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On Action of (σ,τ)- Derivations on Lie Ideals in Prime Semirings

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ABSTRACT: In this paper, we examine the properties of (σ,τ)-derivation in additively regular prime semirings. Additionally, we explore the impact of (σ,τ)-derivations on Lie ideals and derive important consequences, leading to some significant results such as whenever the condition $[(l)^d, l]_{(\sigma,\tau)} \subseteq C_{(\sigma,\tau)}\ \forall\ l \in L$ holds, the lie ideal L lies in the centre of S and generalization of the Posner's commutativity theorems in the framework of prime semirings.

Keywords: Additively regular prime semirings, commutator, derivations, (σ,τ)-derivations, commutativity.

Contents

1. Introduction

The study of additive maps in rings that behaved like derivations originated with Herstein's work (cf. [5,6]). Later in 1957, Posner proved a significant theorem on commutativity for rings with derivations and over the years, numerous researchers have extensively studied the connection between the commutativity of prime rings and derivations (cf. [1,2,4,7]). Derivations with commutators play a crucial role in both ring and semirings theories. However, the absence of additive inverses in semirings adds complexity to their commutator structure, which makes it more challenging to study. So to overcome this, the concept of pseudo inverse, pioneered by Karvellas ([13]), helps the researchers to investigate derivations using commutators in semirings. For further insights into semiring theory, notable contribution can be found in (cf. [10,11,14]). In view of [10], an element s_1 of a semiring S is called additively regular if and only if there exists an element s'_1 of S which satisfies $s_1 + s_1 + s'_1 = s_1$ and $s'_1 + s'_1 + s_1 = s'_1$ and S is called an additively regular semiring if and only if $S = S' = reg(S)$, where $reg(S)$ represents the set of all additively regular elements of S and the element s'_1 is called the pseudo inverse of $s_1 \in S$.

Jacobson introduced the idea of (s_1, s_2) derivations, which are now commonly known as (σ,τ) or (θ,ϕ) derivations. In 1997 Deng, Yenigul et al. studied the (σ,τ)-derivations in prime rings. Afterwards, the research on (σ,τ)-derivations in prime and semiprime rings have been extensive and they have proven instrumental in resolving functional equations and other ring properties. The above discussion motivated us to investigate the concept of (σ,τ)-derivations in semirings, exploring their properties and behaviour in this algebraic structure. By (σ,τ)-derivations we mean an additive map $d : S \to S$ such that $(p_1q_1)^d = (p_1)^d(q_1)^\sigma + (p_1)^\tau(q_1)^d$, $\forall\ p_1, q_1 \in S$, where σ,τ are any two automorphisms on S. For any $p_1, q_1 \in S$, we have the commutator $[p_1, q_1] = p_1q_1 + q_1p'_1$ whereas (σ,τ)-commutator we have $[p_1, q_1]_{(\sigma,\tau)} = p_1(q_1)^\sigma + (q_1)^\tau(p_1)'$. We denote (σ,τ)-centre of S as $C_{(\sigma,\tau)} = \{c \in S \mid c(p_1)^\sigma + (p'_1)^\tau c = 0\ \forall\ p_1 \in S\}$. A simple proof reveals that the $C_{(\sigma,\tau)}$ is a proper subset of $Z(S)$ centre of a semiring S. In addition, d is a (σ,τ)-commuting derivation of S if $[p_1, q_1]_{(\sigma,\tau)} = 0\ \forall\ p_1, q_1 \in S$.

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We examined some basic identities and results regarding (σ,τ)-derivations which are the backbone of this paper in second section. Further, in the next section, we extend and generalize some results of [9,11,12]. Next, we also extend Herstein's result in the framework of prime semirings for non-zero derivations by investigating the implications of $[(p)^d,(q)^d]=0$ on semiring structure. Throughout this paper, S represents an additively regular prime semiring satisfying A_2-condition i.e., $q_1(p_1+p_1')=(p_1+p_1')q_1,\ \forall\ p_1,q_1\in S$, as introduced in [3] for all $p_1\in S$ with $p_1+p_1'=p_o$ and σ, τ are non-zero automorphisms of S. For basic definitions not provided here, one can refer to (cf. [7,8,10]).

2. Some Basic Results

This section presents several fundamental identities and results essential for the development of the paper. We now proceed with some examples:

Example 2.1 *[11] Consider $S=\{0,1,\alpha,\beta\}$, where $0,1,\alpha,\beta$ are additively idempotent elements of S. We define binary operation by Cayley tables:*

$\oplus$	0	1	α	β
0	0	1	α	β
1	1	1	α	β
α	α	α	α	β
β	β	β	β	β

$\otimes$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	α	β
β	0	β	β	β

From above tables, it is clear that $a'=a,\ \forall\ a\in S$ and S is a proper additively regular semiring with A_2-condition.

The next example shows that every additively regular semiring may not satisfy the A_2-condition.

Example 2.2 *[7] Let $S=\{0,1\}$ with operation defined by $a+b=max(a,b)$ and $a\cdot b=min(a,b)$. Then $(S,+,\cdot,0,1)$ is a semiring. This semiring is additively regular, since for any $a\in S$, we have $a+0+a=a$. However, it does not satisfy the A_2-condition, as there exists no $x\in S$ such that $a+x=0$ for $a=1$.*

Proposition 2.1 *[11] Let S be a semiring and satisfies A_2-condition. Then the following statements hold: (i) $p_1''=p_1$; (ii)$p_1'q_1'=(p_1'q_1)'=(p_1q_1')'=(p_1q_1)''=p_1q_1$; (iii) $(p_1q_1)'=p_1'q_1=p_1q_1'$; (iv) $(p_1+q_1)'=p_1'+q_1'$; (v) $p_1+p_1'=p_0=p_0'$; (vi) $p_1+p_0=p_1$; (vii) $p_1'+p_0=p_1'$; (viii)$p_0q_1=p_1q_0=(p_1q_1)_0=p_0q_0=q_0p_0=(q_1p_1)_0,\ \forall\ p_1,q_1\in S$.*

Proposition 2.2 *[11] "If S is a semiring and $p_1,q_1,s_1\in S$, then the following statements hold: (i) $[p_1,q_1s_1]=q_1[p_1,s_1]+[p_1,q_1]s_1$; (ii) $[p_1q_1,s_1]=p_1[q_1,s_1]+[p_1,s_1]q_1$; (iii) $[p_1+q_1,s_1]=[p_1,s_1]+[q_1,s_1]$;(v) $[p_1q_1,p_1]=p_1[q_1,p_1]$; (vi) $[p_1,[q_1,s_1]]+[q_1,[s_1,p_1]]=[[p_1,q_1],s_1]$(Jacobi Identity)".*

Proposition 2.3 *[11] If S is an additively regular prime semiring and $p_1,q_1\in S$, then $p_1+q_1=0$ implies $p_1=q_1'$.*

Proposition 2.4 *Suppose d is a (σ,τ)-derivation of S and a is any arbitrary element in S. If a annihilates the image of d i.e $ap_1^d=0,\ \forall\ p_1\in S$, then either $a=0$ or d is the zero derivation.*

The proof of above Proposition is quite easy so we omit the proof.

Proposition 2.5 *If S satisfies A_2-condition, then the following identities hold:*

(i) $[p_1q_1,s_1]_{(\sigma,\tau)}=p_1[q_1,s_1]_{(\sigma,\tau)}+[p_1,(s_1)^\tau]q_1=p_1[q_1,(s_1)^\sigma]+[p_1,s_1]_{(\sigma,\tau)}q_1$,

(ii) $[p_1,q_1s_1]_{(\sigma,\tau)}=(q_1)^\tau[p_1,s_1]_{(\sigma,\tau)}+[p_1,q_1]_{(\sigma,\tau)}(s_1)^\sigma$.

(iii) $[[p_1,q_1]_{(\sigma,\tau)},s_1]_{(\sigma,\tau)}=[p_1,[q_1,s_1]_{(\sigma,\tau)}]+[[p_1,s_1]_{(\sigma,\tau)},q_1]_{(\sigma,\tau)}\ \forall\ p_1,q_1,s_1\in S$.

Proof: (i) Let $p_1, q_1, s_1 \in S$. Then

$$
\begin{aligned}
[p_1q_1, s_1]_{(\sigma,\tau)} &= p_1q_1(s_1)^\sigma + (s_1)^\tau p_1q_1' \\
&= p_1q_1(s_1)^\sigma + (s_1)^\tau (p_1+p_0)q_1' \ \text{[by Proposition 2.1 } (iv)] \\
&= p_1q_1(s_1)^\sigma + (s_1)^\tau p_1q_1' + (s_1)^\tau p_0q_1' \\
&= p_1q_1(s_1)^\sigma + (s_1)^\tau (p_1+p_1')q_1' + (s_1)^\tau p_1q_1' \ \text{[by Proposition 2.1 } (v)] \\
&= p_1q_1(s_1)^\sigma + (p_1+p_1')(s_1)^\tau q_1' + (s_1)^\tau p_1q_1' \qquad [by\ A_2 - condition] \\
&= p_1q_1(s_1)^\sigma + p_1(s_1)^\tau q_1' + p_1'(s_1)^\tau q_1' + (s_1)^\tau p_1q_1' \\
&= p_1(q_1(s_1)^\sigma + (s_1)^\tau q_1') + p_1(s_1')^\tau q_1' + (s_1)^\tau p_1'q_1 \\
&= p_1[q_1, s_1]_{(\sigma,\tau)} + p_1((s_1)^\tau q_1)'' + (s_1)^\tau p_1'q_1 \\
&= p_1[q_1, s_1]_{(\sigma,\tau)} + p_1((s_1)^\tau q_1) + (s_1)^\tau p_1'q_1 \ \text{[by Proposition 2.1 } (ii)] \\
&= p_1[q_1, s_1]_{(\sigma,\tau)} + (p_1(s_1)^\tau + (s_1)^\tau p_1')q_1 \\
&= p_1[q_1, s_1]_{(\sigma,\tau)} + [p_1, (s_1)^\tau]q_1.
\end{aligned}
$$

For the other equality, the procedure is similar as already followed in the above solved equality.

(ii) Let $p_1, q_1, s_1 \in S$. Then

$$
\begin{aligned}
[p_1, q_1s_1]_{(\sigma,\tau)} &= p_1(q_1)^\sigma(s_1)^\sigma + (q_1)^\tau(s_1)^\tau p_1' \\
&= p_1(q_1+q_0)^\sigma(s_1)^\sigma + (q_1)^\tau(s_1+s_o)^\tau p_1' \ \text{[by Proposition 2.1 } (vi)] \\
&= p_1(q_1)^\sigma(s_1)^\sigma + p_1(q_0)^\sigma(s_1)^\sigma + (q_1)^\tau(s_1)^\tau p_1' + (q_1)^\tau(s_o)^\tau p_1' \\
&= p_1(q_1)^\sigma(s_1)^\sigma + (q_0)^\sigma p_1(s_1)^\sigma + (q_1)^\tau(s_1)^\tau p_1' + (q_1)^\tau p_1'(s_o)^\tau. \qquad [by\ A_2 - condition]
\end{aligned}
$$

Replacing s_0 by $(s_0)^{\tau^{-1}}$ and q_0 by $(q_0)^{\sigma^{-1}}$ in the above equation, we have

$$
[p_1, q_1s_1]_{(\sigma,\tau)} = p_1(q_1)^\sigma(s_1)^\sigma + (q_0)p(s_1)^\sigma + (q_1)^\tau(s_1)^\tau p_1' + (q_1)^\tau p_1'(s_0). \tag{2.1}
$$

Again, replacing s_0 by $(s_1)^\sigma$ and q_0 by $(q_1)^\tau$ in equation (2.1), we get

$$
\begin{aligned}
[p_1, q_1s_1]_{(\sigma,\tau)} &= p_1(q_1)^\sigma(s_1)^\sigma + (q_1)^\tau p_1(s_1)^\sigma + (q_1)^\tau(s_1)^\tau p_1' + (q_1)^\tau p_1'(s_1)^\sigma \\
&= (q_1)^\tau(p_1(s_1)^\sigma + (s_1)^\tau p_1') + (p_1(q_1)^\sigma + (q_1)^\tau p_1')(s_1)^\sigma \\
&= (q_1)^\tau[p_1, s_1]_{(\sigma,\tau)} + [p_1, q_1]_{(\sigma,\tau)}(s_1)^\sigma.
\end{aligned}
$$

(iii) Let $p_1, q_1, s_1 \in S$. Then

$$\begin{aligned}
[[p_1,q_1]_{(\sigma,\tau)},s_1]_{(\sigma,\tau)} &= [p_1,q_1]_{(\sigma,\tau)}(s_1)^\sigma + (s_1)^\tau[p_1,q_1]'_{(\sigma,\tau)} \\
&= (p_1(q_1)^\sigma + (q_1)^\tau p_1')(s_1)^\sigma + (s_1)^\tau(p_1(q_1)^\sigma + (q_1)^\tau p_1')' \\
&= p_1(q_1)^\sigma(s_1)^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma + (s_1)^\tau(q_1')^\tau p_1' \\
&= p_1(q_1 s_1)^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma + (s_1 q_1')^\tau p_1' \\
&= p_1((q_1+q_0)s_1)^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma \\
&\quad + (s_1(q_1'+q_0))^\tau p_1' \;\; \text{[by Proposition 2.1 } (iv)] \\
&= p_1(q_1 s_1)^\sigma + p_1(q_o s_1)^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma + (s_1 q_1')^\tau p_1' \\
&\quad + (s_1 q_o)^\tau p_1' \\
&= p_1(q_1 s_1)^\sigma + p_1(s_1 q_o)^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma + (s_1 q_1')^\tau p_1' \\
&\quad + (q_o s_1)^\tau p_1' \qquad [by\ A_2 - condition] \\
&= p_1(q_1 s_1)^\sigma + p_1(s_1 q_1)^\sigma + p_1(s_1 q_1')^\sigma + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma + (s_1 q_1')^\tau p_1' \\
&\quad + (q_1 s_1)^\tau p_1' + (q_1' s_1)^\tau p_1' \\
&= (p_1(q_1 s_1)^\sigma + (q_1 s_1)^\tau p_1') + (p_1(s_1 q_1')^\sigma + (s_1 q_1')^\tau p_1') + p_1(s_1)^\sigma(q_1)^\sigma \\
&\quad + (q_1')^\tau(s_1)^\tau p_1' + (q_1)^\tau p_1'(s_1)^\sigma + (s_1)^\tau p_1'(q_1)^\sigma \\
&= [p_1, q_1 s_1]_{(\sigma,\tau)} + [p_1, q_1 s_1']_{(\sigma,\tau)} + [p_1(s_1)^\sigma, q_1]_{(\sigma,\tau)} + [(s_1)^\tau p_1', q_1]_{(\sigma,\tau)} \\
&= [p_1, q_1 s_1 + q_1 s_1']_{(\sigma,\tau)} + [p_1(s_1)^\sigma + (s_1)^\tau p_1', q_1]_{(\sigma,\tau)} \\
&= [p_1,[q_1,s_1]]_{(\sigma,\tau)} + [[p_1,s_1]_{(\sigma,\tau)},q_1]_{(\sigma,\tau)}.
\end{aligned}$$

□

Proposition 2.6 *If $L \subseteq S$ is a Lie ideal with $[l,S] = (0)\ \forall\ l \in L$, then $L \subseteq Z(S)$.*

Proof: By hypothesis

$$[[l,m],s] = 0, \qquad \forall\ [l,m] \in L, s \in S.$$

Replace l by ml, we obtain

$$\begin{aligned} 0 &= [[ml,m],s] \\ &= [ml,m]s + s'[ml,m]. \end{aligned} \tag{2.2}$$

As $[l,S] = (0)\ \forall\ l \in L$, therefore $0 = [[l,m],s] = [l,m]s + s[l,m]' = (lm+ml')s + s(lm+ml')'\ \forall\ l,m \in L, s \in S$. This infers

$$(lm+ml')s + s(lm+ml')' + s(lm+ml') = s(lm+ml')$$

$$(lm+ml')s + s'(lm+ml') + s(lm+ml') = s(lm+ml')$$

$$(lm+ml')s + (s'+s)(lm+ml') = s(lm+ml').$$

By using A_2-condition, we have

$$(lm+ml')s + (lm+ml')(s'+s) = s(lm+ml')$$

$$(lm+ml')(s+s'+s) = s(lm+ml')$$

$$(lm+ml')s = s(lm+ml')$$

$$[l,m]s = s[l,m]. \tag{2.3}$$

Using equation (2.3), in equation (2.2), we get

$$\begin{aligned} 0 &= s[ml,m] + s'[ml,m] \\ &= (s+s')[ml,m]. \end{aligned}$$

By Proposition 2.1, we obtain $0 = s_0 m[l,m]\ \forall\ m \in S$, implies $s_0 S[l,m] = (0)$. Now, primeness of S implies that, either $s_0 = 0$ or $[l,m] = 0$. As $s_0 \neq 0$, therefore $0 = [l,m] = lm + ml'$. Thus, by Proposition 2.3, infers $lm = ml$. Hence $\forall\ l \in L$, we get $L \subseteq Z(S)$. □

Proposition 2.7 *If L_1, L_2 are two lie ideals of a semiring S and $[L_1, L_2] \subseteq Z$, then either $L_1 \subseteq Z$ or $L_2 \subseteq Z$.*

Proof: By hypothesis, we have

$$[l_1, l_2] \in Z, \qquad \forall\ l_1 \in L_1, l_2 \in L_2$$

and by the definition of centre of a semiring, we have

$$\begin{aligned} 0 &= [l_1,l_2]s + s'[l_1,l_2] \\ &= (l_1l_2 + l_2l_1')s + s'(l_1l_2 + l_2l_1'), \qquad \forall\ s \in S. \end{aligned} \tag{2.4}$$

Replace l_2 by $l_1 t$ in equation (2.4), we have

$$\begin{aligned} 0 &= (l_1l_1t + l_1tl_1')s + s'(l_1l_1t + l_1tl_1') \\ &= l_1(l_1t + tl_1')s + s'l_1(l_1t + tl_1') \\ &= l_1[l_1,t]s + s'l_1[l_1,t], \qquad \forall\ t \in S. \end{aligned}$$

By Proposition 2.3, we have $l_1[l_1,t]s = sl_1[l_1,t]$ and putting in the above equation, we get

$$\begin{aligned} 0 &= sl_1[l_1,t] + s'l_1[l_1,t] \\ &= (s+s')l_1[l_1,t] \\ &= s_0 l_1[l_1,t] \end{aligned}$$

$\forall\ s_0 l_1 \in L_1$, we have $(0) = L_1[l_1,t]$. Replacing l_1 by tl_1, we get $(0) = L_1[tl_1,t] = L_1 t[l_1,t]\ \forall\ t \in S$, we have $L_1 S[l_1,t] = (0)$. As $L_1 \neq 0$, so the primeness of S gives $[l_1,t] = 0$ and by Proposition 2.4, $L_1 \subseteq Z(S)$. Analogously, it can be shown $L_2 \subseteq Z(S)$. □

Proposition 2.8 *Let L be a non-zero Lie ideal of S and $d : S \to S$ be a (σ,τ) -derivation. If $d = 0$ on L, then $d = 0$ on S.*

Proof: By hypothesis

$$[l_1, s_1]^d = 0, \qquad \forall\ l_1, s_1 \in L. \tag{2.5}$$

Replace l_1 by $s_1 l_1$, in equation (2.5), we have $0 = [s_1l_1, s_1]^d = (s_1[l_1,s_1])^d = (s_1)^d[l_1,s_1]^\sigma + (s_1)^\tau[l_1,s_1]^d = (s_1)^d[l_1,s_1]^\sigma$. Again, replace l by l_1s_1, we have $0 = (s_1)^d[l_1s_1,s_1]^\sigma = (s_1)^d([l_1,s_1]s_1)^\sigma = (s_1)^d[l_1,s_1]^\sigma(s_1)^\sigma\ \forall\ s_1 \in S$. Since $\forall\ [l_1,s_1]^\sigma \in S$, we have $(s_1)^d S(s_1)^\sigma = (0)$. By primeness of S and $\sigma \neq 0$ on S, then $d = 0$ on S. □

3. Results on (σ,τ)-Derivations of Prime Semirings

This section investigates the action of (σ,τ)-derivations on Lie ideals of S. We begin with generalizing some results of [11] for (σ,τ)-derivations on prime semirings.

Theorem 3.1 *Let d be a (σ,τ)-commuting derivation of a semiring S. Then either S is commutative or d is the zero derivation.*

Proof: By hypothesis,

$$0 = [p_1, q_1]_{(\sigma,\tau)}, \qquad \forall\ p_1, q_1 \in S. \tag{3.1}$$

Putting p_1s_1 in place of p_1 in (3.1), we get

$$\begin{aligned} 0 &= [p_1s_1, q_1]_{(\sigma,\tau)} \\ &= p_1[s_1,(q_1)^\sigma] + [p_1,q_1]_{(\sigma,\tau)}s_1 \text{ [by Proposition 2.5 } (i)] \\ &= p_1[s_1,(q_1)^\sigma] = 0, \qquad \forall\, p_1,q_1,s_1 \in S. \end{aligned}$$

Now, again replacing p_1 with $((q_1)^\sigma)^d p_1$, we get

$$\begin{aligned} 0 &= ((q_1)^\sigma)^d p_1[s_1,(q_1)^\sigma] \\ (0) &= ((q_1)^\sigma)^d S[s_1,(q_1)^\sigma], \qquad \forall\, p_1 \in S. \end{aligned}$$

Due to the primality of S, either $((q_1)^\sigma)^d = 0$ or $[s_1,(q_1)^\sigma] = 0$. If $d = 0$, then result holds trivially. If $d \neq 0$, then there exists atleast some $((z_t)^\sigma)^d \neq 0$ and gives $[s_1,(z_t)^\sigma] = 0\ \forall\, s_1 \in S$. We further claim that $[S,(q_1)^\sigma] = (0)\ \forall\, (q_1)^\sigma \in S$. If possible, let $(q_1)^\sigma (\neq (z_t)^\sigma) \in S$ with $[s_1,(q_1)^\sigma] \neq 0$. Then $((q_1)^\sigma)^d = 0$,

$$[s_1,(z_t+q_1)^\sigma] \neq 0$$

and $((z_t+q_1)^\sigma)^d \neq 0$, which is not true. Thus, $[S,(q_1)^\sigma] = (0)\ \forall\, s_1 \in S$. This implies $0 = [p_1,(q_1)^\sigma] = p_1(q_1)^\sigma + (q_1)^\sigma p_1'$. By using Proposition 2.1, we have

$$\begin{aligned} p_1(q_1)^\sigma + (q_1)^\sigma p_1' + (q_1)^\sigma p_1 &= (q_1)^\sigma p_1 \\ p_1(q_1)^\sigma + (q_1)^\sigma(p_1+p_1') &= (q_1)^\sigma p_1 \\ (p_1+p_1+p_1')(q_1)^\sigma &= (q_1)^\sigma p_1 \\ p_1(q_1)^\sigma &= (q_1)^\sigma p_1, \qquad \forall\, p_1,q_1 \in S. \end{aligned}$$

This completes the proof. □

Theorem 3.2 *Suppose d is a (σ,τ)-derivation of additively regular prime semiring S and $L \neq 0$ with $(L)^d = (0)$, then $d = 0$.*

Proof: By hypothesis, we have

$$\begin{aligned} 0 &= [l_1,s_1]^d \\ &= (l_1s_1)^d + (s_1'l_1)^d \\ &= (l_1)^d(s_1)^\sigma + (l_1)^\tau(s_1)^d + (s_1')^d(l_1)^\sigma + (s_1')^\tau(l_1)^d \\ &= (l_1)^\tau(s_1)^d + (s_1')^d(l_1)^\sigma, \qquad \forall\, l_1 \in L, s_1 \in S. \end{aligned}$$

Replacing s_1 by s_1m_1, we get

$$\begin{aligned} 0 &= (l_1)^\tau(s_1m_1)^d + (s_1'm_1)^d(l_1)^\sigma \\ &= (l_1)^\tau(s_1)^d(m_1)^\sigma + (l_1)^\tau(s_1)^\tau(m_1)^d + (s_1')^d(m_1)^\sigma(l_1)^\sigma + (s_1')^\tau(m_1)^d(l_1)^\sigma \\ &= (l_1)^\tau(s_1)^d(m_1)^\sigma + (s_1')^d(m_1)^\sigma(l_1)^\sigma, \qquad \forall\, m_1 \in L, s_1 \in S. \end{aligned}$$

By using Proposition 2.3, we have $(s_1')^d(m_1)^\sigma(l_1)^\sigma = (l_1)^\tau(s_1')^d(m_1)^\sigma$, and the above equation becomes

$$\begin{aligned} 0 &= (l_1)^\tau(s_1')^d(m_1)^\sigma + (l_1)^\tau(s_1)^d(m_1)^\sigma \\ &= (l_1)^\tau(s_1+s_1')^d(m_1)^\sigma. \end{aligned}$$

By using A_2-condition, we have

$$(l_1)^\tau(m_1)^\sigma(s_0)^d = 0.$$

Since S is an additively regular prime semiring, therefore by Proposition 2.4, either $(l_1)^\tau(m_1)^\sigma = 0$ or $(s_0)^d = 0$. As $L \neq 0$, therefore $(l_1)^\tau(m_1)^\sigma \neq 0$, implies $(s_0)^d = 0$ or $d = 0$. □

Theorem 3.3 *Let S be a 2-torsion free additively regular prime semiring, L is a Lie ideal of S. If S admits a (σ,τ)-derivation d, such that $(L)^{d^2}=(0)$ and d commutes with both σ and τ, then d=0.*

Proof: Because $[l_1,s_1]l_1 = l_1(s_1l_1)+(s_1'l_1)\in[L,S]\subseteq L$, therefore

$$\begin{aligned}
0 &= ([l_1,s_1])^{d^2}\\
&= ([l_1,s_1]^d(l_1)^\sigma+[l_1,s_1]^\tau(l_1)^d)^d\\
&= [l_1,s_1]^{d^2}(l_1)^{\sigma^2}+([l_1,s_1]^d)((l_1)^\sigma)^d+([l_1,s_1]^\tau)^d((l_1)^d)^\sigma+[l_1,s_1]^{\tau^2}(l_1)^{d^2}\\
&= ([l_1,s_1]^d)((l_1)^\sigma)^d+([l_1,s_1]^\tau)^d((l_1)^d)^\sigma, \qquad \forall\ l_1\in L, s_1\in S.
\end{aligned}$$

As d commutes with σ and τ, then we get

$$2([l_1,s_1]^\tau)^d((l_1)^d)^\sigma=0.$$

Since S is 2-torsion free and σ,τ are automorphic maps, therefore we have

$$\begin{aligned}
0 &= (([l_1,s_1]^\tau)^d((l_1)^\sigma)^d)^{\sigma^{-1}}\\
&= (([l_1,s_1]^\tau)^d)^{\sigma^{-1}}(l_1)^d
\end{aligned}$$

$\forall\ (([l_1,s_1]^\tau)^d)^{\sigma^{-1}}\in S$ and by Proposition 2.4, either $(([l_1,s_1]^\tau)^d)^{\sigma^{-1}}=0$ or $(l_1)^d=0$. Also, $(([l_1,s_1]^\tau)^d)^{\sigma^{-1}}\neq 0\ \forall\ [l_1,s_1]\in S$, therefore $(L)^d=(0)\ \forall\ l_1\in L$ gives that $d=0$. □

The subsequent theorem broadens the scope of Theorem 1 in [9], which itself is an extension of Posner's theorem.

Theorem 3.4 *Let S be a 2-torsion free additively regular prime semiring. Suppose there exists a (σ,τ)-derivation $d:S\to S$ such that $[(p_1)^d,p_1]_{(\sigma,\tau)}=0\ \forall\ p_1\in S$. Then either $d=0$ or S is commutative.*

Proof: Suppose a map $F(\cdot,\cdot):S\times S\to S$, defined by $(p_1,q_1)^F=[(p_1)^d,q_1]_{(\sigma,\tau)}+[q_1,(p_1)^d]_{(\sigma,\tau)}$, $\forall\ p_1,q_1\in S$ is a symmetric map. Then $\forall\ p_1,q_1,s_1\in S$, we have

$$\begin{aligned}
(p_1q_1,s_1)^F &= [(p_1q_1)^d,s_1]_{(\sigma,\tau)}+[(s_1)^d,p_1q_1]_{(\sigma,\tau)}\\
&= [(p_1)^d(q_1)^\sigma+(p_1)^\tau(q_1)^d,s_1]_{(\sigma,\tau)}+[(s_1)^d,p_1q_1]_{(\sigma,\tau)}\\
&= [(p_1)^d(q_1)^\sigma,s_1]_{(\sigma,\tau)}+[(p_1)^\tau(q_1)^d,s_1]_{(\sigma,\tau)}+[(s_1)^d,p_1q_1]_{(\sigma,\tau)}\\
&= (p_1)^d[(q_1)^\sigma,(s_1)^\sigma]+[(p_1)^d,s_1]_{(\sigma,\tau)}(q_1)^\sigma+(p_1)^\tau[(q_1)^d,s_1]_{(\sigma,\tau)}+[(p_1)^\tau,(s_1)^\tau](q_1)^d\\
&\quad +(p_1)^\tau[(s_1)^d,q_1]_{(\sigma,\tau)}+[(s_1)^d,p_1]_{(\sigma,\tau)}(q_1)^\sigma\ \text{[by Proposition 2.3]}\\
&= (p_1,s_1)^F(q_1)^\sigma+(p_1)^\tau(q_1,s_1)^F+(p_1)^d[q_1,s_1]^\sigma+[p_1,s_1]^\tau(q_1)^d.
\end{aligned}$$

Further, let us define a new map $f:R\to R$, by $(p_1)^f=(p_1,p_1)^F\ \forall\ p_1\in S$ induced by F. Thus $(p_1)^f=2[(p_1)^d,p_1]_{(\sigma,\tau)}\ \forall\ p_1\in S$. On linearizing this equation, we have

$$\begin{aligned}
(p_1+q_1)^f &= 2[(p_1+q_1)^d,(p_1+q_1)]_{(\sigma,\tau)}\\
&= 2[(p_1)^d+(q_1)^d,p_1+q_1]_{(\sigma,\tau)}\\
&= 2[(p_1)^d,p_1]_{(\sigma,\tau)}+2[(p_1)^d,q_1]_{(\sigma,\tau)}+2[(q_1)^d,p_1]_{(\sigma,\tau)}+2[(q_1)^d,q_1]_{(\sigma,\tau)}\\
&= (p_1)^f+2(p_1,q_1)^F+(q_1)^f.
\end{aligned}$$

However, $(p_1)^f=0$ as $[(p_1)^d,p_1]_{(\sigma,\tau)}=0$, and this leads to

$$(p_1,q_1)^F=0, \qquad \forall\ p_1,q_1\in S. \tag{3.2}$$

Replacing q_1 by p_1q_1 in above defined map, we get

$$\begin{aligned}(p_1,p_1q_1)^F &= [(p_1)^d,p_1q_1]_{(\sigma,\tau)}+[(p_1q_1)^d,p_1]_{(\sigma,\tau)}\\ &= (p_1)^\tau[(p_1)^d,q_1]_{(\sigma,\tau)}+[(p_1)^d,p_1]_{(\sigma,\tau)}(q_1)^\sigma+(p_1)^d[(q_1)^\sigma,(p_1)^\sigma]\\ &\quad +[(p_1)^d,p_1]_{(\sigma,\tau)}(q_1)^\sigma+(p_1)^\tau[(q_1)^d,p_1]_{(\sigma,\tau)}+[(p_1)^\tau,(p_1)^\tau](q_1)^d\\ &= (p_1)^\tau([(p_1)^d,q_1]_{(\sigma,\tau)}+[(q_1)^d,p_1]_{(\sigma,\tau)})+2[(p_1)^d,p_1]_{(\sigma,\tau)}(q_1)^\sigma\\ &\quad +(p_1)^d[q_1,p_1]^\sigma+[p_1,p_1]^\tau(q_1)^d\\ &= (p_1)^\tau(p_1,q_1)^F+(p_1)^f(q_1)^\sigma+(p_1)^d[q_1,p_1]^\sigma+[p_1,p_1]^\tau(q_1)^d\\ &= (p_1)^d[q_1,p_1]^\sigma+[p_1,p_1]^\tau(q_1)^d.\end{aligned}$$

From equation (3.2), we obtain

$$\begin{aligned}0 &= (p_1)^d[q_1,p_1]^\sigma+[p_1,p_1]^\tau(q_1)^d\\ &= (p_1)^d[q_1,p_1]^\sigma+(p_1')^d[q_1,p_1]^\sigma, \quad \text{[by Proposition 2.3]}\\ &= (p_1+p_1')^d[q_1,p_1]^\sigma\\ &= (p_0)^d[q_1,p_1]^\sigma, \qquad [\textit{by } A_2-\textit{condition}].\end{aligned}$$

On multiplying with σ^{-1}, we get

$$((p_0)^d)^{\sigma^{-1}}[q_1,p_1]=0. \tag{3.3}$$

Replace q_1 by p_1q_1 in equation (3.3), we have

$$\begin{aligned}0 &= ((p_0)^d)^{\sigma^{-1}}[p_1q_1,p_1]\\ &= ((p_0)^d)^{\sigma^{-1}}p_1[q_1,p_1],\end{aligned}$$

$\forall\ p_1,q_1\in S$, we have $((p_0)^d)^{\sigma^{-1}}S[q_1,p_1]=(0)$. Primeness of S, implies either $((p_0)^d)^{\sigma^{-1}}=0$ or $[q_1,p_1]=0$. This implies $(p_0)^d=0$ i.e. $d=0$ or if $[q_1,p_1]=0$, we have $q_1p_1=p_1q_1,\ \forall\ p_1,q_1\in S$, i.e, S is commutative. □

We now present a generalization of Theorem 2 in [9] which is also an extension of Herstein's Theorem 2 in [4].

Theorem 3.5 *Let S be a 2-torsion free additively regular prime semiring, and L be a non-zero lie ideal of S such that $[(p_1)^d,(q_1)^d]=0$, $\forall\ p_1,q_1\in L$ and d commutes with σ,τ. Then either $d=0$ or S is commutative.*

Proof: By hypothesis, we have

$$[(p_1)^d,(q_1)^d]=0, \qquad \forall\ p_1,q_1\in L.$$

Replace q_1 by p_1q_1, we get

$$\begin{aligned}0 &= [(p_1)^d,(p_1)^d(q_1)^\sigma+(p_1)^\tau(q_1)^d]\\ &= [(p_1)^d,(p_1)^d(q_1)^\sigma]+[(p_1)^d,(p_1)^\tau(q_1)^d]\\ &= (p_1)^d[(p_1)^d,(q_1)^d]+[(p_1)^d,(p_1)^\tau](q_1)^d\\ &= (p_1)^d[(p_1)^d,(q_1)^\sigma]+[(p_1)^d,(p_1)^\tau](q_1)^d.\end{aligned}$$

Replace q_1 by q_1s_1, we get

$$\begin{aligned}0 =&\ (p_1)^d[(p_1)^d,(q_1s_1)^\sigma]+[(p_1)^d,(p_1)^\tau](q_1s_1)^d\\ =&\ (p_1)^d[(p_1)^d,(q_1)^\sigma](s_1)^\sigma+(p_1)^d(q_1)^\sigma[(p_1)^d,(s_1)^\sigma]\\ &+[(p_1)^d,(p_1)^\tau](q_1)^d(s_1)^\sigma+[(p_1)^d,(p_1)^\tau](q_1)^\tau(s_1)^d\\ =&\ ((p_1)^d[(p_1)^d,(q_1)^\sigma]+[(p_1)^d,(p_1)^\tau](q_1)^d)(s_1)^\sigma\\ &+(p_1)^d(q_1)^\sigma[(p_1)^d,(s_1)^\sigma]+[(p_1)^d,(p_1)^\tau](q_1)^\tau(s_1)^d\\ =&\ (p_1)^d(q_1)^\sigma[(p_1)^d,(s_1)^\sigma]+[(p_1)^d,(p_1)^\tau](q_1)^\tau(s_1)^d, \qquad \forall\ q_1\in L, s_1\in S.\end{aligned}$$

Now, replace s_1 by $((s_1)^{\sigma^{-1}})^d$ and we have

$$\begin{aligned} 0 &= (p_1)^d(q_1)^\sigma[(p_1)^d,(s_1)^d]+[(p_1)^d,(p_1)^\tau](q_1)^\tau(((s_1)^{\sigma^{-1}})^d)^d \\ &= [(p_1)^d,(p_1)^\tau](q_1)^\tau(((s_1)^{\sigma^{-1}})^d)^d, \qquad \forall\ s_1\in S,\ s_1\in L. \end{aligned}$$

$\forall\ (q_1)^\tau \in S$, we have $[(p_1)^d,(p_1)^\tau]S((s_1)^{\sigma^{-1}})^{d^2} = (0)$. By the primeness of semiring S, either $[(p_1)^d,(p_1)^\tau]=0$ or $((s_1)^{\sigma^{-1}})^{d^2}=0$. If $((s_1)^{\sigma^{-1}})^{d^2}=0$, we get $d=0\ \forall\ (s_1)^{\sigma^{-1}}\in L$ and if $[(p_1)^d,(p_1)^\tau]=0$, then we will linearize the equation and get

$$\begin{aligned} 0 &= [(p_1+q_1)^d,(p_1+q_1)^\tau] \\ &= [(p_1)^d,(q_1)^\tau]+[(q_1)^d,(p_1)^\tau], \qquad \forall\ p_1,q_1\in L. \end{aligned}$$

Replace p_1 by q_1p_1, we get

$$\begin{aligned} 0 &= [(q_1p_1)^d,(q_1)^\tau]+[(q_1)^d,(q_1p_1)^\tau] \\ &= [(q_1)^d(p_1)^\sigma+(q_1)^\tau(p_1)^d,(q_1)^\tau]+[(q_1)^d,(q_1)^\tau(p_1)^\tau] \\ &= [(q_1)^d(p_1)^\sigma,(q_1)^\tau]+[(q_1)^\tau(p_1)^d,(q_1)^\tau]+[(q_1)^d,(q_1)^\tau](p_1)^\tau+(q_1)^\tau[(q_1)^d,(p_1)^\tau] \\ &= (q_1)^d[(p_1)^\sigma,(q_1)^\tau]+[(q_1)^d,(q_1)^\tau](p_1)^\sigma+(q_1)^\tau[(p_1)^d,(q_1)^\tau]+(q_1)^\tau[(q_1)^d,(p_1)^\tau] \\ &= (q_1)^d[(p_1)^\sigma,(q_1)^\tau]+(q_1)^\tau([(p_1)^d,(q_1)^\tau]+[(q_1)^d,(p_1)^\tau]) \\ &= (q_1)^d[(p_1)^\sigma,(q_1)^\tau]. \end{aligned}$$

By Proposition 2.4, we have either $(q_1)^d=0$ or $[(p_1)^\sigma,(q_1)^\tau]=0$. If $(q_1)^d=0$, then $d=0$ and the result is trivial, whereas if $[(p_1)^\sigma,(q_1)^\tau]=0$, we can put $p_1=p_1(s_1)^{\sigma^{-1}}$ and obtain

$$\begin{aligned} 0 &= (q_1)^d[(p_1)^\sigma s_1,(q_1)^\tau] \\ &= (q_1)^d(p_1)^\sigma[s_1,(q_1)^\tau]+(q_1)^d[(p_1)^\sigma,(q_1)^\tau]s_1 \\ &= (q_1)^d(p_1)^\sigma[s_1,(q_1)^\tau], \end{aligned}$$

$\forall\ (p_1)^\sigma\in S$, we have $(q_1)^dS[s_1,(q_1)^\tau]=(0)$. Therefore, by using primeness of S, either $(q)^d=0$ or $[s_1,(q_1)^\tau]=0$. If $(q_1)^d=0$, then $d=0$ and if $[s_1,(q_1)^\tau]=0$, implies $s_1(q_1)^\tau+(q_1)^\tau s_1'=0$

$$s_1(q_1)^\tau=(q_1)^\tau s_1.$$

Hence S commutative. □

Theorem 3.6 *Let S be a 2-torsion free additively regular prime semiring, and L be a non-zero lie ideal of S. If S admits a non-zero (σ,τ)-derivation d such that $[p_1,q_1]^d+[q_1,p_1]^{d'}=0$, $\forall\ p_1,q_1\in L$, and d commutes with σ,τ, then S is commutative.*

Proof: By hypothesis, we have

$$\begin{aligned} 0 &= [p_1,q_1]^d+[q_1,p_1]^{d'} \\ &= (p_1q_1+q_1p_1')^d+(q_1p_1+p_1q_1')^{d'} \\ &= 2((p_1q_1)^d+(q_1p_1')^d), \qquad \forall\ p_1,q_1\in L. \end{aligned}$$

Since it is a 2-torsion free additively regular prime semiring, we get

$$\begin{aligned} 0 &= (p_1q_1)^d+(q_1p_1')^d \\ &= [(p_1)^d,q_1]_{(\sigma,\tau)}+(p_1)^\tau(q_1)^d+(q_1)^d(p_1')^\sigma. \end{aligned}$$

Replace p_1' by $((p_1')^\tau)^{\sigma^{-1}}$, we obtain

$$[(p_1)^d,q_1]_{(\sigma,\tau)}+(p_1)^\tau(q_1)^d+(q_1)^d(p_1')^\tau=0.$$

Now, replace p_1 by $((p_1)^{\sigma})^{\tau^{-1}}$, we get

$$[(p_1)^d, q_1]_{(\sigma,\tau)} + (p_1)^{\sigma}(q_1)^d + (q_1)^d(p_1')^{\tau} = 0.$$

Since σ, τ commutes with d, thus

$$\begin{aligned} 0 &= [(p_1)^d, q_1]_{(\sigma,\tau)} + (p_1)^d(q_1)^{\sigma} + (q_1)^{\tau}(p_1')^d \\ &= [(p_1)^d, q_1]_{(\sigma,\tau)} + [(p_1)^d, q_1]_{(\sigma,\tau)} \\ &= 2[(p_1)^d, q_1]_{(\sigma,\tau)}. \end{aligned}$$

Since S is 2-torsion free, therefore $[(p_1)^d, q_1]_{(\sigma,\tau)} = 0$, and now by replacing q_1 with $[p_1, q_1]$, we conclude

$$\begin{aligned} 0 &= [(p_1)^d, [p_1, q_1]]_{(\sigma,\tau)} \\ &= (p_1)^d[p_1, q_1]^{\sigma} + [p_1, q_1]^{\tau}(p_1')^d. \end{aligned} \tag{3.4}$$

By using Proposition 2.3, we have $[p_1, q_1]^{\tau}(p_1')^d = (p_1)^d[p_1, q_1]^{\sigma}$ and equation (3.4), becomes

$$\begin{aligned} 0 &= 2(p_1)^d[p_1, q_1]^{\sigma} \\ &= (p_1)^d[p_1, q_1]^{\sigma}. \end{aligned}$$

On multiplying with σ^{-1}, we get

$$((p_1)^d)^{\sigma^{-1}}[p_1, q_1] = 0.$$

As d commutes with σ, therefore

$$((p_1)^{\sigma^{-1}})^d[p_1, q_1] = 0.$$

Using Proposition 2.4, either $[p_1, q_1] = 0$ or $((p_1)^{\sigma^{-1}})^d = 0$. Because d is a non-zero derivation of S, we have $0 = [p_1, q_1] = p_1q_1 + q_1'p_1$ and by Proposition 2.3, we get $p_1q_1 = q_1p_1 \ \forall \ p_1, q_1 \in S$. Hence S is commutative. □

The forthcoming Theorem is an extension to Posner's first theorem.

Theorem 3.7 *Let S be a 2-torsion free additively regular prime semiring and d_1, d_2 be any two (σ,τ)-derivations of S and $(p_1)^{d_1}(p_1)^{d_2} = 0 \ \forall \ p_1 \in S$. Then either $d_1 = 0$ or $d_2 = 0$.*

Proof: By hypothesis, we have

$$(p_1)^{d_1}(p_1)^{d_2} = 0.$$

Linearizing the above equation, we get

$$\begin{aligned} 0 &= ((p_1)^{d_1} + (q_1)^{d_1})((p_1)^{d_2} + (q_1)^{d_2}) \\ &= (p_1)^{d_1}(p_1)^{d_2} + (p_1)^{d_1}(q_1)^{d_2} + (q_1)^{d_1}(p_1)^{d_2} + (q_1)^{d_1}(q_1)^{d_2} \\ &= (p_1)^{d_1}(q_1)^{d_2} + (q_1)^{d_1}(p_1)^{d_2}, \qquad \forall \ p_1, q_1 \in S. \end{aligned}$$

Replacing q_1 by p_1q_1, and using the above equation, we get

$$\begin{aligned} 0 &= (p_1)^{d_1}(p_1q_1)^{d_2} + (p_1q_1)^{d_1}(p_1)^{d_2} \\ &= (p_1)^{d_1}((p_1)^{d_2}(q_1)^{\sigma} + (p_1)^{\tau}(q_1)^{d_2}) + ((p_1)^{d_1}(q_1)^{\sigma} + (p_1)^{\tau}(q_1)^{d_2})(p_1)^{d_2} \\ &= (p_1)^{d_1}(p_1)^{d_2}(q_1)^{\sigma} + (p_1)^{d_1}(p_1)^{\tau}(q_1)^{d_2} + (p_1)^{d_1}(q_1)^{\sigma}(p_1)^{d_2} + (p_1)^{\tau}(q_1)^{d_2}(p_1)^{d_2}. \end{aligned}$$

Since d_2 commutes with τ, we get

$$\begin{aligned} 0 &= (p_1)^{d_1}(p_1)^{d_2}(q_1)^{\sigma} + (p_1)^{d_1}(p_1)^{d_2}(q_1)^{\tau} + (p_1)^{d_1}(q_1)^{\sigma}(p_1)^{d_2} + (p_1)^{\tau}(q_1)^{d_2}(p_1)^{d_2} \\ &= (p_1)^{d_1}(q_1)^{\sigma}(p_1)^{d_2} + (p_1)^{\tau}(q_1)^{d_1}(p_1)^{d_2}, \qquad \forall \ p_1, q_1 \in S. \end{aligned}$$

By using Proposition 2.3, we have $(p_1)^\tau (q_1)^{d_1} (p_1)^{d_2} = (p_1)^{d_1} (q_1')^\sigma (p_1)^{d_2}$, and the above equation changes to

$$\begin{aligned} 0 &= (p_1)^{d_1} (q_1)^\sigma (p_1)^{d_2} + (p_1)^{d_1} (q_1')^\sigma (p_1)^{d_2} \\ &= (p_1)^{d_1} (q_1 + q_1')^\sigma (p_1)^{d_2} \\ &= (p_1)^{d_1} (q_0)^\sigma (p_1)^{d_2}, \end{aligned}$$

$\forall\ (q_0)^\sigma \in S$, we have $(p_1)^{d_1} S (p_1)^{d_2} = (0)$. By primeness of S, we have either $(p_1)^{d_1} = 0$ or $(p_1)^{d_2} = 0$, implies either $d_1 = 0$ or $d_2 = 0$. □

Theorem 3.8 *Let L be a Lie ideal of S such that $[(l_1)^d, l_1]_{(\sigma,\tau)} = 0\ \forall\ l_1 \in L$. Then $L \subseteq Z(S)$.*

Proof: By hypothesis

$$[(l_1)^d, l_1]_{(\sigma,\tau)} = 0 \qquad \forall\ l_1 \in L. \tag{3.5}$$

On Linearizing equation (3.5), we get

$$[(l_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1]_{(\sigma,\tau)} = 0 \qquad \forall\ l_1, m_1 \in L. \tag{3.6}$$

Replacing l_1 by $l_1 m_1$ in equation (3.6) and it gives

$$\begin{aligned} 0 &= [(l_1 m_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1 m_1]_{(\sigma,\tau)} \\ &= [(l_1)^d (m_1)^\sigma, m_1]_{(\sigma,\tau)} + [(l_1)^\tau (m_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1 m_1]_{(\sigma,\tau)}. \end{aligned} \tag{3.7}$$

By using Proposition 2.5, we get

$$\begin{aligned} &(l_1)^d [(m_1)^\sigma, (m_1)^\sigma] + [(l_1)^d, m_1]_{(\sigma,\tau)} (m_1)^\sigma + (l_1)^\tau [(m_1)^d, m_1]_{(\sigma,\tau)} \\ &+ [(l_1)^\tau, (m_1)^\tau] (m_1)^d + (l_1)^\tau [(m_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1]_{(\sigma,\tau)} (m_1)^\sigma = 0. \end{aligned}$$

Using equation (3.6) and (3.5), we get

$$(l_1)^d [(m_1)^\sigma, (m_1)^\sigma] + [(l_1)^\tau, (m_1)^\tau] (m_1)^d = 0.$$

By using Proposition 2.3, $(l_1)^d [(m_1)^\sigma, (m_1)^\sigma] = [(l_1)^\tau, (m_1)^\tau] (m_1')^d$, and the previous equation becomes

$$\begin{aligned} 0 &= [(l_1)^\tau, (m_1)^\tau] (m_1')^d + [(l_1)^\tau, (m_1)^\tau] (m_1)^d \\ &= [l_1, m_1]^\tau (m_1 + m_1')^d \\ &= [l_1, m_1]^\tau (m_0)^d. \end{aligned} \tag{3.8}$$

Replace m_1 by $m_1 l_1$, in equation (3.8), we have

$$\begin{aligned} 0 &= [l_1, m_1]^\tau (l_1)^\tau (m_0)^d \\ &= [l_1, m_1]^\tau (l_1)^\tau (m_0)^d, \qquad \forall\ l_1, m_1 \in L. \end{aligned}$$

As the previous equation holds $\forall\ (l_1)^\tau \in S$, so we deduce that $[l_1, m_1]^\tau S (m_0)^d = (0)$ and by primeness of S, either $[l_1, m_1]^\tau = 0$ or $(m_0)^d = 0$. Since d is a non-zero derivation, we have $[l_1, m_1]^\tau = 0$ or $[l_1, m_1] = 0$ as τ is an automorphism. Hence $L \subseteq Z(S)$. This completes the proof. □

Theorem 3.9 *Let L be a nonzero Lie ideal of S. Suppose S admits a nonzero (σ, τ)-derivation d such that $[(l_1)^d, l_1]_{(\sigma,\tau)} \subseteq C_{(\sigma,\tau)}$, $\forall\ l_1 \in L$. Then $L \subseteq Z(S)$.*

Proof: By hypothesis,

$$[(l_1)^d, l_1]_{(\sigma,\tau)} \subseteq C_{(\sigma,\tau)}. \tag{3.9}$$

By linearizing equation (3.9), we have

$$[(l_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}, \qquad \forall\, l_1, m_1 \in L. \tag{3.10}$$

Replace m_1 by $[m_1, l_1]$ in equation (3.10), we get

$$\begin{aligned}
[(l_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1]_{(\sigma,\tau)} &= [(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} + [[m_1, l_1]^d, l_1]_{(\sigma,\tau)} \\
&= [(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} + [((m_1)^d (l_1)^\sigma + (m_1)^\tau (l_1)^d + \\
&\quad (l_1)^d (m_1')^\sigma + (l_1)^\tau (m_1')^d), l_1]_{(\sigma,\tau)} \\
&= [(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} + [((m_1)^d (l_1)^\sigma + (l_1)^\tau (m_1')^d) + \\
&\quad ((m_1')^\tau (l_1)^d + (l_1)^d (m_1)^\sigma)', l_1]_{(\sigma,\tau)} \\
&= [(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} + [[(m_1)^d, l_1]_{(\sigma,\tau)} + \\
&\quad [(l_1)^d, m_1]'_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} \\
&= [(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} + [[(m_1)^d, l_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} + \\
&\quad [[(l_1)^d, m_1]'_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}.
\end{aligned}$$

By Proposition 2.5 (iii), we have

$$[(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} = [[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} + [[(l_1)^d, l_1]_{(\sigma,\tau)}, m_1]'_{(\sigma,\tau)} \in C_{(\sigma,\tau)}. \tag{3.11}$$

By definition of $C_{(\sigma,\tau)}$ in equation (3.11), we have

$$[(l_1)^d, [m_1, l_1]]_{(\sigma,\tau)} = [[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)}. \tag{3.12}$$

Now, putting equation (3.12) in equation (3.11), we obtain

$$[[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} + [[(m_1)^d, l]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} + [[(l_1)^d, m_1]'_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}. \tag{3.13}$$

Using definition of $C_{(\sigma,\tau)}$ in equation (3.10), we have

$$[[(l_1)^d, m_1]_{(\sigma,\tau)} + [(m_1)^d, l_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} = 0, \qquad \forall\, l_1, m_1 \in L.$$

By using Proposition 2.3, it leads to

$$[[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]'_{(\sigma,\tau)} = [[(m_1)^d, l_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} \tag{3.14}$$

and next, putting equation (3.14) in equation (3.13) and using Proposition 2.3, we get

$$[[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]_{(\sigma,\tau)} + [[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]'_{(\sigma,\tau)} + [[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]'_{(\sigma,\tau)} \in C_{(\sigma,\tau)}. \tag{3.15}$$

This implies that

$$[[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]'_{(\sigma,\tau)} \in C_{(\sigma,\tau)}.$$

Also, $[(l_1)^d, [l_1, m_1]]_{(\sigma,\tau)} = [[(l_1)^d, m_1]_{(\sigma,\tau)}, l_1]'_{(\sigma,\tau)}$. Therefore, by previous equation, we have

$$[(l_1)^d, [l_1, m_1]]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}. \tag{3.16}$$

Replacing m_1 by $l_1 m_1$, in equation (3.16), we have

$$[(l_1)^d, [l_1, l_1 m_1]]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}$$

$$[(l_1)^d, l_1 [l_1, m_1]]_{(\sigma,\tau)} \in C_{(\sigma,\tau)}.$$

Using Proposition 2.5 (iii), we get

$$(l_1)^\tau[(l_1)^d,[l_1,m_1]]_{(\sigma,\tau)}+[(l_1)^d,l_1]_{(\sigma,\tau)}[l_1,m_1]^\sigma\in C_{(\sigma,\tau)}. \tag{3.17}$$

By definition of $C_{(\sigma,\tau)}$ in equation (3.17) and have

$$\begin{aligned} 0 &= [(l_1)^\tau[(l_1)^d,[l_1,m_1]]_{(\sigma,\tau)},(l_1)^\tau]_{(\sigma,\tau)}+[[(l_1)^d,l_1]_{(\sigma,\tau)}[l_1,m_1]^\sigma,(l_1)^\tau]_{(\sigma,\tau)} \\ &= [(l_1)^\tau[(l_1)^d,[l_1,m_1]]_{(\sigma,\tau)},(l_1)^\tau]_{(\sigma,\tau)}+[[(l_1)^d,l_1]_{(\sigma,\tau)}[l_1,m_1]^\sigma,(l_1)^\tau]_{(\sigma,\tau)}, \qquad \forall\ (l_1)^\tau\in S. \end{aligned}$$

Since $[(l_1)^d,[l_1,m_1]]_{(\sigma,\tau)}\in C_{(\sigma,\tau)}$, so by using definition of $C_{(\sigma,\tau)}$, we have

$$[[(l_1)^d,l_1]_{(\sigma,\tau)}[l_1,m_1]^\sigma,(l_1)^\tau]_{(\sigma,\tau)}=0.$$

Using Proposition 2.5, we get

$$[(l_1)^d,l_1]_{(\sigma,\tau)}[[l_1,m_1]^\sigma,((l_1)^\tau)^\sigma]+[[(l_1)^d,l_1]_{(\sigma,\tau)},(l_1)^\tau]_{(\sigma,\tau)}[l_1,m_1]^\sigma=0.$$

Again, by the definition of $C_{(\sigma,\tau)}$, we have

$$[(l_1)^d,l_1]_{(\sigma,\tau)}[[l_1,m_1]^\sigma,((l_1)^\tau)^\sigma]=0.$$

Since σ is an automorphism, thus

$$[(l_1)^d,l_1]_{(\sigma,\tau)}[[l_1,m_1],((l_1)^\tau)]^\sigma=0. \tag{3.18}$$

Now, multiply equation (3.18) by $(n_1)^\tau$, we get

$$(n_1)^\tau[(l_1)^d,l_1]_{(\sigma,\tau)}[[l_1,m_1],(l_1)^\tau]^\sigma=0, \qquad \forall\ (n_1)^\tau\in S.$$

We know that $[(l_1)^d,l_1]_{(\sigma,\tau)}(n_1)^\sigma=(n_1)^\tau[(l_1)^d,l_1]_{(\sigma,\tau)}$ and $[(l_1)^d,l_1]_{(\sigma,\tau)}(n_1)^\sigma[[l_1,m_1],(l_1)^\tau]^\sigma=0\,\forall\, l_1,m_1\in L$, $(n_1)^\sigma,(n_1)^\tau\in S$. This gives $[(l_1)^d,l_1]_{(\sigma,\tau)}S[[l_1,m_1],(l_1)^\tau]^\sigma=(0)$. So by primeness of S, we have either $[(l_1)^d,l_1]_{(\sigma,\tau)}=0$ or $[[l_1,m_1],(l_1)^\tau]^\sigma=0$. If $[(l_1)^d,l_1]_{(\sigma,\tau)}=0$, then by Theorem 3.8, implies $l\subseteq Z(S)$ and if $[[l_1,m_1],(l_1)^\tau]^\sigma=0$, then

$$[[l_1,m_1],(l_1)^\tau]=0$$

$$[l_1,m_1](l_1)^\tau+(l_1')^\tau[l_1,m_1]=0.$$

Using Proposition 2.3, we have

$$[l_1,m_1](l_1)^\tau=(l_1)^\tau[l_1,m_1]$$

$$[l_1,m_1]^{\tau^{-1}}l_1=l_1[l_1,m_1]^{\tau^{-1}}.$$

This infers $L\subseteq Z(S)$ and completes the proof. □

The next Theorem offers an extension to Theorem 3 in [12].

Theorem 3.10 *Let $L\subseteq S$ be a lie ideal and d be a non-zero derivation such that $[(l)^d,a]_{(\sigma,\tau)}=0$ for $a\in S$. Then either $a\in Z(S)$ or $L\subseteq Z(S)$.*

Proof: By hypothesis

$$[(l_1)^d,a]_{(\sigma,\tau)}=0. \tag{3.19}$$

Replace l_1 by l_1m_1, we get

$$\begin{aligned} 0 &= [(l_1m_1)^d,a]_{(\sigma,\tau)} \\ &= [(l_1)^d(m_1)^\sigma,a]_{(\sigma,\tau)}+[(l_1)^\tau(m_1)^d,a]_{(\sigma,\tau)}, \qquad \forall\ l_1,m_1\in L. \end{aligned}$$

Using Proposition 2.5, in previous equation, we get

$$\begin{aligned} 0 &= (l_1)^d[(m_1)^\sigma,(a)^\sigma] + [(l_1)^d,a]_{(\sigma,\tau)}(m_1)^\sigma + (l_1)^\tau[(m_1)^d,a]_{(\sigma,\tau)} + [(l_1)^\tau,(a)^\tau](m_1)^d \\ &= (l_1)^d[(m_1)^\sigma,(a)^\sigma] + [(l_1)^\tau,(a)^\tau](m_1)^d \\ &= (l_1)^d[m_1,a]^\sigma + [l_1,a]^\tau(m_1)^d. \end{aligned}$$

By using Proposition 2.3, we have $(l_1)^d[m_1,a]^\sigma = [l_1,a]^\tau(m_1')^d$ and putting in above equation, we get

$$\begin{aligned} 0 &= [l_1,a]^\tau(m_1')^d + [l_1,a]^\tau(m_1)^d \\ &= [l_1,a]^\tau(m_1'+m_1)^d \\ &= [l_1,a]^\tau(m_0)^d, \ \ \text{[by Proposition 2.1 } \forall\, l_1,m_1 \in L]. \end{aligned}$$

Now by using Proposition 2.4, $[l_1,a]^\tau = 0$ or $(m_0)^d = 0$. As $(m_0)^d \neq 0$, therefore $[l_1,a]^\tau = 0$, gives $[l_1,a] = 0$. This infers that either $a \in Z(S)$ or $L \subseteq Z(S)$ $\forall\, l_1 \in L$. □

4. Conclusion

In this paper, we extended the Posner's and Herstein's theorems for additively regular prime semirings by investigating the action of (σ,τ)-derivations. We identify some conditions under which these derivations vanish and characterize when the semirings become commutative. In addition, it is determined that under the action of (σ,τ)-derivations, Lie ideals contained in centre of prime semirings. These results provided a deeper understanding of the structural properties of prime semirings.

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