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Advances in Neutrosophic Graph Theory: Topological Structures, Neutrosophic Bridges, and Applications via MATLAB

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ABSTRACT: As modern systems become increasingly complex, there is a growing need for advanced mathematical frameworks capable of modeling uncertainty, vagueness, and indeterminacy. This research presents a novel contribution by integrating neutrosophic set theory with graph theory and topology, providing a more flexible model for handling complex networked data. The study investigates the fundamental properties of neutrosophic graphs (NG), introducing new types of arcs along with two innovative constructs: the neutrosophic ψ-bridges and Ξ-bridges, offering deeper insights into the structural dynamics of neutrosophic graphs. A major advancement in this work is the introduction of the neutrosophic Generalized Adjacency Topological Space (NGATTS), a novel topological framework that redefines classical concepts such as T-closure and T-open sets in the context of graph structures. Through a rigorous series of definitions and theorems, the paper proves that every NGATTS can be represented by a graph and vice versa. To validate the theoretical findings, the neutrosophic graphs and their related structures were implemented and analyzed using MATLAB. The practical experiments involved constructing various well-known graph models such as complete graphs, wheel graphs, cycles, stars, and weak graphs environment, successfully visualizing the new types of arcs and bridges, and confirming their compliance with the NGATTS framework. Overall, this research provides a comprehensive and innovative bridge between topology and graph theory, combining robust theoretical development with practical implementation. It opens new directions for the topological analysis of complex networks and offers a valuable reference for researchers in applied mathematics and network science.

Key Words: NGATTS, η_ψ-strong, σ_ψ-strong, ψ-bridge, Ξ-bridge.

Contents

1. Introduction

The mathematical modeling of uncertainty has witnessed significant advancements since the pioneering work of Zadeh in 1965 [15], who introduced the concept of fuzzy sets a breakthrough that laid the groundwork for numerous extensions across diverse branches of mathematics. In 1986, Atanassov [11] [12], expanded this concept by proposing intuitionistic fuzzy sets, wherein each element in a universe is assigned two independent degrees: membership and non-membership. This structure offers a more flexible and expressive framework for representing vagueness and uncertainty.

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Building upon these foundations, fuzzy topology emerged as an extension of classical topology through the work of Chang in 1968, [6]. Over the years, this field has evolved considerably, undergoing multiple reformulations to accommodate various generalizations such as fuzzy topological spaces, [2] [7] [8]. With the introduction of neutrosophic set theory—most notably advanced by Smarandache and colleagues, the notion of uncertainty was further expanded to include three distinct components: truth, falsity, and indeterminacy. This has given rise to neutrosophic topology, a robust platform for exploring complex systems characterized by incomplete or inconsistent information.

In parallel, the study of graphs has undergone similar transformations. Since Rosenfeld [1] introduction of fuzzy graphs in 1975, subsequent contributions particularly from Yeh, Bang, Al-Omeri, and Kaviyarasu, [3] [17] [18] [19] have significantly enriched the field. These developments have led to versatile tools for addressing a broad spectrum of combinatorial problems across algebra, topology, optimization, and computer science. Bhattacharya [16] further extended this area by investigating operations on fuzzy graphs, while Broumi et al [5] incorporated neutrosophic sets into graph theory in 2016, thereby introducing neutrosophic graphs a more expressive model capable of capturing uncertainty, vagueness, and indeterminacy in networked systems.

This paper investigates the essential characteristics and foundational structures of neutrosophic graphs (NG). We introduce and examine several types of arcs in neutrosophic graphs, including η_ψ-strong, σ_ψ-strong, and ρ_ψ-arcs, along with their corresponding η_Ξ-strong, σ_Ξ-strong, and ρ_Ξ-arcs. Additionally, we propose two novel constructs: the neutrosophic ψ-bridge and the neutrosophic Ξ-bridge, providing both theoretical definitions and analyses of their properties. These contributions enrich the theoretical understanding of neutrosophic graph structures and offer new perspectives for future research. A central innovation of this study is the development of a new topological concept known as the neutrosophic Generalized Adjacency Topological Space (NGATTS), which is constructed based on graph structures and vertex adjacency relations. This novel framework redefines the concepts of T-closure and T-open sets within graphs, establishing a close alignment with the properties of classical topological spaces. We present a systematic sequence of definitions, theorems, and propositions that rigorously establish the foundational properties of NGATTS. Notably, the paper proves that every space and every finitely disconnected space can be represented as a NGATTS, and vice versa. Through detailed examples and illustrative figures, the study demonstrates that various well-known graphs, such as complete graphs, wheel graphs, cycles, stars, and weak graphs satisfy the criteria of the NGATTS framework. This underscores the model's flexibility and its potential for broad application. Overall, this research offers a compelling bridge between abstract topological structures and graph theory. It opens promising avenues for further mathematical investigation, particularly in the topological analysis of networks. By balancing rigorous theoretical foundations with practical graphical illustrations, the study remains both accessible and intellectually stimulating for the advanced mathematical reader.

2. Preliminaries

This section introduces fundamental concepts and preliminary results that serve as essential tools for our manuscript.

Definition 2.1. Let $\mathcal{M}$ be a non-empty fixed set. A *neutrosophic set* (abbreviated as NS) on $\mathcal{M}$ is defined as

$$A = \{(\mathfrak{m}, \psi_{*A}(\mathfrak{m}), \Theta_{*A}(\mathfrak{m}), \Xi_{*A}(\mathfrak{m})) : \mathfrak{m} \in \mathcal{M}\},$$

where the functions

$$\psi_{*A},\ \Theta_{*A},\ \Xi_{*A} : \mathcal{M} \to [0,1]$$

denote, respectively, the degree of membership, degree of indeterminacy, and degree of non-membership of the element $\mathfrak{m} \in \mathcal{M}$ in the set A. These degrees satisfy the condition

$$0 \leq \psi_{*A}(\mathfrak{h}) + \Theta_{*A}(\mathfrak{h}) + \Xi_{*A}(\mathfrak{h}) \leq 3 \quad \text{for all } \mathfrak{m} \in \mathcal{M}.$$

Definition 2.2. Let $\mathcal{M}$ be a non-empty fixed set. If $\mathcal{I}$ and $\mathcal{J}$ are neutrosophic sets (NS) defined on $\mathcal{M}$, then the following operations and relationships hold:

1. **Inclusion:** $\mathcal{I} \subseteq \mathcal{J}$ if and only if, for every $\mathfrak{m} \in \mathcal{M}$,

$$\psi_{*\mathcal{I}}(\mathfrak{m}) \leq \psi_{*\mathcal{J}}(\mathfrak{m}), \quad \Theta_{*\mathcal{I}}(\mathfrak{m}) \leq \Theta_{*\mathcal{J}}(\mathfrak{m}), \quad \Xi_{*\mathcal{I}}(\mathfrak{m}) \geq \Xi_{*\mathcal{J}}(\mathfrak{m}).$$

2. **Complement:** The complement of $\mathcal{I}$ is given by

$$\mathcal{I}^c = \{(\mathfrak{m}, \Xi_{*\mathcal{I}}(\mathfrak{m}), \Theta_{*\mathcal{I}}(\mathfrak{m}), \psi_{*\mathcal{I}}(\mathfrak{m})) : \mathfrak{m} \in \mathcal{M}\}.$$

3. **Intersection:**

$$\mathcal{I} \cap \mathcal{J} = \{(\mathfrak{m}, \min\{\psi_{*\mathcal{I}}(\mathfrak{m}), \psi_{*\mathcal{J}}(\mathfrak{m})\}, \min\{\Theta_{*\mathcal{I}}(\mathfrak{m}), \Theta_{*\mathcal{J}}(\mathfrak{m})\}, \max\{\Xi_{*\mathcal{I}}(\mathfrak{m}), \Xi_{*\mathcal{J}}(\mathfrak{m})\}) : \mathfrak{m} \in \mathcal{M}\}.$$

4. **Union:**

$$\mathcal{I} \cup \mathcal{J} = \{(\mathfrak{m}, \max\{\psi_{*\mathcal{I}}(\mathfrak{m}), \psi_{*\mathcal{J}}(\mathfrak{m})\}, \max\{\Theta_{*\mathcal{I}}(\mathfrak{m}), \Theta_{*\mathcal{J}}(\mathfrak{m})\}, \min\{\Xi_{*\mathcal{I}}(\mathfrak{m}), \Xi_{*\mathcal{J}}(\mathfrak{m})\}) : \mathfrak{m} \in \mathcal{M}\}.$$

Definition 2.3. Let $\mathcal{M}$ be a non-empty fixed set, and let (r, t, s) be elements of the extended open interval $]0^-, 1^+[$. A neutrosophic set $x_{r,t,s}$ in $\mathcal{M}$ is called a *neutrosophic point* if it is defined by:

$$x_{r,t,s}(x) = \begin{cases} (r, t, s) & \text{if } x = x_p, \\ (0, 0, 1) & \text{if } x \neq x_p, \end{cases}$$

where $x_p \in \mathcal{M}$ is called the *support* of the neutrosophic point $x_{r,t,s}$, and (r, t, s) respectively represent the degrees of membership, indeterminacy, and non-membership of x to the neutrosophic set.

In this context, a neutrosophic singleton is denoted by

$$p = (\psi_{*p}, \Theta_{*p}, \Xi_{*p}).$$

Definition 2.4. Let $\mathcal{M} \neq \emptyset$, and let

$$A = \{(\mathfrak{m}, \psi_{*A}(\mathfrak{m}), \Theta_{*A}(\mathfrak{m}), \Xi_{*A}(\mathfrak{m})) : \mathfrak{m} \in \mathcal{M}\}$$

be a neutrosophic set defined on $\mathcal{M}$. A neutrosophic singleton $p = (\psi_{*p}, \Theta_{*p}, \Xi_{*p})$ is said to belong to the set A, denoted by $p \in A$, if

$$\psi_{*p} \leq \psi_{*A}, \quad \Theta_{*p} \leq \Theta_{*A}, \quad \Xi_{*A} \leq \Xi_{*p}.$$

Definition 2.5. Let $\mathcal{M} \neq \emptyset$. A *neutrosophic topology* (abbreviated as NTCT) on $\mathcal{H}$ is a collection $\mathcal{T}$ of neutrosophic sets (NS) in $\mathcal{M}$ satisfying the following conditions:

1. The neutrosophic null set
$$0_M = \{(\mathfrak{m}, 0, 0, 1) : \mathfrak{m} \in \mathcal{M}\}$$
and the neutrosophic universal set
$$1_M = \{(\mathfrak{m}, 1, 1, 0) : \mathfrak{m} \in \mathcal{M}\}$$
are elements of $\mathcal{T}$.

2. The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$; that is, if $\mathcal{I}, \mathcal{J} \in \mathcal{T}$, then $\mathcal{I} \cap \mathcal{J} \in \mathcal{T}$.

3. The union of any subfamily of sets in $\mathcal{T}$ belongs to $\mathcal{T}$; that is, if $\{\mathcal{I}_i : i \in \Lambda\} \subseteq \mathcal{T}$, then $\bigcup_{i\in\Lambda} \mathcal{I}_i \in \mathcal{T}$.

The pair $(\mathcal{M}, \mathcal{T})$ is called a *neutrosophic topological space* (NTS). Each element of $\mathcal{T}$ is referred to as a *neutrosophic open set* (NOS). The complement $\mathcal{I}^c$ of a neutrosophic open set $\mathcal{I} \in \mathcal{T}$ is called a *neutrosophic closed set* (NCS).

Definition 2.6. A neutrosophic topological space (NTS) is said to be *neutrosophically connected* if there do not exist neutrosophic open sets (NOS) A and B in the topology such that:

- $A \cap B = 0_M$,
- $A \cup B = 1_M$.

Equivalently, an NTS is neutrosophically connected if it cannot be expressed as the union of two disjoint nontrivial NOS.

Definition 2.7. A *neutrosophic graph* (abbreviated as NG) is a structure $G = (\mathcal{V}^*, E)$, where:

- $\mathcal{V}^* = \{\nu_0, \nu_1, \nu_2, \ldots, \nu_n\}$ is a finite non-empty set of neutrosophic vertices, each associated with three functions:
$$\psi_{*1} : \mathcal{V}^* \to [0, 1], \quad \Theta_{*1} : \mathcal{V}^* \to [0, 1], \quad \Xi_1 : \mathcal{V}^* \to [0, 1],$$
representing the degrees of membership, indeterminacy, and nonmembership, respectively. These functions satisfy the condition
$$0 \le \psi_{*1}(\nu_i) + \Theta_{*1}(\nu_i) + \Xi_1(\nu_i) \le 3 \quad \text{for all} \quad \nu_i \in \mathcal{V}^*.$$

- $E \subseteq \mathcal{V}^* \times \mathcal{V}^*$ is the set of neutrosophic edges. Each edge $(\nu_i, \nu_j) \in E$ is associated with three functions:
$$\psi_{*2} : \mathcal{V}^* \times \mathcal{V}^* \to [0, 1], \quad \Theta_{*2} : \mathcal{V}^* \times \mathcal{V}^* \to [0, 1], \quad \Xi_2 : \mathcal{V}^* \times \mathcal{V}^* \to [0, 1],$$
representing the membership, indeterminacy, and nonmembership degrees of the edge (ν_i, ν_j), respectively, and subject to the constraints:
  (a) $\psi_{*2}(\nu_i, \nu_j) \le \min\{\psi_{*1}(\nu_i), \psi_{*1}(\nu_j)\}$,
  (b) $\Theta_{*2}(\nu_i, \nu_j) \le \min\{\Theta_{*1}(\nu_i), \Theta_{*1}(\nu_j)\}$,
  (c) $\Xi_2(\nu_i, \nu_j) \le \max\{\Xi_1(\nu_i), \Xi_1(\nu_j)\}$,
  (d) $0 \le \psi_{*2}(\nu_i, \nu_j) + \Theta_{*2}(\nu_i, \nu_j) + \Xi_2(\nu_i, \nu_j) \le 3$, for all $(\nu_i, \nu_j) \in E$.

Each vertex $\nu_i \in \mathcal{V}^*$ is represented by the quadruple $(\nu_i, \psi_{*1i}, \Theta_{*1i}, \Xi_{1i})$, and each edge $e_{ij} = (\nu_i, \nu_j) \in E$ is represented by the quadruple $(e_{ij}, \psi_{*2ij}, \Theta_{*2ij}, \Xi_{2ij})$.

Furthermore, based on the predominant degree associated with the edges, we may refer to:

- ψ_*-arcs, if the edges are primarily characterized by membership values,
- Θ_*-arcs, if primarily characterized by indeterminacy values,
- Ξ-arcs, if primarily characterized by nonmembership values.

Definition 2.8. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic graph (NG). A pair $(\acute{\mathcal{V}}^*, \acute{E})$ is called a *neutrosophic subgraph* (NSG) of G if the following conditions hold:

$$\acute{\mathcal{V}}^* \subseteq \mathcal{V}^*, \quad \acute{E} \subseteq E,$$

such that:

- For every $a \in \acute{\mathcal{V}}^*$, if

$$\acute{\psi}_{*1}(a) > 0 \quad \text{or} \quad \acute{\Theta}_{*1}(a) > 0 \quad \text{or} \quad \acute{\Xi}_1(a) > 0,$$

  then

$$\acute{\psi}_{*1}(a) = \psi_{*1}(a), \quad \acute{\Theta}_{*1}(a) = \Theta_{*1}(a), \quad \acute{\Xi}_1(a) = \Xi_1(a).$$

- For every edge $(a, b) \in \acute{E}$, if

$$\acute{\psi}_{*2}(a, b) > 0 \quad \text{or} \quad \acute{\Theta}_{*2}(a, b) > 0 \quad \text{or} \quad \acute{\Xi}_2(a, b) > 0,$$

  then

$$\acute{\psi}_{*2}(a, b) = \psi_{*2}(a, b), \quad \acute{\Theta}_{*2}(a, b) = \Theta_{*2}(a, b), \quad \acute{\Xi}_2(a, b) = \Xi_2(a, b).$$

Moreover, the neutrosophic subgraph $(\acute{\mathcal{V}}^*, \acute{E})$ is called a *neutrosophic spanning subgraph* (NSS) of G if $\acute{\mathcal{V}}^* = \mathcal{V}^*$, i.e., the subgraph includes all vertices of G.

Definition 2.9. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic graph (NG). Then G is said to be:

- **Strong** if for all edges $(\nu_i, \nu_j) \in E$, the neutrosophic degrees of the edge satisfy:

$$\psi_{2ij} = \min\{\psi_{*1i}, \psi_{*1j}\}, \quad \Theta_{2ij} = \min\{\Theta_{*1i}, \Theta_{*1j}\}, \quad \Xi_{2ij} = \max\{\Xi_{1i}, \Xi_{1j}\}.$$

  where $\psi_{2ij}, \Theta_{2ij}, \Xi_{2ij}$ denote the membership, indeterminacy, and nonmembership degrees, respectively, of the edge (ν_i, ν_j), and $\psi_{*1i}, \Theta_{*1i}, \Xi_{1i}$ denote those of the vertex ν_i.

- **Complete** if for every pair of vertices $\nu_i, \nu_j \in \mathcal{V}^*$, the edge (ν_i, ν_j) exists in E and satisfies:

$$\psi_{2ij} = \min\{\psi_{*1i}, \psi_{*1j}\}, \quad \Theta_{2ij} = \min\{\Theta_{*1i}, \Theta_{*1j}\}, \quad \Xi_{2ij} = \max\{\Xi_{1i}, \Xi_{1j}\}.$$

  In other words, a strong neutrosophic graph respects these degree conditions for all existing edges, while a complete neutrosophic graph assumes the existence of all possible edges between vertices and applies the same degree conditions.

Remark 2.1. *If, for some indices i and j, the membership, indeterminacy, and nonmembership degrees of the pair (ν_i, ν_j) satisfy*

$$\psi_{2ij} = \Theta_{2ij} = \Xi_{2ij} = 0,$$

then there is no edge between the vertices ν_i and ν_j in the neutrosophic graph. Conversely, if at least one of these values is nonzero, then an edge between ν_i and ν_j is said to exist.

Definition 2.10. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic graph (NTCG). Then the graph G is called:

- a ψ_***-tree** (denoted by $\psi_* tr$), if the subgraph $G^*_{\psi*}$, determined by the membership function ψ_{*2}, forms a tree.
- an Θ_***-tree** ($\Theta_* tr$) or a Ξ**-tree** (Ξtr), if the subgraphs $G^*_{\Theta*}$ or G^*_{Ξ}, induced by the indeterminacy or nonmembership functions respectively, form trees.
- simply referred to as a **tree** if it satisfies all three conditions, i.e.,

$$G^*_{\psi*} = G^*_{\Theta*} = G^*_{\Xi},$$

and each of these subgraphs is a tree.

Similarly:

- G is called a ψ_***-cycle** ($\psi_* cy$) if $G^*_{\psi*}$ contains a cycle.
- G is called an Θ_***-cycle** ($\Theta_* cy$) or a Ξ**-cycle** (Ξcy) if $G^*_{\Theta*}$ or G^*_{Ξ}, respectively, contains a cycle.
- G is referred to as a **cycle** if all three induced subgraphs satisfy

$$G^*_{\psi*} = G^*_{\Theta*} = G^*_{\Xi},$$

and each forms a cycle.

Remark 2.2. *Let $G = (\mathcal{V}^*, E)$ be a neutrosophic graph (NG).*

- *If G is a $\psi_* cy$, then the weakest ψ_*-arc in G is the arc with the minimum degree of membership.*
- *If G is an $\Theta_* cy$, then the weakest Θ_*-arc is the arc with the minimum degree of indeterminacy.*
- *If G is a Ξcy, then the weakest Ξ-arc is the arc with the maximum degree of nonmembership.*

Example 2.1. Consider the neutrosophic graph $G = (\mathcal{V}^*, E)$ illustrated in Figure 1, where
$\mathcal{V}^* = \{a, b, c, d\}, \quad E = \{(a, b), (b, c), (c, d), (a, d)\}.$

In this case, G satisfies the conditions of a ψ_*-tree, an Θ_*-tree, and a Ξ-tree. However, it does not qualify as a neutrosophic tree in the strict sense, since the corresponding crisp graphs $G^*_{\psi*}$, $G^*_{\Theta*}$, and G^*_{Ξ} are not all identical.

1

Definition 2.11. Let $P : x = \nu_0, \nu_1, \nu_2, \ldots, \nu_n = y$ be a sequence of distinct vertices in a neutrosophic graph (NG) $G = (\mathcal{V}^*, E)$. Then P is called:

- a ψ_***-path** from x to y, if $\psi_{*2}(\nu_{i-1}, \nu_i) > 0$ for all $i = 1, \ldots, n$;
- an Θ_***-path** from x to y, if $\Theta_{*2}(\nu_{i-1}, \nu_i) > 0$ for all $i = 1, \ldots, n$;
- a Ξ**-path** from x to y, if $\Xi_2(\nu_{i-1}, \nu_i) > 0$ for all $i = 1, \ldots, n$.

The sequence P is called simply a **path** from x to y if it simultaneously satisfies all three conditions. Such a path is referred to as an (x-y) **path**. The length of P, denoted by n, is the number of edges in the path.

Moreover, if $x = y$ and $n > 3$, then P is called:

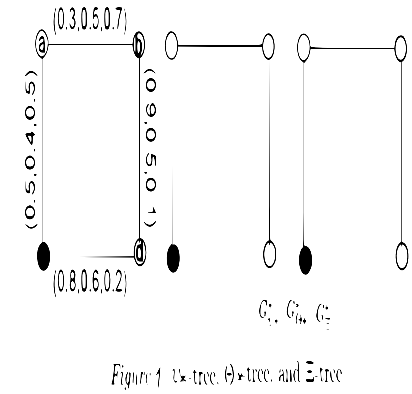

Figure 1: ψ_*-tree, an Θ_*-tree, and a Ξ-tree

- a ψ_***-cycle** ($\psi_* cy$) if it forms a closed ψ_*-path;
- an Θ_***-cycle** ($\Theta_* cy$) if it forms a closed Θ_*-path;
- a Ξ**-cycle** (Ξcy) if it forms a closed Ξ-path;
- simply a **cycle** if it satisfies all three conditions.

Definition 2.12. A neutrosophic graph (NG) $G = (\mathcal{V}^*, E)$ is said to be:

- ψ_***-connected** (abbreviated as ψ_*-cond) if there exists a ψ_*-path between every pair of distinct vertices in G.
- Θ_***-connected** (abbreviated as Θ_*-cond) if there exists an Θ_*-path between every pair of distinct vertices in G.
- Ξ**-connected** (abbreviated as Ξ-cond) if there exists a Ξ-path between every pair of distinct vertices in G.

Moreover, G is said to be **strongly connected** if there exists a (neutrosophic) path connecting every pair of distinct vertices in G, that is, if the graph is simultaneously ψ_*-, Θ_*-, and Ξ-connected.

Remark 2.3. *In Figure 1, the graph G is ψ_*-connected, Θ_*-connected, and Ξ-connected; however, it is not strongly connected in the classical sense.*

Definition 2.13. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic graph (NG), and let $\nu_i, \nu_j \in \mathcal{V}^* \subseteq G$. The ψ_***-strength** (ψ_*-Ş) of connection between ν_i and ν_j, denoted $\mu_{*2}^{\infty}(\nu_i, \nu_j)$, is defined as

$$\psi_{*2}^{\infty}(\nu_i, \nu_j) = \sup\{\psi_{*2}^{w}(\nu_i, \nu_j) : w = 1, 2, \ldots, n\},$$

where $\psi_{*2}^{w}(\nu_i, \nu_j)$ denotes the degree of ψ_*-connection along a path of length w between ν_i and ν_j.

Similarly, the Θ_***-strength** (Θ_*-Ş) and the Ξ**-strength** (Ξ-Ş) of connection between ν_i and ν_j are given by

$$\Theta_{*2}^{\infty}(\nu_i, \nu_j) = \sup\{\Theta_{*2}^{w}(\nu_i, \nu_j) : w = 1, 2, \ldots, n\},$$

$$\Xi_{2}^{\infty}(\nu_i, \nu_j) = \inf\{\Xi_{2}^{w}(\nu_i, \nu_j) : w = 1, 2, \ldots, n\}.$$

For any pair of vertices $u, v \in \mathcal{V}^*$ connected by a ψ_*-path of length L, the ψ_*-strength of connection of length L is defined as

$$\psi_{*2}^{L}(u,v) = \sup\{\psi_{*2}(u,v_1) \cap \psi_{*2}(v_1,v_2) \cap \cdots \cap \psi_{*2}(v_{L-1},v)\},$$

where the supremum is taken over all sequences $u, v_1, \ldots, v_{L-1}, v \in \mathcal{V}^*$.

Analogously, the Θ_*-strength and Ξ-strength of connection of length L are defined by

$$\Theta_{*2}^{L}(u,v) = \sup\{\Theta_{*2}(u,v_1) \cap \Theta_{*2}(v_1,v_2) \cap \cdots \cap \Theta_{*2}(v_{L-1},v)\},$$

$$\Xi_{2}^{L}(u,v) = \sup\{\Xi_{2}(u,v_1) \cap \Xi_{2}(v_1,v_2) \cap \cdots \cap \Xi_{2}(v_{L-1},v)\},$$

where again the supremum is taken over all sequences $u, v_1, \ldots, v_{L-1}, v \in \mathcal{V}^*$.

Let $G - (\nu_i, \nu_j)$ denote the graph obtained from G by removing the arc (ν_i, ν_j). Then, the updated strengths of connection after the deletion are denoted as

$$\psi^{\infty}_{*G-(\nu_i,\nu_j)}(\nu_i,\nu_j), \quad \Theta^{\infty}_{*G-(\nu_i,\nu_j)}(\nu_i,\nu_j), \quad \text{and} \quad \Xi^{\infty}_{G-(\nu_i,\nu_j)}(\nu_i,\nu_j),$$

representing the respective strengths in the modified graph.

Definition 2.14. A *neutrosophic star graph* is a graph in which $n-1$ vertices have degree 1 and a single vertex has degree $n-1$. It is denoted by S_n.

Definition 2.15. A *neutrosophic wheel graph* is formed by joining a center vertex of a star graph S_n to each vertex of a cycle graph C_n. It is denoted by W_n.

Definition 2.16. Given a neutrosophic graph $G = (\mathcal{V}^*, E)$, the *neutrosophic neighborhood set* of a vertex v is $N(v) = \{u \in \mathcal{V}^* \mid (v,u) \in E\}$.

Definition 2.17. Given a neutrosophic graph $G = (\mathcal{V}^*, E)$, a neutrosophic topology τ on $\mathcal{V}^*$ is called a *neutrosophic graph adjacency topology* if it has a subbase consisting of neutrosophic open neighborhoods $N(v)$ for each $v \in \mathcal{V}^*$

Remark 2.4. *A pair $(\mathcal{V}^*, E_\tau)$ is called a neutrosophic graph adjacency topological space, denoted by NGATS.*

Definition 2.18. Let $G = (\mathcal{V}^*, E_\tau)$ be a NGATS and $Q \subseteq \mathcal{V}^*$ Then:

1. The neutrosophic closure of Q is $\overline{Q} = \bigcap\{F \subseteq \mathcal{V}^* \mid Q \subseteq F$ and F is neutrosophic closed set$\}$.
2. The neutrosophic interior of Q is $\mathrm{int}(Q) = \bigcup\{U \subseteq \mathcal{V}^* \mid U \subseteq Q$ and U is neutrosophic open set$\}$.

Definition 2.19. Let $(\mathcal{V}^*, E_\tau)$ be a NGATS. A set $T \subseteq V^*$ is called a *t-set* if $T = \bigcap\{U \subseteq V^* \mid U$ is neutrosophic open and $T \subseteq U\}$.

Definition 2.20. A neutrosophic topological space $(V^*, \mathcal{T})$ is a $\mathcal{T}_1$-space if for any two different vertices $u, v \in V^*$, there are neutrosophic open sets U and V such that $u \in U$, $v \notin U$ and $v \in V$, $u \notin V$.

Definition 2.21. A space is *definitely disconnected* if every neutrosophic open set is also neutrosophic closed (i.e., neutrosophic clopen).

Definition 2.22. Let $(\mathcal{V}^*, E_\tau)$ be NGATS and $Q \subseteq V^*$. A vertex $v \in V^*$ is called a *neutrosophic T-cluster point* of Q if for every t-set T containing v, we have $T \cap Q \neq \emptyset$. The set of all neutrosophic T-cluster points is called the *neutrosophic T-closure* of Q.

Definition 2.23. A set is called *neutrosophic T-closed* if it contains all its neutrosophic T-cluster points. The complement is called a *neutrosophic T-open set*. The family of all neutrosophic T-open sets is denoted by $\mathcal{T}_\tau$.

Definition 2.24. Let $\mathcal{T}_\tau$ be the family of all neutrosophic T-open sets on V^*.Then (V^*,E$_{\mathcal{T}}\tau$) forms a neutrosophic topology, and the pair is called a *Neutrosophic graph adjacency topological space with Neutrosophic T-open topology*, denoted by NGATTS.

Proposition 2.1. *Let* (V^*,$E_{\mathcal{T}}\tau$) *be a NGATS. Then:*

1. *Every discrete NGATS is NGATTS.*
2. *Every T_1-space is NGATTS.*
3. *Every definitely disconnected NGATS is NGATTS.*

Proof : − (1) and (3) are straightforward. (2) Let (V^*,E$_{\mathcal{T}}\tau$) be a T_1 NGATS. Every singleton is Neutrosophic closed, hence Neutrosophic T-open. Since Neutrosophic open sets are unions of Neutrosophic T-open sets, it follows that all Neutrosophic open sets are Neutrosophic T-open. Therefore, the topology $\mathcal{T}$ coincides with $\mathcal{T}\tau$, and so it is a NGATTS.

Proposition 2.2. *Every Neutrosophic complete graph K_n is a NGATTS.*

Proof : − In Neutrosophic complete graph, every Neutrosophic vertex is adjacent to every other vertex. Hence the Neutrosophic neighborhood of any vertex is $V^* \setminus \{v\}$, and the Neutrosophic topology is discrete. Thus, it is a NGATTS by Proposition 2.1(1).

Example 2.2. let $V^* = \{V_{*1}, V_{*2}, V_{*3}, ... , V_{*n}\}$, Then, the subbase of neutrosophic open neighborhoods $S_{*N} = \{V^*, V_{*1}, V^*, V_{*2}, V^*, V_{*3}, ..., V^*, V_{*n}\}$, Then, neutrosophic base $\beta = \{V^*, V_{*1}, V^*, V_{*2}, V^*, V_{*3}, ..., V^*, V_{*n}\} \cup \{V^* V_{*1}, V_{*2}\}$, $\{V^*, V_{*1}, V_{*3}\}$, ... , $\{V^* V_{*1}, V_{*n}\} \cup \{V^* V_{*2}, V_{*3}\}$, , ... , $\{V^* V_{*2}, V_{*n}\}$ for any $V_{*i}, V_{*j} \in V^*$, there exist $\{V^* V_{*i}, V_{*k}\}$, $\{V^* V_{*j}, V_{*k}\} \in \beta$,

Proposition 2.3. *Every neutrosophic cycle graph C_n is a NGATTS.*

Proof : − For $n \leq 4$, C_n is either neutrosophic complete or neutrosophicdiscrete. For $n \geq 5$, each vertex has degree 2, and the neutrosophic neighborhoods generate a neutrosophic discrete topology. Hence, it is a NGATTS.

Example 2.3. Let V^* =$\{V_{*1}, V_{*2}, V_{*3}, V_{*4}, V_{*5}\}$,then the subbase of neutrosophic open neighborhoods $S_{*N} = \{\{V_{*2}, V_{*n}\}, \{V_{*1}, V_{*3}\}, V_{*1}, V_{*4}\}, V_{*3}, V_{*5}\}, ..., \{V_{*1}, V_{*n-1}\}$,then neutrosophic base $\beta = \{\{V_{*2}, V_{*n}\}$, $\{V_{*1}, V_{*3}\}$, $\{V_{*1}, V_{*4}\}$, $\{V_{*3}, V_{*5}\}$, ..., $\{V_{*1}, V_{*n-1}\}$,$\{V_{*1}\}$, $\{V_{*2}\}$, $\{V_{*3}\}$, $\{V_{*n-1}\}$, $\{V_{*n}\}\}$. Now, we have neutrosophic bases that includes all singleton set of a neutrosophic space, thus (V^*,E$_{\mathcal{T}}\tau$) will be neutrosophic discrete topology hence C_n is is a NGATTS by Proposition 2.1(1). (see figure 3)

Proposition 2.4. *Every neutrosophic star graph S_n is a NGATTS.*

Proof : − The center is adjacent to all leaves, and each leaf is adjacent only to the center. The space is neutrosophic definitely disconnected, and hence NGATTS by Proposition 2.1(3).

Example 2.4. let $V^* = \{V_{*1}, V_{*2}, V_{*3}, ..., V_{*n}\}$,then the subbase of neutrosophic open neighborhoods S_{*N} = $\{\{V_{*2}, V_{*3}, ..., V_{*n}\}, \{V_{*1}\}\}$,then neutrosophic base $\beta = \{\{V_{*2}, V_{*3}, ..., V_{*n}\}, \{V_{*1}\}\}$, $E_{\mathcal{T}}\tau = \{0_{*V}, 1_{*V}, \{V_{*2}, V_{*3}, ..., V_{*n}\}$,$\{V_{*1}\}\}$,clear ($V^*$, $E_{\mathcal{T}}\tau$) is neutrosophic disconnected space, so by proposition 2.1.(3) neutrosophic star graph S_n is NGATTS (see figure 4)

Proposition 2.5. *Every neutrosophic wheel graph W_n is a NGATTS.*

Proof : − The structure ensures that each neutrosophic vertex is part of a neutrosophic cycle and connected to the center. It is either neutrosophic definitely disconnected or neutrosophic T_1, hence NGATTS.

Example 2.5. let $V^* = \{V_{*1}, V_{*2}, V_{*3}, V_{*4}..., V_{*n-2}, V_{*n-1}, V_{*n}\}$,then the subbase of neutrosophic open neighborhoods $S_{*N} = \{\{V_{*1}, V_{*2}, V_{*3}, V_{*4}..., V_{*n-2}, V_{*n-1}, V_{*n}\}, \{V_{*n}, V_{*n-1}, V_{*2}\}, \{V_{*n}, V_{*1}, V_{*3}\}, \{V_{*}, V_{*2}, V_{*4}\}, \{V_{*n}, V_{*3}, V_{*5}\}, ..., \{V_{*n}, V_{*1}, V_{*n-2}\}$,then neutrosophic base $\beta = \{\{V_{*1}, V_{*2}, V_{*3}, V_{*4}... , V_{*n-2}, V_{*n-1}, V_{*n}\}, \{V_{*n}, V_{*n-1}, V_{*2}\}, \{V_{*n}, V_{*1}, V_{*3}\}, \{V_{*n}, V_{*2}, V_{*4}\}, \{V_{*n}, V_{*3}, V_{*5}\}, ..., \{V_{*n}, V_{*1}, V_{*n-2}\}, V_{*n-1}, V_{*2}\}, \{V_{*1}, V_{*3}\}, \{V_{*2}, V_{*4}\}, \{V_{*3}, V_{*5}\}, \quad ... \quad ,\{V_{*1}, V_{*n-2}\}, \{V_{*n}, \{V_{*n}, V_{*2}\}, \{V_{*n}, V_{*3}\}, \{V_{*n}, V_{*4}\}, \; ... \; , \{V_{*n}, V_{*n-2}\}\}$,clear $(V^*$,E$\mathcal{T}\tau)$ isneutrosophic T_1, so by proposition 2.1.(2) neutrosophic star graph S_n is NGATTS (see figure 5).

??

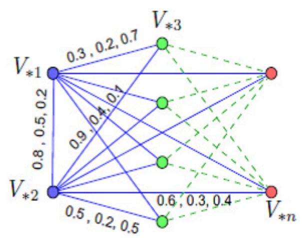

Figure 2

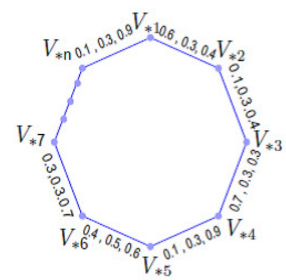

Figure 3

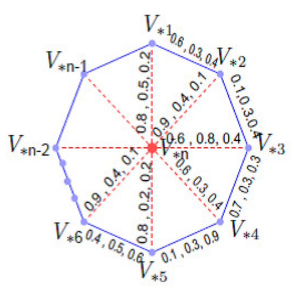

Figure 5

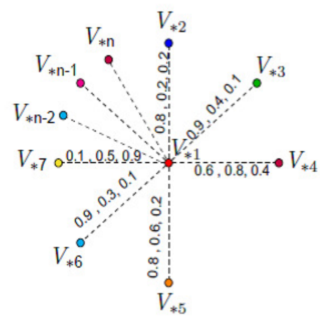

Figure 4

3. On Certain Classes of Arcs and Paths in Neutrosophic Spaces

Definition 3.1. An arc (q, r) in a Neutrosophic topological cognitive graph (NCG) $G = (\mathcal{V}^*, E)$ is said to be:

- ψ_*-**Ş** if $\psi_{*2}(q, r) \geq \acute{\psi}^{\infty}_{*2}(q, r)$;
- Θ_*-**Ş** if $\Theta_{*2}(q, r) \geq \acute{\Theta}^{\infty}_{*2}(q, r)$;
- Ξ-**Ş** if $\Xi_2(q, r) \leq \acute{\Xi}^{\infty}_2(q, r)$.

An arc (q, r) is called **strong** if it satisfies all three of the above conditions; that is, it is simultaneously ψ_*-Ş, Θ_*-Ş, and Ξ-Ş.

Definition 3.2. Let $G = (\mathcal{V}^*, E)$ be a Neutrosophic topological cognitive graph (NCG), and let (q, r) be an arc in G. The arc (q, r) is classified according to the following criteria:

- It is called an $\eta_{\psi*}$**-Ş** arc if $\psi_{*2}(q, r) > \acute{\psi}^{\infty}_{*2}(q, r)$;
  a $\sigma_{\psi*}$**-Ş** arc if $\psi_{*2}(q, r) = \acute{\psi}^{\infty}_{*2}(q, r)$; and
  a $\rho_{\psi*}$ arc (denoted $\rho_{\psi*}$-ar) if $\psi_{*2}(q, r) < \acute{\psi}^{\infty}_{*2}(q, r)$.
- It is called an $\eta_{\Theta*}$**-Ş** arc if $\Theta_{*2}(q, r) > \acute{\Theta}^{\infty}_{*2}(q, r)$;
  a $\sigma_{\Theta*}$**-Ş** arc if $\Theta_{*2}(q, r) = \acute{\Theta}^{\infty}_{*2}(q, r)$; and
  a $\rho_{\Theta*}$ arc (denoted $\rho_{\Theta*}$-ar) if $\Theta_{*2}(q, r) < \acute{\Theta}^{\infty}_{*2}(q, r)$.
- It is called an η_{Ξ}**-Ş** arc if $\Xi_2(q, r) < \acute{\Xi}^{\infty}_2(q, r)$;
  a σ_{Ξ}**-Ş** arc if $\Xi_2(q, r) = \acute{\Xi}^{\infty}_2(q, r)$; and
  a ρ_{Ξ} arc (denoted ρ_{Ξ}-ar) if $\Xi_2(q, r) > \acute{\Xi}^{\infty}_2(q, r)$.

Example 3.1. Consider the neutrosophic graph $G = (\mathcal{V}^*, E)$ depicted in Figur 6, where

$$\mathcal{V}^* = \{A^\circ, \mathcal{B}, \mathcal{L}, \mathcal{W}, \mathcal{Z}\}, \quad E = \{(A^\circ, \mathcal{B}), (A^\circ, \mathcal{L}), (\mathcal{B}, \mathcal{L}), (\mathcal{B}, \mathcal{W}), (\mathcal{L}, \mathcal{Z}), (\mathcal{W}, \mathcal{Z})\}.$$

Then:

- The arc $(A^\circ, \mathcal{B})$ is classified as a $\rho_{\psi*}$-arc and is η_{Ξ}-strong.
- The arc $(A^\circ, \mathcal{L})$ is an $\eta_{\psi*}$-Ş arc and a ρ_{Ξ}-arc.
- The arc $(\mathcal{B}, \mathcal{L})$ is both an $\eta_{\psi*}$-Ş arc and an η_{Ξ}-Ş arc.

Additionally, the arcs $(\mathcal{B}, \mathcal{W})$, $(\mathcal{L}, \mathcal{Z})$, and $(\mathcal{W}, \mathcal{Z})$ are all classified as $\sigma_{\psi*}$-Ş and ρ_{Ξ}-Ş.

??

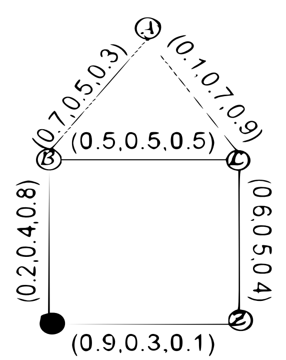

Figure 6: Neutrosophic Graph G

Definition 3.3. Let $P : x = \nu_0, \nu_1, \nu_2, \ldots, \nu_n = y$ be a ψ-path (denoted ψ-pa) from x to y in the neutrosophic graph $G = (\mathcal{V}^*, E)$. Then:

- The path P is called ψ_***-Ş** (respectively, η_{ψ_*}**-Ş**) if for each $i = 1, 2, \ldots, n$, the arc (ν_{i-1}, ν_i) is ψ_*-Ş (respectively, η_{ψ_*}-Ş).

- If P is an Θ_*-path, then P is called **Θ_*-Ş** (respectively, **η_{Θ_*}-Ş**) if for each $i = 1, 2, \ldots, n$, the arc (ν_{i-1}, ν_i) is Θ_*-Ş (respectively, η_{Θ_*}-Ş).
- If P is a Ξ-path, then P is called **Ξ-Ş** (respectively, **η_{Ξ}-Ş**) if for each $i = 1, 2, \ldots, n$, the arc (ν_{i-1}, ν_i) is Ξ-Ş (respectively, η_{Ξ}-Ş).

Finally, the path P is called **strong η-Ş** if it is simultaneously ψ_*-Ş, Θ_*-Ş, and Ξ-Ş (respectively, η_{ψ_*}-Ş, η_{Θ_*}-Ş, and η_{Ξ}-Ş).

Remark 3.1. *Continuing from Example 3.1, consider the path P: $A^\circ, \mathcal{B}, \mathcal{L}$ in Figure* **??**. *This path is an η_{ψ_*}-$Ş_P$ path. Similarly, the path $\acute{P}$: $A^\circ, \mathcal{B}, \mathcal{L}$ is an η_{Ξ}-$Ş_P$ path. Therefore, both P and $\acute{P}$ are η-$Ş_P$ paths.*

Proposition 3.1. *Let $G = (\mathcal{V}^*, E)$ be a neutrosophic topological connectivity graph (NCG). If G is Ξ-connected, then there exists a Ξ-$Ş_P$ path between every pair of vertices in G.*

Proof : $-$ Let $G = (\mathcal{V}^*, E)$ be a neutrosophic topological connectivity graph (NCG), and suppose that G is Ξ-connected. By definition, this means that for every pair of vertices in G, there exists a Ξ-path connecting them. Consider any two vertices $a, b \in \mathcal{V}^*$. If the direct arc (a, b) is not a Ξ-Ş arc, then it must satisfy

$$\Xi_2(a, b) > \acute{\Xi}_2^*(a, b).$$

This implies the existence of an alternative Ξ-path P from a to b whose total Ξ-strength is strictly less than $\Xi_2(a, b)$. If some arcs within P are still not Ξ-Ş, the same reasoning can be applied iteratively to replace each such arc with a subpath of strictly lower Ξ-strength. This process must terminate in a finite number of steps due to the finiteness of the graph and the decreasing nature of Ξ-strength values. Hence, there exists a Ξ-path from a to b composed entirely of Ξ-Ş arcs. That is, a Ξ-Ş path exists between every pair of vertices in G.

Proposition 3.2. *Let $G = (\mathcal{V}^*, E)$ be a neutrosophic topological connectivity graph (NCG). If a path P from vertex x to vertex y is an η_{Ξ}-Ş path, then P is an η-Ş path from x to y; that is, P is an η-$Ş(x - y)$ path.*

Proof : $-$ Let $P : x = \nu_0, \nu_1, \nu_2, \ldots, \nu_n = y$ be an η_{Ξ}-Ş path in the neutrosophic topological connectivity graph $G = (\mathcal{V}^*, E)$. Suppose, for contradiction, that P is not a Ξ-$Ş(x - y)$ path in G. Let $\acute{P} : x = \acute{\nu}_0, \acute{\nu}_1, \acute{\nu}_2, \ldots, \acute{\nu}_n = y$ be a Ξ-$Ş(x - y)$ path in G. Then, for each $i = 1, 2, \ldots, n$, we have

$$\Xi_2(\acute{\nu}_{i-1}, \acute{\nu}_i) < \Xi_P^\infty(x, y),$$

where $\Xi_P^\infty(x, y)$ denotes the maximum Ξ_2-value among the arcs of path P. Now consider the cycle C formed by joining P and $\acute{P}$. By construction, any arc in C that does not belong to both P and $\acute{P}$ must belong to exactly one of them. Let (u, v) be the weakest Ξ-arc on P, i.e., the arc in P with the maximum Ξ_2-value. Consider the path $\acute{\acute{P}} \subset C$, which connects u to v and does not include the arc (u, v). Since (u, v) was the weakest arc in P, it follows that

$$\Xi_2(u, v) \geq \Xi_{\acute{P}}^\infty(u, v),$$

which contradicts the assumption that (u, v) is an η_{Ξ}-Ş arc (because such arcs must be strictly weaker than any alternative connection in the same cycle). This contradiction implies that the initial assumption was false. Hence, P must be a Ξ-$Ş(x - y)$ path, and consequently, P is an η-$Ş(x - y)$ path in G.

Definition 3.4. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic topological connectivity graph (NTCG).

- An arc $(x, y) \in E$ is said to be a neutrosophic ψ_*-bridge (abbreviated as NTCψ_*-bridge) if the removal of (x, y) reduces the ψ_*-Ş connectivity between some pair of vertices in G. Equivalently, there exist vertices $u, v \in \mathcal{V}^*$ such that (x, y) appears in every ψ_*-Ş$(u - v)$ path in G.
- Similarly, an arc $(x, y) \in E$ is called a neutrosophic Ξ-bridge (abbreviated as NTCΞ-bridge) if the removal of (x, y) reduces the Ξ-Ş connectivity between some pair of vertices. That is, there exist vertices $u, v \in \mathcal{V}^*$ such that (x, y) belongs to every Ξ-Ş$(u - v)$ path in G.
- An arc $(x, y) \in E$ is called a neutrosophic bridge (abbreviated as NB) if it is either a NCψ_*-bridge or a NCξ-bridge.

Definition 3.5. Let $G = (\mathcal{V}^*, E)$ be a neutrosophic topological connectivity graph (NCG).

- A vertex $x \in \mathcal{V}^*$ is called a neutrosophic ψ_*-cut vertex (abbreviated as NTCψ_*-cv) if the removal of x decreases the ψ_*-Ş connectivity between some pair of vertices in G. Equivalently, there exist vertices $u, v \in \mathcal{V}^*$ such that every ψ_*-Ş$(u - v)$ path passes through x.
- A vertex $x \in \mathcal{V}^*$ is called a neutrosophic Ξ-cut vertex (abbreviated as Nξ-cv) if the removal of x decreases the Ξ-Ş connectivity between some pair of vertices in G. Equivalently, there exist vertices $u, v \in \mathcal{V}^*$ such that every Ξ-Ş$(u - v)$ path passes through x.
- A vertex $x \in \mathcal{V}^*$ is called a neutrosophic cut vertex (abbreviated as N$\mathcal{V}^*$) if it is either a Nψ_*-cv or a NΞ-cv.

Example 3.2. Consider the neutrosophic topological connectivity graph (NCG) $G = (\mathcal{V}^*, E)$, where

$$\mathcal{V}^* = \{a, b, c, w, z\}, \quad E = \{(a, b), (a, c), (b, c), (b, w), (v, z), (w, z)\}.$$

The following properties hold for the arcs in G:

- The arcs (a, b) and (a, c) are both σ_{ψ_*}-Ş and η_Ξ-Ş.
- The arcs (b, c) and (b, w) are σ_{ψ_*}-Ş and ρ_Ξ-arcs.
- The arcs (c, z) and (w, z) are η_{ψ_*}-Ş and η_Ξ-Ş.
- All arcs in the graph are considered strong.

Additionally, the arcs (c, z) and (w, z) are Nψ_*-bridges, and the arcs (a, b), (a, c), (c, z), and (w, z) are Nξ-bridges. Thus, all arcs in G, except (b, c), qualify as neutrosophic bridges (NB). Furthermore, the vertex z is a NTCψ_*-cut vertex, while both a and z are NΞ-cut vertices.

Theorem 3.1. *Let $(\mathfrak{h}, \mathbb{B})$ be an arc in a neutrosophic topological connectivity graph (NCG) $G = (\mathcal{V}^*, E)$. Then:*

(i) *The arc $(\mathfrak{h}, \mathbb{B})$ is a Nψ_*-bridge if and only if*

$$\psi_{*2}(\mathfrak{h}, \mathbb{B}) > \acute{\psi}_{*2}^{\infty}(\mathfrak{h}, \mathbb{B}).$$

(ii) *The arc $(\mathfrak{h}, \mathbb{B})$ is a NΞ-bridge if and only if*

$$\Xi_2(\mathfrak{h}, \mathbb{B}) < \acute{\Xi}_2^{\infty}(\mathfrak{h}, \mathbb{B}).$$

(iii) *The arc* $(\mathfrak{h},\mathbb{B})$ *is a neutrosophic bridge (NB) if and only if*

$$\psi_{*2}(\mathfrak{h},\mathbb{B}) > \acute{\mu}^{\infty}_{*2}(\mathfrak{h},\mathbb{B}) \quad or \quad \Xi_2(\mathfrak{h},\mathbb{B}) < \acute{\Xi}^{\infty}_2(\mathfrak{h},\mathbb{B}).$$

Proof : −

(i) The result is immediate from the definition.

(ii) Assume that the arc $(\mathfrak{h},\mathbb{B})$ is a NΞ-bridge. By definition, there exist vertices $u, v \in \mathcal{V}^*$ such that every Ξ-Ş$(u-v)$ path contains the arc $(\mathfrak{h},\mathbb{B})$. Let P denote such a path. Suppose, for contradiction, that there exists a Ξ-path $\acute{P}$ from u to v that does not include the arc $(\mathfrak{h},\mathbb{B})$, and let $\acute{P}$ be chosen such that its Ξ-strength $\Xi_{\acute{P}}$ is minimal among all such alternative paths. Then, $P \cup \acute{P}$ forms a cycle C, and the subpath $C \setminus (\mathfrak{h},\mathbb{B})$ is denoted by $\acute{\acute{P}}$, which constitutes a $\mathbb{B}$-path between $\mathfrak{h}$ and $\mathbb{B}$. This implies that $P \cup \acute{P}$ provides a valid alternative Ξ-path avoiding $(\mathfrak{h},\mathbb{B})$, contradicting the assumption that all Ξ-paths from u to v must include $(\mathfrak{h},\mathbb{B})$. Moreover, by construction, we have

$$\Xi_2(\mathfrak{h},\mathbb{B}) < \acute{\acute{\Xi}}^{\infty}_{\acute{P}}(\mathfrak{h},\mathbb{B}) = \acute{\acute{\Xi}}^{\infty}_2(\mathfrak{h},\mathbb{B}).$$

Hence, $\Xi_2(\mathfrak{h},\mathbb{B}) < \acute{\acute{\Xi}}^{\infty}_2(\mathfrak{h},\mathbb{B})$. Conversely, suppose that $\Xi_2(\mathfrak{h},\mathbb{B}) < \acute{\acute{\Xi}}^{\infty}_2(\mathfrak{h},\mathbb{B})$. Then removing the arc $(\mathfrak{h},\mathbb{B})$ from the graph increases the Ξ-strength of connectivity between $\mathfrak{h}$ and $\mathbb{B}$, which by definition means that $(\mathfrak{h},\mathbb{B})$ is a NΞ-bridge.

(iii) The result follows directly from parts (i) and (ii).

4. Application

In this section, the proposed study is implemented using MATLAB, where random distributions of stations are utilized to validate the theoretical concepts and derive novel insights. This implementation not only reinforces the theoretical framework of neutrosophic graphs but also contributes to their practical applicability, thereby advancing both the conceptual understanding and real-world utility of neutrosophic graph models.

```
        [language=Matlab]
        % Neutrosophic Graph Representation
        % Initialize the number of nodes (stations)
        numNodes = 20;

        % Generate random positions for nodes (x, y coordinates in 2D space)
        x = rand(1, numNodes) * 100; % x-coordinates
        y = rand(1, numNodes) * 100; % y-coordinates

        % Create adjacency matrix to represent the neutrosophic graph
        adjMatrix = zeros(numNodes); % Initialize adjacency matrix
        thresholdDistance = 30; % Define threshold distance for connection

        % Define edge matrices
        eta_psi_strong = zeros(numNodes); % eta_psi-strong edges
        sigma_psi_strong = zeros(numNodes); % \sigma_\psi-strong edges
        rho_psi_arc = zeros(numNodes); % rho_psi-arc edges
        eta_Xi_strong = zeros(numNodes); % eta_Xi-strong edges
```

```

% Generate edges and classify them based on distance and strength
for i = 1:numNodes
for j = 1:numNodes
if i ~= j
% Calculate Euclidean distance between nodes
distance = sqrt((x(i) - x(j))^2 + (y(i) - y(j))^2);
% If distance is below the threshold, create an edge
if distance <= thresholdDistance
adjMatrix(i, j) = 1; % General edge
% Classify edge types based on arbitrary rules (for demonstration)
if distance <= thresholdDistance * 0.5
eta_psi_strong(i, j) = 1; % \eta_\psi-strong edge
elseif distance > thresholdDistance * 0.5 && distance <=
    ↪ thresholdDistance * 0.75
sigma_psi_strong(i, j) = 1; % \sigma_\psi-strong edge
else
rho_psi_arc(i, j) = 1; % \rho_\mu-arc edge
end
% Assign \eta_\psi-strong edges based on additional condition
if mod(i + j, 2) == 0
eta_xi_strong(i, j) = 1;
end
end
end
end
end

% Plot the graph
figure;
hold on;
% Plot nodes (stations)
plot(x, y, 'o', 'MarkerSize', 8, 'MarkerFaceColor', 'b', 'DisplayName',
    ↪ 'Stations');

% Add labels to nodes
for i = 1:numNodes
text(x(i) + 1, y(i) + 1, num2str(i), 'FontSize', 8);
end

% Plot edges based on types
for i = 1:numNodes
for j = 1:numNodes
if eta_psi_strong(i, j) == 1
plot([x(i), x(j)], [y(i), y(j)], 'r-', 'LineWidth', 1.5, 'DisplayName',
    ↪ '\eta_\psi-strong');
elseif sigma_psi_strong(i, j) == 1
plot([x(i), x(j)], [y(i), y(j)], 'g-', 'LineWidth', 1.5, 'DisplayName',
```

```
        'sigma_psi-strong');
    elseif rho_psi_arc(i, j) == 1
    plot([x(i), x(j)], [y(i), y(j)], 'b--', 'LineWidth', 1.5, 'DisplayName', 'rho_psi-arc');
    elseif eta_xi_strong(i, j) == 1
    plot([x(i), x(j)], [y(i), y(j)], 'k-.', 'LineWidth', 1.5, 'DisplayName', '\eta_\Xi-strong');
    end
    end
    end

    % Set graph title and labels
    title('Neutrosophic Graph with Edge Classifications');
    xlabel('X-Coordinate');
    ylabel('Y-Coordinate');
    legend show;
    grid on;
    axis equal;
    hold off;

    % Save the adjacency matrices for further analysis
    save('NeutrosophicGraph_AdjMatrices.mat', 'adjMatrix', 'eta_psi_strong', 'sigma_psi_strong', 'rho_psi_arc', 'eta_Xi_strong');
```

5. Conclusion

Extension of Graph Theory to the Neutrosophic Domain

This research demonstrates that integrating neutrosophic logic into graph theory provides a more comprehensive framework for modeling systems characterized by varying degrees of membership, non-membership, and indeterminacy. This allows for a richer representation of uncertainty within complex networks.

Enrichment of Neutrosophic Graph Structures

By introducing new types of arcs and the novel constructs of the ψ-bridge Ξ-bridge, the study significantly enhances the structural modeling capabilities of neutrosophic graphs, enabling more precise descriptions of node relationships in uncertain environments.

Development of the neutrosophic Generalized Adjacency Topological Space (NGATTS)

The proposed NGATTS model successfully bridges classical topological concepts with graph-theoretic structures, providing a robust mathematical framework for analyzing adjacency relationships in a topological context. This advancement opens new pathways for topological graph analysis.

Demonstration of Model Generality

The study proves that several well-known graph classes including complete graphs, cycles, wheel graphs, star graphs, and weak graphs naturally fit within the NGATTS framework. This confirms the generality and versatility of the proposed model.

Practical Implementation Using MATLAB

The theoretical contributions were validated through practical implementation in MATLAB, where various neutrosophic graph structures were constructed, visualized, and analyzed. This practical approach highlights the feasibility of applying the proposed models to real-world problems in network science, biology, computer networks, and uncertain data analysis.

Opening New Research Directions

This work lays a strong foundation for future research in fields such as network topology, graph algorithm optimization, and neutrosophic data analysis. The flexibility of the proposed framework invites further exploration into applications where classical graph models are insufficient to capture uncertainty and indeterminacy.

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