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An Approximate Mixed Type Quadratic and Quintic Functional Equation in Various Banach Spaces

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ABSTRACT: In this paper, the general solutions of a mixed type quadratic and quintic functional equation are obtained. In fact, it is shown under which conditions (the evenness and oddness properties), the former functional equation can be either quadratic or quintic. Moreover, some Hyers-Ulam-Rassias stability results of a mixed quadratic and quintic functional equation are established in various Banach spaces by using (Hyers) direct method and fixed point technique. In other words, by applying a fixed point theorem, we give a control function to investigate Hyers, Găvruţa and Rassias stability to approximate the solution of the mentioned equation with quality and certainty of the approximation by using the concept of generalized Z-numbers.

Keywords: Quadratic functional equation, quintic functional equation, mixed type functional equation, Hyers-Ulam stability, fixed point.

Contents

1. Introduction

The story of stability in functional equations has been initially posed by Ulam [35] regarding the challenge of the approximate stability of group homomorphisms and then the notion of stability in functional equations has grown into a significant field of research. Recall that a functional equation $\mathcal{F}$ is said to be *stable* if any function f satisfying the equation $\mathcal{F}$ approximately, must be near to an exact solution.The question of Ulam answered by Hyers [23] for Banach algebras. Next, the stability problem for functional equations were extended and generalized for miscellaneous mappings and equations which are available in many articles and books; see for instance [8], [21], [30] and [32].

The well known that the quadratic functional equation

$$Q(x+y)+Q(x-y)=2[Q(x)+Q(y)] \tag{1.1}$$

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was introduced by Skof [34]. Then, its solutions and miscellaneous stabilities in various Banach spaces was discussed in many articles and books; we refer to [1], [2], [18], [25], [26]. Also, Cho et. al. [17] introduced and investigate the Hyers-Ulam-Rassias stability of the Quintic functional equation

$$2[Q(2x+y)+Q(2x-y)]+Q(x+2y)+Q(x-2y)=20[Q(x+y)+Q(x-y)]+90Q(x) \tag{1.2}$$

in quasi-β-normed spaces. Moreover, The generalized Hyers-Ulam stability of (1.1)in RN-space in the sense of Scherstnev under the minimum $t-$norm T_M has been studied by Abdou et al. [4]. Several other types of quadratic and quintic functional equation in various normed spaces were discussed in [9,11,12,24,31,33,36] and references therein. More information about the structures, characterizations and the stability results for various multi-quadratic mappings and equation on miscellaneous spaces are available in [10], [13], [14] and [15].

The theory of fuzzy which is a powerful hand set for modeling uncertainty and vagueness in various problems arising was introduced by Zadeh [38] in 1965. Next, he presented the concept of Z-numbers which is relates to the issue of reliability of information in [37]. Recall that a Z-number, Z, is denoted as $Z=(A,B)$. The first component, A, is a restriction (constraint) on the values which a real-valued uncertain variable, X, is allowed to take. The second one, namely B, is a measure of reliability (certainty) of A; for more information and new results on Z-numbers, we refer to [6] and [7]. Recently, Ahadi et al. [5] applied the notion of Z-numbers and introduce a special matrix of the form $\mathrm{diag}(A,B,C)$, say, the generalized Z-number, where A is a fuzzy time-stamped set, B is the probability distribution function, and C is a degree of reliability of A that is described as a value of $A*B$. Then, they indicated a novel control function to investigate the stability to approximate the solution of a special quadratic functional equation with quality and certainty of the approximation by using generalized Z-numbers. The modeling random and non-random decision uncertainty in ratings data as a fuzzy beta model was studied in [16].

In this paper, we obtain the general solutions of the mixed type quadratic and quintic functional equation

$$\begin{aligned}2[Q(2x+y)+Q(2x-y)]+Q(x+2y)+Q(x-2y)+14[Q(y)+Q(-y)]\\ =20[Q(x+y)+Q(x-y)]+2[17Q(x)-28Q(-x)].\end{aligned} \tag{1.3}$$

We prove that under what conditions, functional equation (1.3) is either quadratic or quintic. Moreover, we provide the generalized Hyers-Ulam-Rassias stability of functional equation (1.3) in various Banach spaces by applying direct and fixed point manners.

2. Solutions of Quadratic-Quintic Functional Equations

In this section, we explore the solutions of functional equation (1.3). For that, we assume V_1 and V_2 are vector spaces.

Proposition 2.1 *Given a mapping $Q:V_1\longrightarrow V_2$.*

(i) *If Q is an odd mapping satisfies (1.3), then it is quintic;*

(ii) *If Q is an even mapping satisfies (1.3), then it is quadratic.*

Proof: (i) The result of this part follows immediately from the oddness of Q. In fact, Q fulfills (1.1).
(ii) By our assumption, it is clear that $Q(0)=0$ and $Q(2x)=4Q(x)$. Moreover, (1.3) can be reduce as

$$\begin{aligned}2[Q(2x+y)+Q(2x-y)]+Q(x+2y)+Q(x-2y)+28Q(y)\\ =20[Q(x+y)+Q(x-y)]-22Q(x),\end{aligned} \tag{2.1}$$

for all $x,y\in V_1$. Replacing (x,y) by (y,x) in (2.1) and using the evenness of Q, we have

$$\begin{aligned}2[Q(x+2y)+Q(x-2y)]+Q(2x+y)+Q(2x-y)+28Q(x)\\ =20[Q(x+y)+Q(x-y)]-22Q(y),\end{aligned} \tag{2.2}$$

for all $x, y \in V_1$. Plugging (2.1) into (2.2), we obtain

$$\begin{aligned} 3[Q(x+2y)+Q(x-2y)+Q(2x+y)+Q(2x-y)]+28[Q(x)+Q(y)] \\ =40[Q(x+y)+Q(x-y)]-22[Q(x)+Q(y)], \end{aligned} \tag{2.3}$$

for all $x, y \in V_1$. It follows from (2.1) and (2.3) that

$$3[Q(2x+y)+Q(2x-y)]=20[Q(x+y)+Q(x-y)]-16Q(x)-34Q(y), \tag{2.4}$$

for all $x, y \in V_1$. One can show that (2.4) is equivalent to

$$Q(2x+y)+Q(2x-y)=Q(x+y)+Q(x-y)+6Q(x), \tag{2.5}$$

for all $x, y \in V_1$. Now, Proposition 3.1 of [28] implies that (2.5) is a representation of quadratic functional equations. □

Note that by the above proposition, the mapping $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\Psi(r) = \alpha r^2 + \beta r^5$ is a solution of functional equation (1.3).

3. Stability Analysis of Quadratic-Quintic Functional Equations in Banach Space

In this section, we firstly investigate the Găvruţa stability of functional equation (1.3) in the setting of Banach spaces by two classical methods and then by means of obtained results, we establish various Hyers-Ulam-Rassias stabilities as some corollaries. For doing it, let us take $\mathcal{V}_1$ be a normed space and $\mathcal{V}_2$ be a Banach space.

Here and subsequently, for a mapping $\Phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$, we remark the difference operator

$$\begin{aligned} \mathbf{D}\Phi(x,y) := 2[Q_2(2x+y)+Q_2(2x-y)]+Q_2(x+2y)+Q_2(x-2y)+14[Q_2(y)+Q_2(-y)] \\ -20[Q_2(x+y)+Q_2(x-y)]-2[17Q_2(x)-28Q_2(-x)]. \end{aligned}$$

3.1. Quadratic case stability analysis: Hyers method

This subsection is devoted of the stability of functional equation (1.3) in the even case.

Theorem 3.1 *Let $Q_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an even mapping fulfilling the inequality*

$$\|\mathbf{D}Q_2(x,y)\| \le W(x,y), \tag{3.1}$$

for all $x, y \in \mathcal{V}_1$, where $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0,\infty)$ is a function with the condition

$$\lim_{D\to\infty} \frac{1}{9^{DE}} W\Big(3^{DE}x,\ 3^{DE}y\Big) = 0. \tag{3.2}$$

Then, there exists a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying (1.3) and

$$\|\mathcal{T}_2(y) - Q_2(y)\| \le \frac{1}{27} \sum_{C=\frac{1-E}{2}}^{\infty} \frac{W_{25}\Big(3^{CE}y\Big)}{9^{CE}}, \tag{3.3}$$

for all $y \in \mathcal{V}_1$, where $W_{25}\Big(3^{CE}y\Big) = W\Big(3^{CE}y, 3^{CE}y\Big) + 5W\Big(3^{CE}y, 0\Big)$ and $E = \pm 1$. Moreover, the mapping $\mathcal{T}_2$ is defined by

$$\mathcal{T}_2(y) = \lim_{D\to\infty} \frac{Q_2(3^{DE}y)}{9^{DE}},$$

for all $y \in \mathcal{V}_1$.

Proof: At first, by using the evenness of Q_2 in (3.1), we get

$$\begin{aligned}\Big\|2[Q_2(2x+y)+Q_2(2x-y)]+Q_2(x+2y)+Q_2(x-2y)+28Q_2(y)\\ -20[Q_2(x+y)+Q_2(x-y)]+22Q_2(x)\Big\| \le W(x,y),\end{aligned} \tag{3.4}$$

for all $x, y \in \mathcal{V}_1$. Putting $x = y$ in (3.4) and using the evenness of Q_2, we find

$$\Big\|3Q_2(3y)-20Q_2(2y)+53Q_2(y)\Big\| \le W(x,y), \tag{3.5}$$

for all $y \in \mathcal{V}_1$. Replacing (x, y) by $(y, 0)$ in (3.4), we have

$$\Big\|4Q_2(2y)-16Q_2(y)\Big\| \le W(y,0), \tag{3.6}$$

for all $y \in \mathcal{V}_1$. Combining (3.5) and (3.6), we obtain

$$\Big\|3Q_2(3y)-27Q_2(y)\Big\| \le W(y,y)+5W(y,0)=W_{25}(y),$$

for all $y \in \mathcal{V}_1$. The above inequality can be rewritten as

$$\left\|\frac{Q_2(3y)}{9}-Q_2(y)\right\| \le \frac{W_{25}(y)}{27},$$

for all $y \in \mathcal{V}_1$. Repeating the above process, one can find for a positive integer D that

$$\left\|\frac{Q_2(3^Dy)}{9^D}-Q_2(y)\right\| \le \frac{1}{27}\sum_{C=0}^{D-1}\frac{W_{25}\left(3^Cy\right)}{9^C}, \tag{3.7}$$

for all $y \in \mathcal{V}_1$. Hence, $\left\{\frac{Q_2(3^Dy)}{9^D}\right\}$ is a Cauchy sequence and it converges to a point $\mathcal{T}_2(y) \in \mathcal{V}_2$. Indeed, replacing y by $3^{D_1}y$ and dividing the resultant by 9^{D_1} in (3.7), we get

$$\left\|\frac{Q_2(3^{D+D_1}y\ )}{9^{D+D_1}}-\frac{Q_2(3^{D_1}y\ )}{9^{D_1}}\right\| \le \frac{1}{27}\sum_{C=0}^{D-1}\frac{1}{9^{C+D_1}}W_{25}\left(3^{C+D_1}y\right), \tag{3.8}$$

for all $y \in \mathcal{V}_1$. The right hand of inequality (3.8) goes toward zero when D_1 intends to infinity. Here, we define the mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ via

$$\mathcal{T}_2(y)=\lim_{D\longrightarrow\infty}\frac{Q_2(3^Dy\ )}{9^D}, \qquad (y \in \mathcal{V}_1).$$

Letting limit $D \to \infty$ in (3.7) and using the definition of $\mathcal{T}_2$, we get

$$\left\|\lim_{D\longrightarrow\infty}\frac{Q_2(3^Dy)}{9^D}-Q_2(y)\right\| \le \frac{1}{27}\sum_{C=0}^{\infty}\frac{W_{25}\left(3^Cy\right)}{9^C},$$

for all $y \in \mathcal{V}_1$. Thus, (3.3) holds for $E = 1$. Next, we show that $\mathcal{T}_2$ satisfies (1.3). Switching (x, y) into $(3^Dx,\ 3^Dy)$ and dividing by 9^D in (3.1), we arrive at

$$\begin{aligned}\frac{1}{9^D}\Big\|2\Big[Q_2\Big(3^D(2x+y)\Big)+Q_2\Big(3^D(2x-y)\Big)\Big]+Q_2\Big(3^D(x+2y)\Big)+Q_2\Big(3^D(x-2y)\Big)\\ +14\Big[Q_2\Big(3^D(y)\Big)+Q_2\Big(3^D(-y)\Big)\Big]-20\Big[Q_2\Big(3^D(x+y)\Big)\\ +Q_2\Big(3^D(x-y)\Big)\Big]-2\Big[17Q_2\Big(3^D(x)\Big)-28Q_2\Big(3^D(-x)\Big)\Big]\Big\| \le \frac{1}{9^D}W\Big(3^Dx,3^Dy\Big),\end{aligned}$$

for all $x, y \in \mathcal{V}_1$. Approaching $D \to \infty$ and using the definition of $\mathcal{T}_2$ and (3.2) in the above inequality, we see that $\mathcal{T}_2$ fulfills (1.3) for all $x, y \in \mathcal{V}_1$. In order to prove the uniqueness of $\mathcal{T}_2(\ y\ )$, assume $\mathcal{T}_2'$ is another quadratic mapping satisfying (1.3) and (3.3). We have

$$\begin{aligned}
\|\mathcal{T}_2(y) - \mathcal{T}_2'(y)\| &= \frac{1}{9^{D_1}} \left\|\mathcal{T}_2\left(3^{D_1}y\right) - \mathcal{T}_2'\left(3^{D_1}y\right)\right\| \\
&= \frac{1}{9^{D_1}} \left\|\mathcal{T}_2\left(3^{D_1}y\right) - Q_2\left(3^{D_1}y\right) + Q_2\left(3^{D_1}y\right) - \mathcal{T}_2'\left(3^{D_1}y\right)\right\| \\
&\le \frac{1}{9^{D_1}} \left\{\left\|\mathcal{T}_2\left(3^{D_1}y\right) - Q_2\left(3^{D_1}y\right)\right\| + \left\|\mathcal{T}_2'\left(3^{D_1}y\right) - Q_2\left(3^{D_1}y\right)\right\|\right\} \\
&\le \frac{2}{27} \sum_{C=0}^{\infty} \frac{1}{9^{C+D_1}} W_{25}\left(3^{C+D_1}y\right),
\end{aligned}$$

for all $y \in \mathcal{V}_1$. Letting $D_1 \to \infty$, we reach the uniqueness of $\mathcal{T}_2$. Therefore, the result holds for the case $E = 1$. The other case can be obtained similarly and hence the proof is complete. □

The following corollary is a direct consequence of Theorem 3.1 concerning the miscellaneous stabilities of (1.3). The proof is routine and we do not include it.

Corollary 3.1 *Given positive numbers M, N, N_1 and N_2. Let $Q_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an even mapping fulfilling the inequality*

$$\|\mathbf{D}Q_2(x,y)\| \le \begin{cases} M \\ M\left[||x||^N + ||y||^N\right] & N \neq 2, \\ M\left[||x||^{N_1} + ||y||^{N_2}\right] & N_1, N_2 \neq 2, \\ M||x||^N||y||^N & N \neq 1, \\ M||x||^{N_1}||y||^{N_2} & N_1 + N_2 \neq 2, \\ M\left[||x||^{2N} + ||y||^{2N} + ||x||^N||y||^N\right] & N \neq 1, \\ M\left[||x||^{N_1+N_2} + ||y||^{N_1+N_2} + ||x||^{N_1}||y||^{N_2}\right] & N_1 + N_2 \neq 2, \end{cases} \tag{3.9}$$

for all $x, y \in \mathcal{V}_1$. Then, there exists a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying (1.3) and

$$\|\mathcal{T}_2(\ y\ ) - Q_2(\ y\ )\| \le \begin{cases} \frac{M}{|4|} \\ \frac{7M||y||^N}{3|9-3^N|} & N \neq 2, \\ \frac{6M||y||^{N_1}}{3|9-3^{N_1}|} + \frac{M||y||^{N_2}}{3|9-3^{N_2}|} & N_1, N_2 \neq 2, \\ \frac{M||y||^{2N}}{|27-3^{2N+1}|} & N \neq 1, \\ \frac{M||y||^{N_1+N_2}}{|27-3^{N_1+N_2+1}|} & N_1 + N_2 \neq 2, \\ \frac{8M||y||^{2N}}{3|9-3^{2N}|} & N \neq 1, \\ \frac{8M||y||^{N_1+N_2}}{|27-3^{N_1+N_2+1}|} & N_1 + N_2 \neq 2, \end{cases} \tag{3.10}$$

for all $y \in \mathcal{V}_1$.

3.2. Quintic case stability analysis: direct method

In this subsection, we study the stability of functional equation (1.3) in the odd case.

Theorem 3.2 *Suppose that $Q_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is an odd mapping satisfying*

$$\|\mathbf{D}Q_5(x,y)\| \leq W(x,y), \tag{3.11}$$

for all $x, y \in \mathcal{V}_1$, where $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0,\infty)$ is a function fulfills

$$\lim_{D\to\infty} \frac{1}{243^{DE}} W\left(3^{DE}x,\ 3^{DE}y\right) = 0, \tag{3.12}$$

for all $y \in \mathcal{V}_1$. Then, there exists a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying (1.3) and

$$\|\mathcal{T}_5(y) - Q_5(y)\| \leq \frac{1}{729} \sum_{C=\frac{1-E}{2}}^{\infty} \frac{W_{25}\left(3^{CE}y\right)}{243^{CE}},$$

for all $y \in \mathcal{V}_1$, where $W_{25}\left(3^{CE}y\right) = W\left(3^{CE}y, 3^{CE}y\right) + 5W\left(3^{CE}y, 0\right)$ and $E = \pm 1$.

Proof: Using the oddness of Q_5 in (3.11), we have

$$\Big\|2[Q_5(2x+y) + Q_5(2x-y)] + Q_5(x+2y) + Q_5(x-2y) - 20[Q_5(x+y) + Q_5(x-y)] - 90Q_5(x)\Big\| \leq W(x,y), \tag{3.13}$$

for all $x, y \in \mathcal{V}_1$. Putting $x = y$ in (3.13) and using again the oddness of Q_5, we get

$$\Big\|3Q_5(3y) - 20Q_5(2y) - 89Q_5(y)\Big\| \leq W(y,y), \tag{3.14}$$

for all $y \in \mathcal{V}_1$. Interchanging (x, y) by $(y, 0)$ in (3.13), we obtain

$$\Big\|4Q_5(2y) - 128Q_5(y)\Big\| \leq W(y,0), \tag{3.15}$$

for all $y \in \mathcal{V}_1$. Plugging (3.14) into (3.15), we arrive to

$$\Big\|3Q_5(3y) - 729Q_5(y)\Big\| \leq W(y,y) + 5W(y,0) = W_{25}(y),$$

for all $y \in \mathcal{V}_1$. The rest of the proof is similar to that of Theorem 3.1. □

A consequence of Theorem 3.2 concerning Hyers-Ulam-Rassias stability of (1.3) is given as follows.

Corollary 3.2 *Given positive numbers M, N, N_1 and N_2. Let $Q_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an odd mapping fulfilling the inequality*

$$\|\mathbf{D}Q_5(x,y)\| \leq \begin{cases} M & \\ M\left[||x||^N + ||y||^N\right] & N \neq 5, \\ M\left[||x||^{N_1} + ||y||^{N_2}\right] & N_1, N_2 \neq 5, \\ M||x||^N||y||^N & 2N \neq 5, \\ M||x||^{N_1}||y||^{N_2} & N_1 + N_2 \neq 5, \\ M\left[||x||^{2N} + ||y||^{2N} + ||x||^N||y||^N\right] & 2N \neq 5, \\ M\left[||x||^{N_1+N_2} + ||y||^{N_1+N_2} + ||x||^{N_1}||y||^{N_2}\right] & N_1 + N_2 \neq 5, \end{cases} \tag{3.16}$$

for all $x, y \in \mathcal{V}_1$. *Then, there exists a unique quintic mapping* $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *satisfying* (1.3) *and*

$$\|\mathcal{T}_5(y) - Q_5(y)\| \leq \begin{cases} \frac{M}{121} & \\ \frac{7M||y||^N}{3|243-3^N|} & N \neq 5, \\ \frac{6\ M||y||^{N_1}}{3|243-3^{N_1}|} + \frac{M||y||^{N_2}}{3|243-3^{N_2}|} & N_1, N_2 \neq 5, \\ \frac{M||y||^{2N}}{|729-3^{2N+1}|} & 2N \neq 5, \\ \frac{M||y||^{N_1+N_2}}{|729-3^{N_1+N_2+1}|} & N_1 + N_2 \neq 5, \\ \frac{8M||y||^{2N}}{3|243-3^{2N}|} & 2N \neq 5, \\ \frac{8M||y||^{N_1+N_2}}{|729-3^{N_1+N_2+1}|} & N_1 + N_2 \neq 5, \end{cases} \tag{3.17}$$

for all $y \in \mathcal{V}_1$.

3.3. Quadratic-quintic case stability analysis: direct method

In this subsection, we establish the stability of functional equation (1.3) when the general case occurs.

Theorem 3.3 *Given* $E \in \{-1, 1\}$. *Suppose that* $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *is a mapping satisfying*

$$\|\mathbf{D}Q(x, y)\| \leq W(x, y), \tag{3.18}$$

for all $x, y \in \mathcal{V}_1$ *in which* $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0, \infty)$ *is a function fulfills* (3.2) *and* (3.12) *for all* $y \in \mathcal{V}_1$. *Then, there exist a unique quadratic mapping* $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *and a unique quintic mapping* $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *satisfying* (1.3) *and*

$$\begin{aligned} &\|\mathcal{T}_2(y) + \mathcal{T}_5(y) - Q(y)\| \\ &\leq \frac{1}{54} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W\Big(3^{CE}y, 3^{CE}y\Big) + W\Big(3^{CE}y, 0\Big)}{27} + \frac{W\Big(-3^{CE}y, -3^{CE}y\Big) + W\Big(-3^{CE}y, 0\Big)}{27} \right] \\ &+ \frac{1}{1458} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W\Big(3^{CE}y, 3^{CE}y\Big) + W\Big(3^{CE}y, 0\Big)}{243} + \frac{W\Big(-3^{CE}y, -3^{CE}y\Big) + W\Big(-3^{CE}y, 0\Big)}{243} \right], \end{aligned}$$

for all $y \in \mathcal{V}_1$.

Proof: Set

$$Q_e(y) := \frac{Q_2(y) + Q_2(-y)}{2}, \tag{3.19}$$

for all $y \in \mathcal{V}_1$. It follows from (3.19) that Q_e is an even mapping satisfies (3.18). Hence, by Theorem 3.1, there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ we have

$$\|\mathcal{T}_2(y) - Q_e(y)\| \leq \frac{1}{54} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W_{25}\Big(3^{CE}y\Big)}{9^{CE}} + \frac{W_{25}\Big(-3^{CE}y\Big)}{9^{CE}} \right], \tag{3.20}$$

for all $y \in \mathcal{V}_1$. Similarly, consider

$$Q_o(y) := \frac{Q_5(y) - Q_2(-y)}{2}, \qquad (y \in \mathcal{V}_1). \tag{3.21}$$

Relation (3.21) implies that Q_e is an odd mapping satisfies (3.18) and so by Theorem 3.2there exists a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ such that

$$\|\mathcal{T}_5(y) - Q_o(y)\| \leq \frac{1}{1458} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W_{25}\Big(3^{CE}y\Big)}{243^{CE}} + \frac{W_{25}\Big(-3^{CE}y\Big)}{243^{CE}} \right], \tag{3.22}$$

for all $y \in \mathcal{V}_1$. Obviously,

$$Q(y) = Q_e(y) + Q_o(-y) \tag{3.23}$$

for all $y \in \mathcal{V}_1$. It now follows from (3.20), (3.22), (3.23) that

$$\begin{aligned} \|\mathcal{T}_2(y) + \mathcal{T}_5(y) - Q(y)\| &= \|\mathcal{T}_2(y) - Q_e(y)\| + \|\mathcal{T}_5(y) - Q_o(y)\| \\ &\leq \frac{1}{54} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W_{25}\Big(3^{CE}y\Big)}{9^{CE}} + \frac{W_{25}\Big(-3^{CE}y\Big)}{9^{CE}} \right] \\ &\quad + \frac{1}{1458} \sum_{C=\frac{1-E}{2}}^{\infty} \left[\frac{W_{25}\Big(3^{CE}y\Big)}{243^{CE}} + \frac{W_{25}\Big(-3^{CE}y\Big)}{243^{CE}} \right], \end{aligned}$$

for all $y \in \mathcal{V}_1$. □

The upcoming corollary is a consequence of Theorem 3.3 regarding various stabilities of (1.3) in the general case.

Corollary 3.3 *Let M, N, N_1, and N_2 be positive numbers. Let also $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be a mapping with*

$$\|\mathbf{D}Q(x,y)\| \leq \begin{cases} M & \\ M\Big[||x||^N + ||y||^N\Big] & N \neq 2,5, \\ M\Big[||x||^{N_1} + ||y||^{N_2}\Big] & N_1, N_2 \neq 2,5, \\ M||x||^N||y||^N; & 2N \neq 2,5, \\ M||x||^{N_1}||y||^{N_2} & N_1 + N_2 \neq 2,5, \\ M\Big[||x||^{2N} + ||y||^{2N} + ||x||^N||y||^N\Big] & 2N \neq 2,5, \\ M\Big[||x||^{N_1+N_2} + ||y||^{N_1+N_2} + ||x||^{N_1}||y||^{N_2}\Big] & N_1 + N_2 \neq 2,5, \end{cases} \tag{3.24}$$

for all $x, y \in \mathcal{V}_1$. Then, there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ and a unique quintic *mapping* $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *satisfying* (1.3) *and*

$$\|\mathcal{T}_2(y) + \mathcal{T}_5(y) - Q(y)\| \leq 2 \times \begin{cases} \frac{M}{4} + \frac{M}{121} & \\ \frac{7M||y||^N}{3|9-3^N|} + \frac{7M||y||^N}{81|243-3^N|} & N \neq 2,5, \\ \frac{6\ M||y||^{N_1}}{3|9-3^{N_1}|} + \frac{M||y||^{N_2}}{3|9-3^{N_2}|} + \frac{6\ M||y||^{N_1}}{81|243-3^{N_1}|} + \frac{M||y||^{N_2}}{81|243-3^{N_2}|} & N_1, N_2 \neq 2,5, \\ \frac{M||y||^{2N}}{|27-3^{2N+1}|} + \frac{M||y||^{2N}}{|729-3^{2N+1}|} & 2N \neq 2,5, \\ \frac{M||y||^{N_1+N_2}}{|27-3^{N_1+N_2+1}|} + \frac{M||y||^{N_1+N_2}}{|729-3^{N_1+N_2+1}|} & N_1+N_2 \neq 2,5, \\ \frac{8M||y||^{2N}}{3|9-3^{2N}|} + \frac{8M||y||^{2N}}{3|243-3^{2N}|} & 2N \neq 2,5, \\ \frac{8M||y||^{N_1+N_2}}{|27-3^{N_1+N_2+1}|} + \frac{8M||y||^{N_1+N_2}}{|729-3^{N_1+N_2+1}|} & N_1+N_2 \neq 2,5, \end{cases} \tag{3.25}$$

for all $y \in \mathcal{V}_1$.

3.4. Quadratic case stability analysis: fixed point technique

In this subsection, we prove the stability of mixed type quadratic and quintic mapping by means of a known fixed point theorem. We now present the result due to Margolis, Diaz [27] and Radu [29] for fixed point theory. Some applications of fixed point techniques can be found in [19], [20] and [22].

Theorem 3.4 [27,29] *Let* (Ω, Δ) *be a complete generalized metric space and* $\Xi : \Omega \longrightarrow \Omega$ *such that for all* $x, y \in \Omega$,

$$d(\Xi x, \Xi y) \leq L d(x,y), L \in (0,1).$$

Then, for each given $x \in \Omega$, *either*

$$d(\Xi^n x, \Xi^{n+1} x) = \infty, \qquad (\forall\ n \geq 0),$$

or there exists a natural number n_0 *such that*
(FPC1) $d(\Xi^n x, \Xi^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(FPC2) $\lim_{n\to\infty} \Xi^n x = y^*$;
(FPC3) y^* *is the unique fixed point of* Ξ *in the set* $\Omega^* = \{y \in \Omega : d(\Xi^{n_0} x, y) < \infty\}$;
(FPC4) $d(y, y^*) \leq \frac{1}{1-L} d(y, \Xi y)$ *for all* $Y \in \Omega^*$.

Theorem 3.5 *Let* $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0,\infty)$ *be a function with the condition*

$$\lim_{D\to\infty} \frac{1}{F_I^{2D}} W\left(F_I^D y, F_I^D y\right) = 0, \tag{3.26}$$

in which

$$F_I = \begin{cases} 3 & I = 0, \\ \frac{1}{3} & I = 1. \end{cases} \tag{3.27}$$

Suppose that there exists $\mathcal{L} = \mathcal{L}(I)$ *such that the function* W_{25} *has the property*

$$W_{25}\Big(y\Big) = \frac{1}{3}\ W_{25}\left(\frac{y}{3}\right), \qquad \frac{1}{F_I^2}\ W_{25}\left(F_I\ y\right) = \mathcal{L}\ W_{25}(y), \tag{3.28}$$

for all $y \in \mathcal{V}_1$, where $W_{25}(y) = W(y, y) + 5W(y, 0)$. If Let $Q_2 : \mathcal{V}_1 \to \mathcal{V}_2$ is an even mapping satisfying inequality (3.1) for all $x, y \in \mathcal{V}_1$, then there exists a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ fulfilling (1.3) and

$$\|Q_2(y) - \mathcal{T}_2(y)\| \leq \left(\frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}\right) W_{25}(y), \tag{3.29}$$

for all $y \in \mathcal{V}_1$.

Proof: Consider the set $\mathcal{X} = \{p | \, p : \mathcal{V}_1 \to \mathcal{V}_2, \; p(0) = 0\}$ and introduce the generalized metric on $\mathcal{X}$ as

$$d(p, q) = \inf\{K \in (0, \infty) : \|p(\; y \;) - q(\; y \;)\| \leq K \; W_{25}(y), y \in \mathcal{V}_1\}.$$

It is easy to see that $(\mathcal{X}, d)$ is complete. Define the mapping $T : \mathcal{X} \to \mathcal{X}$ through

$$Tp(\; y \;) = \frac{1}{F_I^2} p(F_I \; y \;), \quad (y \in \mathcal{V}_1).$$

Assume that $p, q \in \mathcal{X}$ and $y \in \mathcal{V}_1$. Thus, $d(p, q) \leq K$ and so $\|p(y) - q(y)\| \leq K \; W_{25}(y)$. This implies that $\left\| \frac{1}{F_I^2} p(F_I y) - \frac{1}{F_I} q(F_I y) \right\| \leq \frac{1}{F_I^2} K \; W_{25}(F_I y)$ and hence $\left\| \frac{1}{F_I^2} p(F_I y) - \frac{1}{F_I^2} q(F_I y) \right\| \leq \mathcal{L} \; K \; W_{25}(y)$. Therefore, $\|Tp(\; y \;) - Tq(\; y \;)\| \leq \mathcal{L} \; K \; W_{25}(y)$. The last relation necessitates that $d(Tp, Tq) \leq \mathcal{L} \; K$ i.e., T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant $\mathcal{L}$. It follows from the proof of Theorem 3.1 and (3.28) for both cases $I = 0$ and $I = 1$ that $d(TQ_2, Q_2) \leq \mathcal{L} = \mathcal{L}^{1-I} < \infty$. Thus, condition (FPC1) of Theorem 3.4 holds. On the other hand, by (FPC2) of Theorem 3.4, it follows that there exists a fixed point $\mathcal{T}_2$ of T in $\mathcal{X}$ such that

$$\mathcal{T}_2(y) = \lim_{D \to \infty} \frac{Q_2(F_I^D y \;)}{F_I^{2D}}, \qquad (y \in \mathcal{V}_1). \tag{3.30}$$

The proof of being quadratic $\mathcal{T}_2$ is a similar way to that of Theorem 3.1. Moreover, by (FPC3) of Theorem 3.4, $\mathcal{T}_2$ is the unique fixed point of T in the set

$$\mathcal{Y} = \{\mathcal{T}_2 \in \mathcal{X} : d(Q_2, \mathcal{T}_2) < \infty\}.$$

Hence, $\mathcal{T}_2$ is the unique mapping such that

$$\|Q_2(y) - \mathcal{T}_2(y)\| \leq K \; W(y, y),$$

for all $y \in \mathcal{V}_1$ and $K > 0$. Finally, condition (FPC4) of Theorem 3.4 implies that $d(Q_2, \mathcal{T}_2(y)) \leq \frac{1}{1-\mathcal{L}} \; d(Q_2, \mathcal{T}_2(y))$ and so $d(Q_2, \mathcal{T}_2(y)) \leq \frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}$, which yields

$$\|Q_2(y) - \mathcal{T}_2(y)\| \leq \frac{\mathcal{L}^{1-j}}{1-\mathcal{L}} \; W_{25}(y),$$

for all $y \in \mathcal{V}_1$. This completes the proof. □

Next, we bring a consequence of Theorem 3.5 concerning some stabilities of (1.3) in the even case.

Corollary 3.4 *Under assumptions of Corollary 3.1, let $Q_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an even mapping fulfilling inequality (3.9) for all $x, y \in \mathcal{V}_1$. Then, there exists a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ such that relations (1.3) and (3.10) are valid for all $y \in \mathcal{V}_1$.*

Proof: In Theorem 3.5, take

$$W(x,y)=\begin{cases} M & \\ M\Big[||x||^N+||y||^N\Big] & N\neq 2,\\ M\Big[||x||^{N_1}+||y||^{N_2}\Big] & N_1,N_2\neq 2\\ M||x||^N||y||^N & N\neq 1\\ M||x||^{N_1}||y||^{N_2} & N_1+N_2\neq 2\\ M\Big[||x||^{2N}+||y||^{2N}+||x||^N||y||^N\Big] & N\neq 1\\ M\Big[||x||^{N_1+N_2}+||y||^{N_1+N_2}+||x||^{N_1}||y||^{N_2}\Big] & N_1+N_2\neq 2\end{cases} \tag{3.31}$$

for all $x,y\in\mathcal{V}_1$. Replacing (x,y) by $\left(F_I^D x, F_I^D y\right)$ and dividing the resultant by F_I^{2D} in (3.31), one can see that (3.26) holds. In the rest of the proof, we ignore all conditions for $W(x,y)$, do not write them and assume that they are true, respectively. From (3.28), we have

$$W_{25}\Big(y\Big)=\frac{1}{3}\Big[W\left(\frac{y}{3},\frac{y}{3}\right)+5W\left(\frac{y}{3},0\right)\Big]=\frac{1}{3}\times\begin{cases}6M,\\ 7M||\frac{y}{3}||^N,\\ 6M||\frac{y}{3}||^{N_1}+M||\frac{y}{3}||^{N_2},\\ M||\frac{y}{3}||^{2N},\\ M||\frac{y}{3}||^{N_1+N_2},\\ 8M||\frac{y}{3}||^{2N},\\ 8M||\frac{y}{3}||^{N_1+N_2}.\end{cases}$$

and

$$\frac{1}{F_I^2}\,W_{25}\left(F_I\ y\right)=\frac{1}{F_I^2}\,\Big[W\left(F_I\ y,F_I\ y\right)+5W\left(F_I\ y,0\right)\Big]=\begin{cases}F_I^{-2}\ 6M,\\ F_I^{N-2}\ 7M||y||^N,\\ F_I^{N_1-2}\ 6M||y||^{N_1}+F_I^{N_2-2}M||y||^{N_2},\\ F_I^{2N-2}\ M||y||^{2N},\\ F_I^{N_1+N_2-2}\ M||y||^{N_1+N_2},\\ F_I^{2N-2}\ 8M||y||^{2N},\\ F_I^{N_1+N_2-2}\ 8M||y||^{N_1+N_2}.\end{cases}$$

for all $y\in\mathcal{V}_1$. From (3.28) for the case $I=0$, we have $L=F_I^{-2}=3^{-2}$ and from (3.29), we get

$$\|Q_2(y)-\mathcal{T}_2(y)\|\leq\frac{L^{1-I}}{1-L}W_{25}(y)=\left(\frac{3^{-2}}{1-3^{-2}}\right)\ \frac{1}{3}\times 6M=\frac{M}{4}.$$

In the case $I=1$, (3.28) implies that $L=F_I^{-2}=\frac{1}{3^{-2}}=3^2$ and moreover by (3.29), we obtain

$$\|Q_2(y)-\mathcal{T}_2(y)\|\leq\frac{L^{1-I}}{1-L}W_{25}(y)=\left(\frac{1}{1-3^{2}}\right)\ \frac{1}{3}\times 6M=-\frac{M}{4}.$$

For the second inequality and for the case $I=0$, we have $L=F_I^{N-2}=3^{N-2}$ and

$$\|Q_2(y)-\mathcal{T}_2(y)\|\leq\frac{L^{1-I}}{1-L}W_{25}(y)=\left(\frac{3^{N-2}}{1-3^{N-2}}\right)\ \frac{1}{3}\times 7M||\frac{y}{3}||^N=\frac{7M\ ||y||^N}{3(9-3^N)}.$$

Once more, from (3.28) for the case $I = 1$, we have $L = F_I^{N-2} = \frac{1}{3^{N-2}} = 3^{2-N}$ and from (3.29), we find

$$\|Q_2(y) - \mathcal{T}_2(y)\| \leq \frac{L^{1-I}}{1-L} W_{25}(y) = \left(\frac{1}{1-3^{2-N}}\right) \frac{1}{3} \times 7M||\frac{y}{3}||^N = \frac{7M\ ||y||^N}{3(3^N - 9)}.$$

Other cases can be obtained similarly. □

3.5. Quintic case stability analysis: fixed point way

We prove the stability of equation (1.3) when the corresponding mapping is odd.

Theorem 3.6 *Let $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0, \infty)$ be a function with the condition*

$$\lim_{D\to\infty} \frac{1}{F_I^{5D}} W\left(F_I^D y, F_I^D y\right) = 0, \tag{3.32}$$

where F_I is given in (3.27). Suppose that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function W_{25} has the property

$$W_{25}\Big(y\Big) = \frac{1}{3}\ W_{25}\left(\frac{y}{3}\right), \qquad \frac{1}{F_I^5}\ W_{25}\left(F_I\ y\right) = \mathcal{L}\ W_{25}\Big(y\Big), \tag{3.33}$$

for all $y \in \mathcal{V}_1$, where $W_{25}(y) = W(y, y) + 5W(y, 0)$. If Let $Q_5 : \mathcal{V}_1 \to \mathcal{V}_2$ is an odd mapping satisfying inequality (3.1) for all $x, y \in \mathcal{V}_1$, then there exists a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ fulfilling (1.3) and

$$\|Q_5(y) - \mathcal{T}_5(y)\| \leq \frac{\mathcal{L}^{1-I}}{1-\mathcal{L}} W_{25}(y),$$

for all $y \in \mathcal{V}_1$.

Proof: The proof is a similar manner to that of Theorem 3.5. □

Corollary 3.5 *Under hypotheses of Corollary 3.2, suppose that $Q_5 : \mathcal{V}_1 \to \mathcal{V}_2$ be an odd function fulfilling inequality (3.16) for all $x, y \in \mathcal{V}_1$. Then, there exists a unique quintic mapping $\mathcal{T}_5(y) : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying relations (1.3) and (3.17) for all $y \in \mathcal{V}_1$.*

3.6. Quadratic-quintic type stability analysis: fixed point manner

In the present subsection, we investigate the stability of equation (1.3) when the corresponding mapping is arbitrary and without any condition.

Theorem 3.7 *Let $W : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow [0, \infty)$ be a function with conditions (3.26) and (3.32) where F_I is defined in (3.27). Suppose that there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function W_{25} has the properties (3.28) and (3.33) for all $y \in \mathcal{V}_1$, where $W_{25}(y) = W(y, y) + 5W(y, 0)$. If Let $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is a mapping satisfying inequality (3.18) for all $x, y \in \mathcal{V}_1$, then there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ and a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying (1.3) and*

$$\|\mathcal{T}_2(y) + \mathcal{T}_5(y) - Q(y)\| \leq \frac{2\ \mathcal{L}^{1-I}}{1-\mathcal{L}} \left[W_{25}(y) + W_{25}(-y)\right], \tag{3.34}$$

for all $y \in \mathcal{V}_1$.

Proof: Assume that $Q_e, Q_o : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ as defined in the proof of Theorem 3.3. Since Q_e satisfies (3.18), then it follows from (3.19) and Theorem 3.5 that

$$\|Q_2(y) - \mathcal{T}_2(y)\| \leq \frac{\mathcal{L}^{1-I}}{1-\mathcal{L}} \left[W_{25}(y) + W_{25}(-y)\right], \tag{3.35}$$

for all $y \in \mathcal{V}_1$. Similarly, Q_o fulfills (3.18) and hence (3.21) and Theorem 3.6 necessitate that

$$\|Q_5(y) - \mathcal{T}_5(y)\| \leq \frac{\mathcal{L}^{1-I}}{1-\mathcal{L}} \left[W_{25}(y) + W_{25}(-y)\right], \tag{3.36}$$

for all $y \in \mathcal{V}_1$. It now concludes from (3.23), (3.35) and (3.36) that (3.34) holds, as desired. □

The following corollary is the immediate consequence of Theorem 3.7 concerning the stabilities of (1.3).

Corollary 3.6 *Let $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be a function fulfilling inequality (3.24) for all $x, y \in \mathcal{V}_1$. Then, there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ and a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying functional equation (1.3) and (3.25) for all $y \in \mathcal{V}_1$.*

4. More Stability Results in Z-Numbers

4.1. Basics of generalized Z-numbers (GZ-N)

In many practical problems, the fuzzy probability approach can be an important component of decision making. In the real world, we consider various aspects of uncertainty that are not always well represented in fuzzy sets of information uncertainty. To overcome this problem, Zadeh introduced the Z-number (Z-N) in 2011 [37] ; for more on the subject, see Aliev et al. [6] and Allahviranloo et al. [7]. A Z-N is an ordered binary of the form (A,B) where the first component shows the fuzzy value and the second shows the uncertainty of the first.

Let $\Theta_1 = [0, 1]$, and let x_{Θ_1} be given as follows:

$$x_{\Theta_1} = \left\{ \text{diag } \Theta_1 = \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n \end{bmatrix} = \text{diag}[\theta_1, \cdots, \theta_n], \theta_1, \cdots, \theta_n \in \Theta_1 \right\}.$$

We write $\text{diag}[\theta_1, \cdots, \theta_n] \preceq \text{diag}[\kappa_1, \cdots, \kappa_n]$, when $\theta_i \leq \kappa_i$ for every $i = 1, \cdots, n$.

Definition 4.1 *A mapping $\circledast : x_{\Theta_1} \times x_{\Theta_1} \to x_{\Theta_1}$ is called a generalized continuous t-norm* (GCTN) *if for each $\rho, \kappa, \varpi, y, \kappa_n, \varpi_n \in x_{\Theta_1}, \mathbf{1} = diag[1, \cdots, 1]$, the following conditions are satisfied:*

(GCT1) $\varpi \circledast \mathbf{1} = \varpi$;

(GCT2) $\varpi \circledast \kappa = \kappa \circledast \varpi$;

(GCT3) $\varpi \circledast (\kappa \circledast \varpi) = (\varpi \circledast \kappa) \circledast \varpi$;

(GCT4) $\rho \preceq \kappa$ *and* $\varpi \preceq y$ *imply that* $\rho \circledast \varpi \preceq \kappa \circledast y$;

(GCT5) *If* $\lim_{n\to\infty} \kappa_n = \kappa$ *and* $\lim_{n\to\infty} \varpi_n = \varpi$, *we have* $\lim_{n\to\infty}(\kappa_n \circledast \varpi_n) = \kappa \circledast \varpi$.

In this section, we choose the minimum t-norm $\circledast_M = x_{\Theta_1} \times x_{\Theta_1} \to x_{\Theta_1}$ which is defined as follows:

$$\varpi \circledast_M \kappa = \text{diag}[\varpi_1, \cdots, \varpi_n] \circledast_M \text{diag}[\kappa_1, \cdots, \kappa_n] = \text{diag}[\min\{\varpi_1, \kappa_1\}, \cdots, \min\{\varpi_n, \kappa_n\}]. \tag{4.1}$$

Definition 4.2 *Let $\wp \in \mathbb{R}$ and $\wp \in (0, 1]$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a vector space over $\mathbb{K}$. A fuzzy set $\aleph_\wp : X \times (0, \infty) \longrightarrow \Theta_1$ is a $\wp-$ fuzzy norm ($\wp-$FN) on X if*

($\wp-$FN1) $\aleph_\wp(x, \zeta) = 1$, *if and only if* $x = 0$ *for* $\zeta \in (0, \infty)$;

($\wp-$FN2) $\aleph_\wp(\gamma x, \zeta) = \aleph_\wp\left(x, \frac{\zeta}{\gamma}\right)$ *for each* $\gamma \neq 0 \in \mathbb{K}$, *all* $x \in X$ *and for* $\zeta \in (0, \infty)$;

($\wp-$FN3) $\aleph_\wp(x + y, \zeta + \delta) \geq \aleph_\wp(x, \zeta) \circledast \mu(y, \delta)$ *for all* $x, y \in X$ *and for* $\zeta, \delta \in (0, \infty)$;

($\wp-$FN4) $\lim_{\zeta\to+\infty} \aleph_\wp(x, \zeta) = 1$ *for any* $\zeta \in (0, \infty)$.

Recall that a $\wp-$Banach FN space is a complete $\wp-$FN space. Now, we use the concept of probability distribution functions to measure the certainty of a vector [3], where we put

$$\epsilon_0(\zeta) = \begin{cases} 0, & if \ \ \zeta \le 0, \\ 1, & if \ \ \zeta > 0. \end{cases}$$

Definition 4.3 *Let $\wp \in (0,1)$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A $\wp-$random normed space ($\wp-$RNS) is a triple $(X, {}^{\wp}\mu, \circledast')$, where X is a vector space over $\mathbb{K}$, $\circledast'$ is a continuous $t-$norm and ${}^{\wp}\mu$ is a mapping from X into $(0,1]$ such that*

($\wp-$RNS1) ${}^{\wp}\mu_x(\zeta) = \varepsilon_0(\zeta)$ *for all $\zeta > 0$ if and only if $x = 0$;*

($\wp-$RNS2) ${}^{\wp}\mu_{\alpha\ x}(\zeta) = {}^{\wp}\mu_x\left(\frac{\zeta}{|\alpha|^{\wp}}\right)$ *for all $x \in X$ and $\alpha \neq 0$;*

($\wp-$RNS3) ${}^{\wp}\mu_{x+y}(\zeta + \delta) \geq {}^{\wp}\mu_x(\zeta) \circledast' {}^{\wp}\mu_y(\delta)$ *for all $x, y \in X$ and $\zeta, \delta \geq 0$,where ${}^{\wp}\mu_x$ denotes the value of ${}^{\wp}\mu$ at a point $x \in X$.*

Definition 4.4 *Let $\wp \in \mathbb{R}$ and $\wp \in (0,1)$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We define a matrix-valued function $\widetilde{Z} : X \times \mathbb{R}^+ \to x_{\Theta_1}$ with*

$$\widetilde{Z}(x,\zeta) = diag[\aleph_{\wp}(x,\zeta), {}^{\wp}\mu_x(\zeta), \aleph_{\wp}(x,\zeta) \circledast {}^{\wp}\mu_x(\zeta)]$$

and call it a generalized Z-number (GZ-N)*, when for all $x, y \in X$ and $\zeta, \delta \geq 0$, $\alpha \neq 0$, the following conditions are satisfied:*

(GZN1) $\widetilde{Z}(x,\zeta) = diag[1,1,1] = 1$ *if and only if $x = 0$;*

(GZN2) $\widetilde{Z}(\alpha\ x, \zeta) = \widetilde{Z}\left(\frac{\zeta}{|\alpha|^{\wp}}\right)$ *for all $x \in X$ and $\alpha \neq 0$;*

(GZN3) $\widetilde{Z}(x + y, \zeta + \delta) \succeq \widetilde{Z}(x,\zeta) \circledast_M \widetilde{Z}(y,\delta)$.

Definition 4.5 *Let $(X, \widetilde{Z})$ be a GZ-N. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\widetilde{Z}(x_n - x, \zeta) > 1 - \lambda$ for all $n \geq N$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\widetilde{Z}(x_n - x_m, \zeta) > 1 - \lambda$ for all $n \geq m \geq N$. Moreover, A GZ-N $(X, \widetilde{Z})$ is said to be complete if every Cauchy sequence in X is convergent to a point in X.*

In order to investigate the generalized Hyers-Ulam stability of functional equation (1.3) in GZ-N, from now on, it assumed that $\mathcal{V}_1$ is a $\wp - N$ left $\mathbb{C}-$ module and $\mathcal{V}_2$ is a $\wp - N$ left Banach $\mathbb{C}-$module.

4.2. Quadratic case GZ-N stability analysis

In the current subsection, we deal with Hyers-Ulam stability of functional equation (1.3) for quadratic case.

Theorem 4.1 *Let $W : \mathcal{V}_1 \times \mathcal{V}_1 \times \mathbb{R}^+ \longrightarrow x_{\Theta_1}$ be a function with the condition*

$$\lim_{D\to\infty} W\left(F_I^D x, F_I^D y, F_I^{2D\wp}\zeta\right) = 1, \tag{4.2}$$

where

$$F_I = \begin{cases} 3 & I = 0, \\ \frac{1}{3} & I = 1, \end{cases} \tag{4.3}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Moreover, there exists $\mathcal{L} = \mathcal{L}(I)$ such that the function W_{25} has the property

$$W_{25}(y,\zeta) = W_{25}\left(\frac{y}{3}, 3^{\wp}\ \zeta\right), \quad W_{25}\left(F_I y, F_I^{2\wp}\ \zeta\right) = W_{25}\left(y, \frac{1}{\mathcal{L}}\zeta\right), \tag{4.4}$$

for all $y \in \mathcal{V}_1$ *and all* $\zeta > 0$, *where* $W_{25}(y,\zeta) = W\left(y, y, \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(y, 0, \frac{\zeta}{1+5^{\wp}}\right)$. *Suppose that* $Q_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *be an even function satisfying the inequality*

$$\widetilde{Z}\Big(\mathbf{D}Q_2(x,y), \zeta\Big) \succeq W(x,y,\zeta), \tag{4.5}$$

where

$$W(x,y,\zeta) = diag\Big[W_1(x,y,\zeta), W_2(x,y,\zeta), W_1(x,y,\zeta) \circledast W_2(x,y,\zeta)\Big], \tag{4.6}$$

for all $x, y \in \mathcal{V}_1$ *and all* $\zeta > 0$, *in which* $W_1 : \mathcal{V}_1 \times \mathcal{V}_1 \times \mathbb{R}^+ \longrightarrow \Theta_1$ *and* $W_2 : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow (0,1]$ *are functions. Then, there exists a unique quadratic mapping* $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *satisfying equation* (1.3) *and*

$$\widetilde{Z}\Big(Q_2(y) - \mathcal{T}_2(y), \zeta\Big) \succeq W_{25}\left(y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\zeta\right), \tag{4.7}$$

for all $y \in \mathcal{V}_1$ *and all* $\zeta > 0$.

Proof: Consider the set $\mathcal{X} = \{q_2 |\, q_2 : \mathcal{V}_1 \to \mathcal{V}_2,\ q_2(0) = 0\}$ and define the generalized metric on $\mathcal{X}$ as

$$d(q_2, q_5) = \inf\Big\{K \in (0,\infty) : \widetilde{Z}(q_2(y) - q_5(y), \zeta) \succeq W_{25}\Big(y, \frac{1}{K}\,\zeta\Big), y \in \mathcal{V}_1,\ \zeta > 0\Big\}.$$

The proof of completeness of $(\mathcal{X}, d)$ is routine. Define a mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ through

$$Tq_2(y) = \frac{1}{F_I^2} q_2(F_I\ y), \quad (y \in \mathcal{V}_1).$$

On the other hand, one can show that T is a strictly contractive mapping on $\mathcal{X}$ with Lipschitz constant $\mathcal{L}$ (see [27]). Using the evenness of Q_2 in (4.5), we get

$$\begin{aligned}\widetilde{Z}\,(2[Q_2(2x+y) + Q_2(2x-y)] + Q_2(x+2y) + Q_2(x-2y) + 28Q_2(y)\\ -20[Q_2(x+y) + Q_2(x-y)] + 22Q_2(x), \zeta) \succeq W(x,y,\zeta),\end{aligned} \tag{4.8}$$

for all $x, y \in \mathcal{V}_1$ and all $\zeta > 0$. Putting $x = y$ in (3.4) and using again the evenness of Q_2, we obtain

$$\widetilde{Z}\,(3Q_2(3y) - 20Q_2(2y) + 53Q_2(y), \zeta) \succeq W(x,y,\zeta), \tag{4.9}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Replacing (x,y) by $(y,0)$ in (4.8) and using property (GZN2), we have

$$\widetilde{Z}\,(4Q_2(2y) - 16Q_2(y), \zeta) \succeq W(y,0,\zeta)$$

and so

$$\widetilde{Z}\,(20Q_2(2y) - 80Q_2(y), 5^{\wp}\zeta) \succeq W(y,0,\zeta), \tag{4.10}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Combining (4.9) and (4.10), we find

$$\widetilde{Z}\,(3Q_2(3y) - 27Q_2(y), \zeta) \succeq W\left(y, y, \frac{1}{1+5^{\wp}}\zeta\right) \circledast_M W\left(y, 0, \frac{1}{1+5^{\wp}}\zeta\right) = W_{25}(y,\zeta), \tag{4.11}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Using (GZN2), the above inequality can be rewritten by as follows:

$$\widetilde{Z}\left(\frac{Q_2(3y)}{9} - Q_2(y), \zeta\right) \succeq W_{25}\left(y, 27^{\wp}\ \zeta\right), \tag{4.12}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. For the case $I = 0$, by applying (4.4) and (4.12), we arrive at

$$\widetilde{Z}\,(TQ_2(y) - Q_2(y), \zeta) \succeq W_{25}\left(y, \frac{1}{\mathcal{L}}\zeta\right)$$

and hence $d(TQ_2, Q_2) \leq \mathcal{L} = \mathcal{L}^{1-I} < \infty$, for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Furthermore, replacing y by $\frac{y}{3}$ in (4.11) and using (GZN2), we have

$$\widetilde{Z}\left(Q_2(y) - 9Q_2\left(\frac{y}{3}\right), \frac{1}{3^{\wp}}\zeta\right) \succeq W_{25}\left(\frac{y}{3}, \zeta\right)$$

and so

$$\widetilde{Z}\left(Q_2(y) - 9Q_2\left(\frac{y}{3}\right), \zeta\right) \succeq W_{25}\left(\frac{y}{3}, 3^{\wp}\zeta\right), \tag{4.13}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. In a similar way, it deduces by (4.4) and (4.13) for the case $I = 1$ that $d(Q_2, TQ_2) \leq 1 = \mathcal{L}^{1-I} < \infty$. Summing up: $d(Q_2, TQ_2) \leq \mathcal{L}^{1-I}$. Therefore, the validity of (FPC1) is proved from Theorem 3.4. By (FPC2) of Theorem 3.4, it follows that there exists a unique fixed point $\mathcal{T}_2$ of T in $\mathcal{X}$ such that

$$\widetilde{Z}\left(\mathcal{T}_2(y) - \lim_{D\to\infty} \frac{Q_2(F_I^D y\ )}{F_I^{2D}}, \zeta\right) = 1, \qquad (y \in \mathcal{V}_1,\ \zeta > 0).$$

We now claim that $\mathcal{T}_2$ satisfies (1.3). Switching (x, y) by $(F_I^D x,\ F_I^D y)$ and using (GZN2) in (4.5), we obtain

$$\begin{aligned}\widetilde{Z}\Big(\frac{1}{F_I^{2D}}\Big\{2\Big[Q_2\Big(F_I^D(2x+y)\Big) + Q_2\Big(F_I^D(2x-y)\Big)\Big] + Q_2\Big(F_I^D(x+2y)\Big) + Q_2\Big(F_I^D(x-2y)\Big)\\ + 14\Big[Q_2\Big(F_I^D(y)\Big) + Q_2\Big(F_I^D(-y)\Big)\Big] - 20\Big[Q_2\Big(F_I^D(x+y)\Big)\\ + Q_2\Big(F_I^D(x-y)\Big)\Big] - 2\Big[17Q_2\Big(F_I^D(x)\Big) - 28Q_2\Big(F_I^D(-x)\Big)\Big]\Big\}, \zeta\Big)\\ \succeq W\Big(\Big(F_I^D\ x, F_I^D\ y\Big), F_I^{2D\wp}\zeta\Big)\end{aligned}$$

for all $x, y \in \mathcal{V}_1$ and all $\zeta > 0$. Letting $D \to \infty$ in the above inequality, using the definition of $\mathcal{T}_2$, (4.2) and (GZN1), we see that $\mathcal{T}_2$ satisfies (1.3) for all $x, y \in \mathcal{V}_1$. Once more, by (FPC3) of Theorem 3.4, it concludes that $\mathcal{T}_2$ is the unique fixed point of T in the set

$$\mathcal{Y} = \{\mathcal{T}_2(y) \in \mathcal{X} : d(Q_2, \mathcal{T}_2(y)) < \infty\},$$

such that

$$d(Q_2(y) - \mathcal{T}_2(y)) \leq K\ W(y, y, \zeta),$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Eventually, part (FPC4) of Theorem 3.4 implies that $d(Q_2, \mathcal{T}_2(y)) \leq \frac{1}{1-\mathcal{L}}\ d(Q_2, \mathcal{T}_2(y))$ and therefore $d(Q_2, \mathcal{T}_2(y)) \leq \frac{\mathcal{L}^{1-I}}{1-\mathcal{L}}$, which yields

$$\widetilde{Z}\Big(Q_2(y) - \mathcal{T}_2(y), \zeta\Big) \succeq W_{25}\left(y, \frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}}\zeta\right),$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. This finishes the proof. □

A corollary of the mentioned theorem is given as follows.

Corollary 4.1 *Given positive numbers M and N. Let $Q_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an even mapping fulfilling the inequality*

$$\widetilde{Z}\Big(\mathbf{D}Q_2(x,y), \zeta\Big) \succeq \begin{cases} diag\left(\frac{\zeta}{M}, e^{-\frac{1}{\frac{\zeta}{M}}}, \frac{\zeta}{M} \circledast e^{\frac{1}{\frac{\zeta}{M}}}\right), \\ diag\left(\frac{\frac{\zeta}{M}}{\frac{\zeta}{M} + \{||x||_{\wp}^N + ||y||_{\wp}^N\}}, e^{-\frac{\{||x||_{\wp}^N + ||y||_{\wp}^N\}}{\frac{\zeta}{M}}}, \frac{\frac{\zeta}{M}}{\frac{\zeta}{M} + \{||x||_{\wp}^N + ||y||_{\wp}^N\}} \circledast e^{-\frac{\{||x||_{\wp}^N + ||y||_{\wp}^N\}}{\frac{\zeta}{M}}}\right), \end{cases}$$

for all $x, y \in \mathcal{V}_1$. *Then, there exists a unique quadratic mapping* $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ *satisfies functional equation* (1.3) *and*

$$\widetilde{Z}\Big(\mathcal{T}_2(\ y\ ) - Q_2(\ y\ ), \zeta\Big) \succeq \begin{cases} diag\left(\frac{3^{\wp}|(9)^{\wp}-1|\zeta}{(1+5^{\wp})M}, e^{-\frac{1}{\frac{3^{\wp}|(9)^{\wp}-1|\zeta}{(1+5^{\wp})M}}}, \frac{3^{\wp}|(9)^{\wp}-1|\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}|(9)^{\wp}-1|\zeta}{(1+5^{\wp})M}}}\right), \\ diag\left(\frac{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}}{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}+||y||_{\wp}^{N}}, e^{-\frac{||y||_{\wp}^{N}}{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}}}, \right. \\ \qquad \left. \frac{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}}{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}+||y||_{\wp}^{N}} \circledast e^{-\frac{||y||_{\wp}^{N}}{\frac{(3)^{\wp}|(3)^{2\wp}-(3)^{N}|\zeta}{(1+5^{\wp})M}}}\right), \end{cases}$$

for all $y \in \mathcal{V}_1$ *and all* $\zeta > 0$.

Proof: One can routinely verify that all hypotheses in Theorem 4.2 hold. From (4.1) and (4.4), we have

$$\begin{aligned} W_{25}(y,\zeta) &= W_{25}\left(\frac{y}{3}, 3^{\wp}\ \zeta\right) \\ &\succeq W\left(\frac{y}{3}, \frac{y}{3}, 3^{\wp}\ \zeta\right) \circledast_M W\left(\frac{y}{3}, 0, 3^{\wp}\ \zeta\right) \\ &= \begin{cases} \mathrm{diag}\left(\frac{3^{\wp}\zeta}{M}, e^{-\frac{1}{\frac{3^{\wp}\zeta}{M}}}, \frac{3^{\wp}\zeta}{M} \circledast e^{\frac{1}{\frac{3^{\wp}\zeta}{M}}}\right) \\ \mathrm{diag}\left(\frac{\frac{3^{\wp}\zeta}{M}}{\frac{3^{\wp}\zeta}{M}+\frac{1}{3^N}||y||_{\wp}^{N}}, e^{-\frac{\frac{1}{3^N}||y||_{\wp}^{N}}{\frac{3^{\wp}\zeta}{M}}}, \frac{\frac{3^{\wp}\zeta}{M}}{\frac{3^{\wp}\zeta}{M}+\frac{1}{3^N}||y||_{\wp}^{N}} \circledast e^{-\frac{\frac{1}{3^N}||y||_{\wp}^{N}}{\frac{3^{\wp}\zeta}{M}}}\right) \end{cases} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &W_{25}\left(F_I y, F_I^{2\wp}\ \zeta\right) \\ &\succeq W\left(F_I y, F_I y, F_I^{2\wp}\ \zeta\right) \circledast_M W\left(F_I y, 0, F_I^{2\wp}\ \zeta\right) \\ &= \begin{cases} \mathrm{diag}\left(\frac{F_I^{2\wp}\zeta}{M}, e^{-\frac{1}{\frac{F_I^{2\wp}\zeta}{M}}}, \frac{F_I^{2\wp}\zeta}{M} \circledast e^{\frac{1}{\frac{F_I^{2\wp}\zeta}{M}}}\right) \\ \mathrm{diag}\left(\frac{\frac{F_I^{2\wp}\zeta}{M}}{\frac{F_I^{2\wp}\zeta}{M}+F_I^N\ ||y||_{\wp}^{N}}, e^{-\frac{F_I^N\ ||y||_{\wp}^{N}}{\frac{F_I^{2\wp}\zeta}{M}}}, \frac{\frac{F_I^{2\wp}\zeta}{M}}{\frac{F_I^{2\wp}\zeta}{M}+F_I^N\ ||y||_{\wp}^{N}} \circledast e^{-\frac{F_I^N\ ||y||_{\wp}^{N}}{\frac{F_I^{2\wp}\zeta}{M}}}\right) \end{cases} = \begin{cases} W_{25}\left(y, \frac{1}{\mathcal{L}}\zeta\right), \\ W_{25}\left(y, \frac{1}{\mathcal{L}}\zeta\right). \end{cases} \end{aligned} \tag{4.15}$$

The following cases can be obtained from (4.14) and (4.15). For the cases

$$\begin{cases} I = 0;\ \mathcal{L} = \frac{1}{F_I{}^{2\wp}} = \frac{1}{3^{2\wp}} = 3^{-2\wp};\ \frac{1-\mathcal{L}}{\mathcal{L}^{1-0}} = \frac{1-3^{-2\wp}}{3^{-2\wp}} = 3^{2\wp} - 1, \\ I = 1;\ \mathcal{L} = \frac{1}{F_I{}^{2\wp}} = \frac{1}{\frac{1}{3^{2\wp}}} = 3^{2\wp};\ \frac{1-\mathcal{L}}{\mathcal{L}^{1-1}} = 1 - 3^{2\wp}, \end{cases}$$

from (4.7), we have

$$\begin{aligned} \widetilde{Z}\Big(Q_2(y) - \mathcal{T}_2(y), \zeta\Big) &\succeq W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\ &= \mathrm{diag}\left(\frac{3^{\wp}[9^{\wp}-1]\zeta}{(1+5^{\wp})M}, e^{-\frac{1}{\frac{3^{\wp}[9^{\wp}-1]\zeta}{(1+5^{\wp})M}}}, \frac{3^{\wp}[9^{\wp}-1]\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}[9^{\wp}-1]\zeta}{(1+5^{\wp})M}}}\right), \\ \widetilde{Z}\Big(Q_2(y) - \mathcal{T}_2(y), \zeta\Big) &\succeq W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\ &= \mathrm{diag}\left(\frac{3^{\wp}[1-9^{\wp}]\zeta}{(1+5^{\wp})M}, e^{-\frac{1}{\frac{3^{\wp}[1-9^{\wp}]\zeta}{(1+5^{\wp})M}}}, \frac{3^{\wp}[1-9^{\wp}]\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}[1-9^{\wp}]\zeta}{(1+5^{\wp})M}}}\right). \end{aligned}$$

In addition, for the cases

$$\begin{cases} I=0;\ \mathcal{L}=\frac{1}{F_I^{2\wp-N}}=\frac{1}{3^{2\wp-N}}=3^{N-2\wp};\ \frac{1-\mathcal{L}}{\mathcal{L}^{1-0}}=\frac{1-3^{N-2\wp}}{3^{N-2\wp}}=\frac{3^{2\wp}-3^N}{3^N}, \\ I=1;\ \mathcal{L}=\frac{1}{\left(\frac{1}{F_I}\right)^{2\wp-N}}=\frac{1}{\left(\frac{1}{3}\right)^{2\wp-N}}=3^{2\wp-N};\ \frac{1-\mathcal{L}}{\mathcal{L}^{1-1}}=1-3^{2\wp-N}=\frac{3^N-3^{2\wp}}{3^N}, \end{cases}$$

it follows from (4.7) that

$$\begin{aligned} &\widetilde{Z}\Big(Q_2(y)-\mathcal{T}_2(y),\zeta\Big) \\ &\succeq W\left(y,y,\frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\cdot\frac{\zeta}{1+5^\wp}\right)\circledast_M W\left(y,0,\frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\cdot\frac{\zeta}{1+5^\wp}\right) \\ &= \operatorname{diag}\left(\frac{\frac{3^\wp[3^{2\wp}-3^N]\zeta}{(1+5^\wp)M}}{\frac{3^\wp[3^{2\wp}-3^N]\zeta}{(1+5^\wp)M}+||y||_\wp^N}, e^{-\frac{||y||_\wp^N}{\frac{3^\wp[3^{2\wp}-3^N]\zeta}{(1+5^\wp)M}}}, \frac{\frac{3^\wp[3^{2\wp}-3^N]\zeta}{(1+5^\wp)M}}{\frac{3^\wp[3^{2\wp}-3^N]\zeta}{(1+5^\wp)M}+||y||_\wp^N}\circledast e^{-\frac{||y||_\wp^N}{\frac{3^\wp[3^{2\wp}-(3)^N]\zeta}{(1+5^\wp)M}}}\right), \\ &\widetilde{Z}\Big(Q_2(y)-\mathcal{T}_2(y),\zeta\Big) \\ &\succeq W\left(y,y,\frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\cdot\frac{\zeta}{1+5^\wp}\right)\circledast_M W\left(y,0,\frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\cdot\frac{\zeta}{1+5^\wp}\right) \\ &= \operatorname{diag}\left(\frac{\frac{3^\wp[3^N-3^{2\wp}]\zeta}{(1+5^\wp)M}}{\frac{3^\wp[3^N-3^{2\wp}]\zeta}{(1+5^\wp)M}+||y||_\wp^N}, e^{-\frac{||y||_\wp^N}{\frac{3^\wp[3^N-3^{2\wp}]\zeta}{(1+5^\wp)M}}}, \frac{\frac{3^\wp[3^N-3^{2\wp}]\zeta}{(1+5^\wp)M}}{\frac{3^\wp[3^N-3^{2\wp}]\zeta}{(1+5^\wp)M}+||y||_\wp^N}\circledast e^{-\frac{||y||_\wp^N}{\frac{3^\wp[3^N-3)^N]\zeta}{(1+5^\wp)M}}}\right). \end{aligned}$$

The proof in now complete. □

4.3. Quintic case GZ-N stability analysis

This subsection is devoted to establish Hyers-Ulam stability of functional equation (1.3) for quintic case. A similar method of the proof of Theorem 4.1 can be used for the odd case of (1.3) as follows and hence we do not include its proof.

Theorem 4.2 *Let $W:\mathcal{V}_1\times\mathcal{V}_1\times\mathbb{R}^+\longrightarrow x_{\Theta_1}$ be a function with the condition*

$$\lim_{D\to\infty} W\left(F_I^D x, F_I^D y, F_I^{5D\wp}\zeta\right)=1, \tag{4.16}$$

for all $y\in\mathcal{V}_1$ and all $\zeta>0$, where F_I is defined in (4.3) Moreover, there exists $\mathcal{L}=\mathcal{L}(I)$ such that the function W_{25} has the property

$$W_{25}(y,\zeta)=W_{25}\left(\frac{y}{3},3^\wp\ \zeta\right),\quad W_{25}\left(F_I y, F_I^{5\wp}\ \zeta\right)=W_{25}\left(y,\frac{1}{\mathcal{L}}\zeta\right), \tag{4.17}$$

for all $y\in\mathcal{V}_1$ and all $\zeta>0$, where $W_{25}(y,\zeta)=W\left(y,y,\frac{\zeta}{1+5^\wp}\right)\circledast_M W\left(y,0,\frac{\zeta}{1+5^\wp}\right)$. Suppose that $Q_5:\mathcal{V}_1\longrightarrow\mathcal{V}_2$ be an odd function satisfying the inequality

$$\widetilde{Z}\Big(\mathbf{D}Q_5(x,y),\zeta\Big)\succeq W(x,y,\zeta),$$

where

$$W(x,y,\zeta)=diag\Big[W_1(x,y,\zeta),W_2(x,y,\zeta),W_1(x,y,\zeta)\circledast W_2(x,y,\zeta)\Big],$$

for all $x,y\in\mathcal{V}_1$ and all $\zeta>0$, in which $W_1:\mathcal{V}_1\times\mathcal{V}_1\times\mathbb{R}^+\longrightarrow\Theta_1$ and $W_2:\mathcal{V}_1\times\mathcal{V}_1\longrightarrow(0,1]$ are functions. Then, there exists a unique quintic mapping $\mathcal{T}_5:\mathcal{V}_1\longrightarrow\mathcal{V}_2$ satisfying equation (1.3) and

$$\widetilde{Z}\Big(Q_5(y)-\mathcal{T}_5(y),\zeta\Big)\succeq W_{25}\left(y,\frac{1-\mathcal{L}}{\mathcal{L}^{1-I}}\zeta\right),$$

for all $y\in\mathcal{V}_1$ and all $\zeta>0$.

Corollary 4.2 *Given positive numbers M and N. Let $Q_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be an odd mapping fulfilling the inequality*

$$\widetilde{Z}\Big(\mathbf{D}Q_5(x,y),\zeta\Big) \succeq \begin{cases} diag\left(\frac{\zeta}{M}, e^{-\frac{1}{\frac{\zeta}{M}}}, \frac{\zeta}{M} \circledast e^{\frac{1}{\frac{\zeta}{M}}}\right) \\ diag\left(\frac{\frac{\zeta}{M}}{\frac{\zeta}{M}+\{||x||_{\wp}^{N}+||y||_{\wp}^{N}\}}, e^{-\frac{\{||x||_{\wp}^{N}+||y||_{\wp}^{N}\}}{\frac{\zeta}{M}}}, \frac{\frac{\zeta}{M}}{\frac{\zeta}{M}+\{||x||_{\wp}^{N}+||y||_{\wp}^{N}\}} \circledast e^{-\frac{\{||x||_{\wp}^{N}+||y||_{\wp}^{N}\}}{\frac{\zeta}{M}}}\right), \end{cases}$$

for all $x, y \in \mathcal{V}_1$. Then, there exists a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying equation (1.3) and

$$\widetilde{Z}\Big(\mathcal{T}_5(y) - Q_5(y),\zeta\Big) \succeq \begin{cases} diag\left(\frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}, e^{-\frac{1}{\frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}}}, \frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}}}\right) \\ diag\left(\frac{\frac{3^{\wp}|3^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}+||y||_{\wp}^{N}}, e^{-\frac{||y||_{\wp}^{N}}{\frac{3^{\wp}|3^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}}}, \right. \\ \qquad \left. \frac{\frac{3^{\wp}|(3)^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}+||y||_{\wp}^{N}} \circledast e^{-\frac{||y||_{\wp}^{N}}{\frac{(3)^{\wp}|(3)^{5\wp}-3^{N}|\zeta}{(1+5^{\wp})M}}}\right), \end{cases}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$.

4.4. Quadratic-quintic case GZ-N stability analysis

In this subsection, we prove the Hyers-Ulam stability of functional equation (1.3) when the corresponding mapping has no any condition.

Theorem 4.3 *Let $W : \mathcal{V}_1 \times \mathcal{V}_1 \times \mathbb{R}^+ \longrightarrow x_{\Theta_1}$ be a function with conditions (4.2) and (4.16). Moreover, there exists $\mathcal{L} = \mathcal{L}(I)$ such that the properties (4.4) and (4.17) hold for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Suppose that $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be a mapping satisfying the inequality*

$$\widetilde{Z}\Big(\mathbf{D}Q(x,y),\zeta\Big) \succeq W(x,y,\zeta), \tag{4.18}$$

where

$$W(x,y,\zeta) = diag\Big[W_1(x,y,\zeta), W_2(x,y,\zeta), W_1(x,y,\zeta) \circledast W_2(x,y,\zeta)\Big],$$

for all $x, y \in \mathcal{V}_1$ and all $\zeta > 0$, in which $W_1 : \mathcal{V}_1 \times \mathcal{V}_1 \times \mathbb{R}^+ \longrightarrow \Theta_1$ and $W_2 : \mathcal{V}_1 \times \mathcal{V}_1 \longrightarrow (0,1]$ are functions. Then, there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ and a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfying (1.3) and

$$\begin{aligned} &\widetilde{Z}\Big(Q(y) - \mathcal{T}_2(y) - \mathcal{T}_5(y), 8\zeta\Big) \\ &\succeq W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, -y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\ &\quad \circledast_M W\left(y, 0, \frac{(1-\mathcal{L})}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\ &\quad \circledast_M W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, -y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\ &\quad \circledast_M W\left(y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right), \end{aligned} \tag{4.19}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$.

Proof: Assume that $Q_e, Q_o : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ as defined in the proof of Theorem 3.3. The mapping Q_e satisfies (4.18) and so it follows from (3.19) and Theorem 4.1 that

$$\begin{aligned}
&\widetilde{Z}\Big(Q_e(y) - \mathcal{T}_5(y), \frac{4\zeta}{1+5^{\wp}}\Big) \\
&\succeq W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, -y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{(1+5^{\wp})}\right) \\
&\quad \circledast_M W\left(y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right),
\end{aligned} \tag{4.20}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. Moreover, since $Q_o : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ satisfies (4.18), relation (3.21) and Theorem 4.2 necessitate that

$$\begin{aligned}
&\widetilde{Z}\Big(Q_o(y) - \mathcal{T}_5(y), \frac{4\zeta}{1+5^{\wp}}\Big) \\
&\succeq W\left(y, y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, -y, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \\
&\quad \circledast_M W\left(y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right) \circledast_M W\left(-y, 0, \frac{1-\mathcal{L}}{\mathcal{L}^{1-I}} \cdot \frac{\zeta}{1+5^{\wp}}\right),
\end{aligned} \tag{4.21}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$. It deduces from (3.20), (4.20) and (4.21) that relation (4.19) is valid. □

The following corollary is a immediate consequence of Theorem 4.3 concerning the stabilities of (1.3) in the general case.

Corollary 4.3 *Let M and N be positive numbers. Suppose that $Q : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is a mapping satisfies*

$$\begin{aligned}
&\widetilde{Z}\Big(\mathbf{D}Q(x,y), \zeta\Big) \\
&\succeq \begin{cases} diag\left(\frac{\zeta}{M}, e^{-\frac{1}{\frac{\zeta}{M}}}, \frac{\zeta}{M} \circledast e^{\frac{1}{\frac{\zeta}{M}}}\right) \\ diag\left(\frac{\frac{\zeta}{M}}{\frac{\zeta}{M} + \{||x||_{\wp}^N + ||y||_{\wp}^N\}}, e^{-\frac{\{||x||_{\wp}^N + ||y||_{\wp}^N\}}{\frac{\zeta}{M}}}, \frac{\frac{\zeta}{M}}{\frac{\zeta}{M} + \{||x||_{\wp}^N + ||y||_{\wp}^N\}} \circledast e^{-\frac{\{||x||_{\wp}^N + ||y||_{\wp}^N\}}{\frac{\zeta}{M}}}\right), \end{cases}
\end{aligned}$$

for all $x, y \in \mathcal{V}_1$ and all $\zeta > 0$. Then, there exist a unique quadratic mapping $\mathcal{T}_2 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ and a unique quintic mapping $\mathcal{T}_5 : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ fulfills equation (1.3) and

$$\begin{aligned}
&\widetilde{Z}\Big(Q(y) - \mathcal{T}_2(y) - \mathcal{T}_5(y), 8\zeta\Big) \\
&\succeq \begin{cases} diag\Bigg(\min\left[\frac{3^{\wp}|9^{\wp}-1|\zeta}{(1+5^{\wp})M}, \frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}\right], \min\left[e^{-\frac{1}{\frac{3^{\wp}|9^{\wp}-1|\zeta}{(1+5^{\wp})M}}}, e^{-\frac{1}{\frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}}}\right], \\ \qquad \min\left[\frac{3^{\wp}|9^{\wp}-1|\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}|9^{\wp}-1|\zeta}{(1+5^{\wp})M}}}, \frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M} \circledast e^{\frac{1}{\frac{3^{\wp}|243^{\wp}-1|\zeta}{(1+5^{\wp})M}}}\right]\Bigg) \\ diag\Bigg(\min\left[\frac{\frac{3)^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M} + ||y||_{\wp}^N}, \frac{\frac{3^{\wp}|3^{5\wp}-3^N|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{5\wp}-3^N|\zeta}{(1+5^{\wp})M} + ||y||_{\wp}^N}\right], \\ \qquad \min\left[e^{-\frac{||y||_{\wp}^N}{\frac{(3)^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M}}}, e^{-\frac{||y||_{\wp}^N}{\frac{3^{\wp}|3^{5\wp}-3^N|\zeta}{(1+5^{\wp})M}}}\right], \\ \qquad \min\left[\frac{\frac{3^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M} + ||y||_{\wp}^N} \circledast e^{-\frac{||y||_{\wp}^N}{\frac{3^{\wp}|3^{2\wp}-3^N|\zeta}{(1+5^{\wp})M}}}, \right. \\ \qquad \left. \frac{\frac{3^{\wp}|(3)^{5\wp}-3^N|\zeta}{(1+5^{\wp})M}}{\frac{3^{\wp}|3^{5\wp}-(3)^N|\zeta}{(1+5^{\wp})M} + ||y||_{\wp}^N} \circledast e^{-\frac{||y||_{\wp}^N}{\frac{3^{\wp}|3^{5\wp}-3^N|\zeta}{(1+5^{\wp})M}}}\right]\Bigg), \end{cases}
\end{aligned}$$

for all $y \in \mathcal{V}_1$ and all $\zeta > 0$.

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