal{H}$ denotes the Sombor hypermatrix, then the characteristic polynomial of $\mathcal{SO}_\mathcal{H}$ is given by*

$$\Phi_{\mathcal{SO}_\mathcal{H}}(\lambda) = \lambda^{r(r-1)^{r-1} - r^{r-1}} (\lambda^r - \sqrt{r})^{r^{r-2}}.$$

3. Main Results

The following result is a simple generalization of the result [8] from adjacency hypermatrix to degree based extended adjacency hypermatrix of the (regular) hypergraph.

Theorem 3.1 *Let G be a d-regular graph and f be an r-variable, symmetric, positive valued function defined on the positive octant. The complex number λ is an eigenvalue of $\mathcal{F}(G^{(r)})$ if and only if*

1. *some* signed induced subgraph *of the graph G has an (adjacency) eigenvalue β such that $f(d, d, 1, \dots, 1)^r \beta^2 = \lambda^r$, for $r = 3$;*

2. *some* signed subgraph *of the graph G has an (adjacency) eigenvalue β such that $f(d, d, 1, \dots, 1)^r \beta^2 = \lambda^r$, for $r \geq 4$.*

Proof: Let $\beta \neq 0$ be an eigenvalue of the adjacency matrix of some signed (induced, if $r = 3$) subgraph of G with corresponding eigenvector $\mathbf{y}$. Also, let λ be a complex number such that $\lambda^r = f(d, d, 1, \dots, 1)^r \beta^2$. With the same eigenvector $\mathbf{x}$ defined in [8], we can see that $(\lambda, \mathbf{x})$ is an eigenpair of the degree based extended adjacency hypermatrix $\mathcal{F}$ (whose corresponding degree function is f) of the power hypergraph $G^{(r)}$. The proof of the direct part is also similar to the case of adjacency hypermatrix of the power hypergraph. □

Theorem 3.2 *Let $C_m^{(r)}$ be an r-uniform $(r \geq 3)$ hypercycle with m hyperedges. Then,* $\Phi_{\mathcal{F}_{C_m^{(r)}}}(\lambda) = \sum_{t=0}^{\lfloor \frac{s}{r} \rfloor} b_t \lambda^{s-rt}$ *is the characteristic polynomial of the degree-based extended adjacency hypermatrix of $C_m^{(r)}$, where*

$$b_t = \sum_{k=1}^{t} \sum_{\substack{t_1+\cdots+t_k=tr,\\ t_i>0}} \frac{1}{r^k k!} \prod_{i=1}^{k} \left(-\frac{Tr_{t_i r}(\mathcal{F})}{t_i} \right). \tag{3.1}$$

Proof: First we show that an r-uniform $(r \geq 3)$ hypercycle is r partite. Then from [12], we have that the co-degree d coefficient of $\Phi_{\mathcal{F}(\mathcal{H})}(\lambda)$ is non-zero only if it is a multiple of r. Let $C_m := v_0 e_0 v_1 e_1 \cdots v_{m-1} e_{m-1} v_0$ be a cycle of length m, and $u_1^{i,j}, \ldots, u_{r-2}^{i,j}$ be the $r-2$ vertices of $C_m^{(r)}$ such that $\{v_i, u_1^{i,j}, \ldots, u_{r-2}^{i,j}, v_j\}$ is a hyperedge of $C_m^{(r)}$. Also, let $V_t, 0 \leq t \leq r-1$ be the partition of the vertices of $C_m^{(r)}$ satisfying the following conditions.

- Include v_i in V_i and v_{i+1} in V_{i+1}, for $0 \leq i \leq r-2$, and include $u_k^{i,i+1}, 1 \leq k \leq r-2$ in the remaining $r-2$ (distinct) partite sets (other than V_i and V_{i+1}).

- If $r \geq m$, then by above construction we can see that an r-uniform $(r \geq 3)$ hypercycle is r-partite. Hence, assume that $r < m$. Include v_i in $V_{i \mod r}$, for $r \leq i \leq m-1$ and $m \not\equiv 0 \mod r$ (and the $r-2$ (pendant) vertices of each hyperedge are included in the $r-2$ partite sets as stated earlier).

- Since $r, m \geq 3$, if $m \equiv 0 \mod r$, then include v_{m-1} to V_1, and otherwise include v_{m-1} to $V_{(m-1) \mod r}$.

By this construction, we can see that an r-uniform hypercycle is r-partite for $r \geq 3$. □

In the following theorem we give the formula for the order $tr, t \geq 1$, trace of the extended adjacency matrix $\mathcal{F}$ of an r-uniform hypercycle $C_m^{(r)}$ and hence by using the above theorem we can obtain the characteristic polynomial of $\mathcal{F}_{C_m^{(r)}}$.

Given a multi-set A, let $\mathcal{P}'(A)$ denote the multi-set of all sub(multi)-sets of A. That is, $\mathcal{P}'(A)$ is the power set of A by treating A as a set. For example, let $A = \{1, 1, 2\}$ be a multi set. Then, $\mathcal{P}'(A) = \{\{\}, \{1\}, \{1\}, \{2\}, \{1,1\}, \{1,2\}, \{1,2\},$
$\{1,1,2\}\}$. For a multi-set B, we denote by B^π, the product of all the elements of B. That is, if $B = \{2,2,3\}$ then $B^\pi = 12$.

Theorem 3.3 *Let $C_m^{(r)}$ be an r-uniform hypercycle with m hyperedges and $\mathcal{F}$ denotes the degree based extended adjacency hypermatrix with corresponding r-variable symmetric vertex degree function f. Then*

$$Tr_r(\mathcal{F}_{C_m^{(r)}}) = mh^r r^{r-1} (r-1)^{(m-1)r-m}.$$

For $2 \leq t \leq m-1$,

$$Tr_{tr}(\mathcal{F}_{C_m^{(r)}}) = mh^{tr} \left[r^{r-1}(r-1)^{(m-1)r-m} + t \sum_{q=2}^{t} (D_{t,q}) r^{q(r-2)+1} (r-1)^{(m-q)r-(m-q+1)} \right].$$

$$\begin{aligned} Tr_{mr}(\mathcal{F}_{C_m^{(r)}}) = mh^{mr} \Big[& r^{r-1}(r-1)^{(m-1)r-m} + 2m(m+1) r^{m(r-2)} \\ & + m \sum_{q=2}^{m-1} (D_{m,q}) r^{q(r-2)+1} (r-1)^{(m-q)r-(m-q+1)} \Big]. \end{aligned}$$

For $t \geq m$,

$$Tr_{tr}(\mathcal{F}_{C_m^{(r)}}) = h^{tr}\left[mr^{r-1}(r-1)^{(m-1)r-m} + mt\sum_{q=2}^{m-1}(D_{t,q})r^{q(r-2)+1}(r-1)^{(m-q)r-(m-q+1)} + t(C_t + 2C'_t)r^{m(r-2)}\right].$$

Here $D_{t,q} = \sum_{\substack{(a_1,\dots,a_q)\in[t]^q,\\ \sum a_i = t}} \frac{1}{a_1}\prod_{i=1}^{q-1}\left(a_i\frac{(a_i+a_{i+1}-1)!}{a_i!a_{i+1}!}\right)$,

$$C_t = \left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-1}} B^\pi\right)\left(\sum_{\substack{(a_1,\dots,a_m)\in[t]^m\\ \sum a_i=t}}\left(\prod_{i=1}^{m}\frac{(a_i+a_{i+1}-1)!}{a_i!a_{i+1}!}\right)\right),$$

$$C'_t = \sum_{d=1}^{z}\left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-2p-1,\\ p=0,1,\dots,\lceil\frac{m}{2}\rceil-1}} d^{2p}B^\pi\right)\left(\sum_{\substack{(a_1,\dots,a_m)\in[t]^m\\ \sum a_i=t}}\left(\prod_{i=1}^{m}\frac{(a_i+a_{i+1}-1)!}{(a_i+d)!(a_{i+1}-d)!}\right)\right),$$ $A = \{a_1,\dots,a_m\}$ *is a multi-set and* $h := f(2,2,1,\dots,1)$.

Proof: By Theorem 2.2, we need to consider all $S = (j_1\beta_1,\dots,j_{tr}\beta_{tr}) \in \mathcal{S}'_{tr}, 1 \leq t \leq m$, for which both $\pi_S(\mathcal{F}_{C_m^{(r)}})$ and $|\mathbf{W}(S)|$ are non-zero. Since S is r-valent, all the elements in the components of S occurs the times that is a multiple of r. If $\pi_S(\mathcal{F}_{C_m^{(r)}}) \neq 0$, then each component of S correspond to a hyperedge in $C_m^{(r)}$.

Case 1: If all the components of S corresponds to a fixed hyperedge of $C_m^{(r)}$.

The multi-digraph $\vec{G}(S)$ corresponding to S is a complete multi-digraph on r vertices and the multiplicity of each arc is t. The total number of such S is $[(r-1)!]^{tr}$ and

$$b(S) = (t!)^{r(r-1)},\ c(S) = [(t(r-1))!]^r,\ \pi_S(\mathcal{F}_{C_m^{(r)}}) = \left[\frac{h}{(r-1)!}\right]^{tr}.$$

By using Matrix-tree theorem [15], we have the number of spanning trees in $\vec{G}(S)$ is equal to $t^{r-1}r^{r-2}$. Also by using the BEST theorem [38], we have the number Eulerian circuits in $\vec{G}(S)$ is equal to

$$t^{r-1}r^{r-2}[(t(r-1)-1)!]^r.$$

Since $\mathbf{W}(S)$ (is a set) does not contain repeated elements and if we label each edge of $\vec{G}(S)$, we have

$$|\mathbf{W}(S)| = \frac{tr(r-1)t^{r-1}r^{r-2}[(t(r-1)-1)!]^r}{(t!)^{r(r-1)}}.$$

In total, there are m such hyperedges and hence, the total contribution of all such S to $Tr_{tr}(\mathcal{F}_{C_m^{(r)}})$ is

$$m(r-1)^{mr-m-1}[(r-1)!]^{tr}\frac{(t!)^{r(r-1)}}{[(t(r-1))!]^r}\left[\frac{h}{(r-1)!}\right]^{tr}\frac{tr(r-1)t^{r-1}r^{r-2}[(t(r-1)-1)!]^r}{(t!)^{r(r-1)}} = mh^{tr}r^{r-1}(r-1)^{(m-1)r-m}.$$

Case 2: All components of S corresponds to $q, (2 \leq q \leq t$ and $q < m)$ fixed hyperedges of the $C_m^{(r)}$. That is, S is an appropriate ordering of $(j_1\beta_1^{(1)},\dots,j_1\beta_1^{(a_1)},\dots,j_r\beta_r^{(1)},\dots,j_r\beta_r^{(a_1)},$ $j_r\beta_{r+1}^{(1)},\dots,j_r\beta_{r+1}^{(a_2)},\dots,j_{2r-1}\beta_{2r}^{(1)},\dots,j_{2r-1}\beta_{2r}^{(a_2)},j_{2r-1}\beta_{2r+1}^{(1)},\dots,j_{2r-1}\beta_{2r+1}^{(a_3)},\dots,j_{q(r-1)+1}$ $\beta_{qr}^{(1)},\dots,j_{q(r-1)+1}\beta_{qr}^{(a_q)})$.

Since $\mathbf{W}(S)$ is non-empty, we can assume that the q considered hyperedges of the hypergraph will induce a linear hyperpath of length q (A linear hyperpath of length q is an r power hypergraph of the path graph of length q). That is, the q edges are of the form $e_1 - e_2 - \cdots - e_q$. The number of orderings of the primary elements of all components of S is

$$\sum_{\substack{(a_1,\ldots,a_q)\in[t]^q \\ \sum a_i = t}} \prod_{i=1}^{q-1} \left(\frac{(a_i + a_{i+1})!}{a_i! a_{i+1}!} \right),$$

and the number of orderings of the secondary elements is $[(r-1)!]^{tr}$. Hence, the total number such S is

$$\sum_{\substack{(a_1,\ldots,a_q)\in[t]^q \\ \sum a_i = t}} \prod_{i=1}^{q-1} \left(\frac{(a_i + a_{i+1})!}{a_i! a_{i+1}!} \right) [(r-1)!]^{tr}.$$

Let $D := \vec{G}(S)$ denote the multi-digraph corresponding to S. Since S is r-valent, let $a_1 r, \ldots, a_q r$ be the number of components of S corresponding to the hyperedges $e_1, \ldots, e_q$, respectively. Figure 1 depicts the multi-digraph corresponds to S when $t < m$.

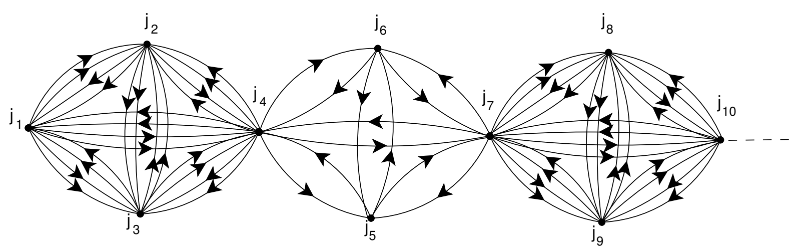

Figure 1: The multi-digraph D, when $r = 4$ and $(a_1, a_2, a_3, \ldots) = (2, 1, 2, \ldots)$

Also,

$$b(S) = \prod_{i=1}^{q} (a_i!)^{r(r-1)}, \ \pi_S(\mathcal{F}_{C_m^{(r)}}) = \left[\frac{h}{(r-1)!} \right]^{tr} \text{ and}$$

$$c(S) = (a_1(r-1))![(a_q(r-1))!] \prod_{i=1}^{q-1} [(a_i(r-1))!]^{r-2} [((a_i + a_{i+1})(r-1))!].$$

By using Matrix-tree theorem [15], we have the number of spanning trees in $\vec{G}(S)$ is equal to $r^{q(r-2)} \prod_{i=1}^{q} a_i^{r-1}$. Also, by using the BEST theorem [38], the number of Eulerian circuits in $\vec{G}(S)$ is $Z :=$

$$\left(r^{q(r-2)} \prod_{i=1}^{q} a_i^{r-1} \right) (a_1(r-1)-1)![(a_q(r-1)-1)!] \prod_{i=1}^{q-1} [(a_i(r-1)-1)!]^{r-2} [((a_i + a_{i+1})(r-1)-)!].$$

Since $\mathbf{W}(S)$ does not contain repeated elements and if we label each edge of $\vec{G}(S)$, we get $|\mathbf{W}(S)|$ is equal to

$$\frac{tr(r-1)Z}{\prod_{i=1}^{q} (a_i!)^{r(r-1)}}.$$

Now, by substituting these in Equation (2.1) and on simplification we get the desired result.
Case 3: All components $j_i \beta_i$ of S correspond to all the hyperedges of the hypergraph ($1 \leq i \leq tr$).

Subcase 3.1: S is an appropriate ordering of $(j_1\beta_1^{(1)},\dots,j_1\beta_1^{(a_m)},j_1\beta_2^{(1)},\dots,j_1\beta_2^{(a_1)},\dots,$ $j_r\beta_{r+1}^{(1)},\dots,j_r\beta_{r+1}^{(a_1)},j_r\beta_{r+2}^{(1)},\dots,j_r\beta_{r+2}^{(a_2)},\dots,j_{2r-1}\beta_{2r+1}^{(1)},\dots,j_{2r-1}\beta_{2r+1}^{(a_2)},$ $j_{2r-1}\beta_{2r+2}^{(1)},\dots,j_{2r-1}\beta_{2r+2}^{(a_3)},\dots,j_{mr-m}\beta_{mr}^{(1)},\dots,j_{mr-m}\beta_{mr}^{(a_m)})$. Throughout the chapter, let $a_{m+1}:=a_1$. Let $D_1:=\vec{G}(S)$ be the multi-digraph corresponding to S. Figure 2 depicts the symmetric (Subcase 3.1) multi-digraph D_1 corresponds to S when $t\geq m$.

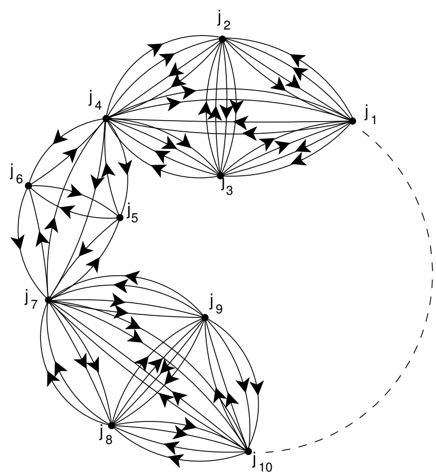

Figure 2: The multi-digraph D_1, when $r=4$ and $(a_1,a_2,a_3,\dots)=(2,1,2,\dots)$.

Then, we can see that the number of orderings of the primary elements of S is

$$\sum_{\substack{(a_1,\dots,a_m)\in[t]^m\\ \sum a_i=t}}\prod_{i=1}^{m}\frac{(a_i+a_{i+1})!}{a_i!a_{i+1}!},\ \text{where } a_{m+1}:=a_1.$$

(It is equal to 2^m, when all a_i's are equal to one. That is, when $t=m$). The number of orderings of the secondary elements is $[(r-1)!]^{tr}$. Hence, the total number of such S is

$$[(r-1)!]^{tr}\left[\sum_{\substack{(a_1,\dots,a_m)\in[t]^m\\ \sum a_i=t}}\prod_{i=1}^{m}\frac{(a_i+a_{i+1})!}{a_i!a_{i+1}!}\right].$$

Also,

$$b(S)=\prod_{i=1}^{m}(a_i!)^{r(r-1)},\ c(S)=\prod_{i=1}^{m}[((a_i+a_{i+1})(r-1))!][a_i(r-1)!]^{(r-2)},(a_{m+1}:=a_1),$$
$$\pi_S(\mathcal{F}_{C_m^{(r)}})=\left[\frac{h}{(r-1)!}\right]^{tr}.$$

By using Matrix-tree theorem [15], we have the number of spanning trees in D_1 is equal to

$$2\prod_{i=1}^{m}(a_i^{r-2})\left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-1}}B^{\pi}\right)r^{m(r-2)+1},\ \text{where } A=\{a_1,\dots,a_m\} \text{ is a multiset.}$$

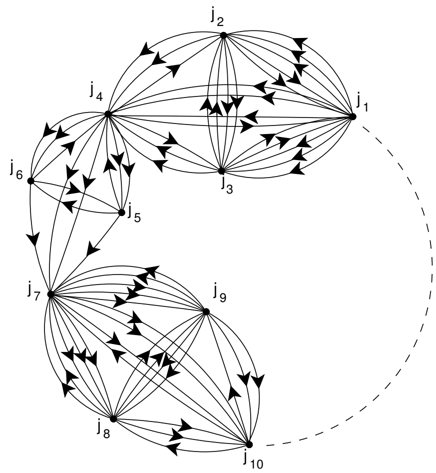

Figure 3: The multi-digraph D_2, when $r = 4$, $(a_1, a_2, a_3, \ldots) = (2, 1, 2, \ldots)$ and $d = 1$.

(when $t = m$, it is equal to $2mr^{m(r-2)-1}$), and hence by using the BEST theorem [38], the number of Eulerian circuits in D_1 is given by

$$\mathcal{E}_1 = 2\prod_{i=1}^{m}(a_i^{r-2}[((a_i + a_{i+1})(r-1) - 1)!][(a_i(r-1) - 1)!]^{(r-2)}) \left(\sum_{\substack{B\in\mathcal{P}'(A)\\|B|=m-1}} B^{\pi}\right) r^{m(r-2)+1}$$

(when $t = m$, it is equal to $2mr^{m(r-2)-1}[(2(r-1)-1)!]^m[(r-2)!]^{m(r-2)}$). Since $\mathbf{W}(S)$ does not contain repeated elements and if we label each arcs of D_1, we have

$$|\mathbf{W}(S)| = \frac{tr(r-1)\mathcal{E}_1}{\prod_{i=1}^{m}(a_i!)^{r(r-1)}}.$$

(when $t = m$, $|\mathbf{W}(S)| = 2m^2(r-1)r^{m(r-2)}[(2(r-1)-1)!]^m[(r-2)!]^{m(r-2)}$). By using Theorem 2.2, the total contribution of such S to $Tr_{tr}(\mathcal{F}_{C_m^{(r)}})$ is

$$2th^{tr}r^{m(r-2)}\left(\sum_{\substack{(a_1,\ldots,a_m)\in[t]^m\\ \sum a_i=t}} \frac{(a_i + a_{i+1} - 1)!}{a_i!a_{i+1}!}\right)\left(\sum_{\substack{B\in\mathcal{P}'(A)\\|B|=m-1}} B^{\pi}\right)$$

(when $t = m$, it is equal to $2m^2h^{mr}r^{m(r-2)}$).

Subcase 3.2: For a fixed $d(1 \le d \le z$, where $z = \min_{1\le i\le m} a_i)$, S is an appropriate ordering of $(\underbrace{j_1\beta_1,\ldots,j_1\beta_1}_{a_m-d\text{ times}},$ $\underbrace{j_1\beta_2,\ldots,j_1\beta_2}_{a_1+d\text{ times}},\ldots,\underbrace{j_r\beta_{r+1},\ldots,j_r\beta_{r+1}}_{a_1-d\text{ times}}\underbrace{j_r\beta_{r+2},\ldots,j_r\beta_{r+2}}_{a_2+d\text{ times}},\ldots,\underbrace{j_{2r-1}\beta_{2r+1},\ldots,j_{2r-1}\beta_{2r+1}}_{a_2-d\text{ times}}$ $\underbrace{j_{2r-1}\beta_{2r+2},\ldots,j_{2r-1}\beta_{2r+2}}_{a_3+d\text{ times}},\ldots,j_{mr-m}\beta_{mr})$.

Let $D_2 := \vec{G}(S)$ be the multi-digraph corresponding to S. Figure 3 depicts the asymmetric (Subcase 3.2) multi-digraph D_2 corresponds to S with $d = 1$, when $t \ge m$.
Then, we can see that the number of orderings of the primary elements of S is

$$2\sum_{d=1}^{z}\sum_{\substack{(a_1,\ldots,a_m)\in[t]^m\\ \sum a_i=t}}\prod_{i=1}^{m}\left(\frac{(a_i + a_{i+1})!}{(a_i - d)!(a_{i+1} - d)!}\right), \text{ where } z = \min_{1\le i\le m} a_i,$$

and the number of orderings of the secondary elements of S is $[(r-1)!]^{tr}$. Hence, the total number of such S is

$$2[(r-1)!]^{tr}\sum_{d=1}^{z}\sum_{\substack{(a_1,\ldots,a_m)\in[t]^m\\ \sum a_i=t}}\prod_{i=1}^{m}\left(\frac{(a_i+a_{i+1})!}{(a_i+d)!(a_{i+1}-d)!}\right).$$

Also,

$$\begin{aligned} b(S) &= \prod_{i=1}^{m}\left[((a_i+d)!)^{(r-1)}(a_i!)^{(r-2)(r-3)+2(r-2)}((a_i-d)!)^{r-1}\right] \\ &= \prod_{i=1}^{m}\left[((a_i+d)!)^{(r-1)}(a_i!)^{(r-2)(r-1)}((a_i-d)!)^{r-1}\right], \end{aligned}$$

$$c(S)=\prod_{i=1}^{m}[((a_i+a_{i+1})(r-1))!][a_i(r-1)!]^{(r-2)}, (a_{m+1}:=a_1),\ \pi_S(\mathcal{F}_{C_m^{(r)}})=\left[\frac{h}{(r-1)!}\right]^{tr}.$$

By using Matrix-tree theorem [15], we have the number of spanning trees in D_2 is equal to

$$2\prod_{i=1}^{m}(a_i^{r-2})\left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-2p-1\\ p=0,1,\ldots\lceil\frac{m}{2}\rceil-1}} d^{2p}B^{\pi}\right)r^{m(r-2)-1}, \text{ where } A=\{a_1,\ldots,a_m\} \text{ is a multiset.}$$

(When $t=m$, it is equal to $2^m r^{m(r-2)-1}$). Hence by using the BEST theorem [38], the number of Eulerian circuits in D_2 is given by

$$\mathcal{E}_2=2\prod_{i=1}^{m}(a_i^{r-2}[((a_i+a_{i+1})(r-1)-1)!][(a_i(r-1)-1)!]^{(r-2)})\left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-2p-1\\ p=0,1,\ldots,\lceil\frac{m}{2}\rceil-1}} d^{2p}B^{\pi}\right)r^{m(r-2)-1}.$$

Since $\mathbf{W}(S)$ does not contain repeated elements, and if we label each arcs of D_2, we have

$$|\mathbf{W}(S)|=\frac{tr(r-1)\mathcal{E}_2}{\prod_{i=1}^{m}\left[((a_i+d)!)^{(r-1)}(a_i!)^{(r-2)(r-1)}((a_i-d)!)^{r-1}\right]}.$$

By using Theorem 2.2, the total contribution of such S to $Tr_{tr}(\mathcal{F}_{C_m^{(r)}})$ is

$$2t\left(\sum_{\substack{(a_1,\ldots,a_m)\in[t]^m\\ \sum a_i=t}}\frac{(a_i+a_{i+1}-1)!}{a_i!a_{i+1}!}\right)\left(\sum_{\substack{B\in\mathcal{P}'(A)\\ |B|=m-2p-1\\ p=0,1,\ldots,\lceil\frac{m}{2}\rceil-1}} d^{2p}B^{\pi}\right)h^{tr}r^{m(r-2)}.$$

Now, on combining the above two subcases, we get the desired result. □

In the following corollary, we give the expression for the characteristic polynomial of $\mathcal{F}_{C_5^{(r)}}$ by using Theorems 3.1 and 3.3.

Corollary 3.1 *Let $C_5^{(r)}$ be an r-uniform hypercycle of length 5 and $\mathcal{F}_{C_5^{(r)}}$ be its degree-based extended adjacency hypermatrix. Then, the characteristic polynomial of $\mathcal{F}_{C_5^{(r)}}$ is given by*

$$\lambda^{c_0}(\lambda^r - h^r)^{c_1}(\lambda^r - 2h^r)^{c_2}(\lambda^r - 3h^r)^{c_3}(\lambda^r - 4h^r)^{c_4}\left(\lambda^r - \frac{3+\sqrt{5}}{2}h^r\right)^{c'}\left(\lambda^r - \frac{3-\sqrt{5}}{2}h^r\right)^{c'},$$

where $c_0 = 5(r-1)^{5r-5} - 5r^{r-1}(r-1)^{4r-5} + 5r^{2r-3}(r-1)^{3r-4} + 35r^{3r-5}(r-1)^{2r-3} + 5r^{4r-7}(r-1)^{r-2} - 85r^{5r-10}$
$c_1 = 5r^{r-2}(r-1)^{4r-5} - 10r^{2r-4}(r-1)^{3r-4} - 35r^{3r-6}(r-1)^{2r-3} + 5r^{4r-8}(r-1)^{r-2} + 130r^{5r-11}$
$c_2 = 5r^{2r-4}(r-1)^{3r-4} - 10r^{3r-6}(r-1)^{2r-3} + 5r^{4r-8}(r-1)^{r-2} - 180r^{5r-11}$, $c_3 = 5r^{4r-8}(r-1)^{r-2} + 10r^{5r-11}$
$c_4 = 5r^{5r-11}$, $c' = 5r^{3r-6}(r-1)^{2r-3} - 10r^{4r-8}(r-1)^{r-2} + 60r^{5r-11}$, *and* $h = f(2,2,1,\ldots,1)$.

Proof: From Theorem 3.3, we have $Tr_r(\mathcal{F}_{C_5^{(r)}}) = 5h^r r^{r-1}(r-1)^{4r-9}$
$Tr_{2r}(\mathcal{F}_{C_5^{(r)}}) = h^{2r}(5r^{r-2}(r-1)^{4r-5} + 10r^{2r-4}(r-1)^{3r-4})$
$Tr_{3r}(\mathcal{F}_{C_5^{(r)}}) = h^{3r}(5r^{r-2}(r-1)^{4r-5} + 30r^{2r-4}(r-1)^{3r-4} + 15r^{3r-6})(r-1)^{2r-3})$
$Tr_{4r}(\mathcal{F}_{C_5^{(r)}}) = h^{4r}(5r^{r-2}(r-1)^{4r-5} + 70r^{2r-4}(r-1)^{3r-4} + 80r^{3r-6}(r-1)^{2r-3} + 20r^{4r-8}(r-1)^{r-2})$
$Tr_{5r}(\mathcal{F}_{C_5^{(r)}}) = h^{5r}(5r^{r-2}(r-1)^{4r-5} + 150r^{2r-4}(r-1)^{3r-4} + 300r^{3r-6}(r-1)^{2r-3} + 150r^{4r-8}(r-1)^{r-2} + 300r^{5r-11})$
By using Theorem 3.1, we can see that $\lambda^r \in \{0, h^r, 2h^r, 3h^r, 4h^r, \frac{3+\sqrt{5}}{2}h^r, \frac{3-\sqrt{5}}{2}h^r\}$. Also, from Theorem 2.3, we have $Tr_r(\mathcal{F}_{C_5^{(r)}}) = rh^r(c_1 + 2c_2 + 3c_3 + 4c_4 + 3c')$
$Tr_{2r}(\mathcal{F}_{C_5^{(r)}}) = rh^{2r}(c_1 + 4c_2 + 9c_3 + 16c_4 + 7c')$
$Tr_{3r}(\mathcal{F}_{C_5^{(r)}}) = rh^{3r}(c_1 + 8c_2 + 27c_3 + 64c_4 + 18c')$
$Tr_{4r}(\mathcal{F}_{C_5^{(r)}}) = rh^{4r}(c_1 + 16c_2 + 81c_3 + 256c_4 + 47c')$
$Tr_{5r}(\mathcal{F}_{C_5^{(r)}}) = rh^{5r}(c_1 + 32c_2 + 243c_3 + 1024c_4 + 123c')$.
Also, the total number of eigenvalues of a hypermatrix corresponding to $C_m^{(r)}$ is $c_0 + r(c_1 + c_2 + c_3 + c_4 + 2c') = m(r-1)^{m(r-1)}$. By solving the above set of equation, we get the desired result. □

4. Conclusion

It is certain to be challenging to compute the characteristic polynomial of the hypermatrices associated with the uniform hypergraphs. This article considers the degree-based extended adjacency (DBEA) hypermatrix associated with the hypergraph and provides the expression for the coefficients of the characteristic polynomial of the DBEA hypermatrix of an r-uniform hypercycle (by combining Theorems 3.2 and 3.3). It is also evident that the characteristic polynomial of the hypercycle can be derived by replacing $f(d,d,1,\ldots,1) = 1$ in Theorem 3.1 and $h = 1$ in Theorem 3.3.

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Shashwath S. Shetty,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal,
India.
E-mail address: shashwathsshetty01334@gmail.com

and

K. Arathi Bhat,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal,
India.
E-mail address: arathi.bhat@manipal.edu