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Fixed Point Results in Perturbed Metric Spaces

Kajal Kharb and Sanjay Kumar

ABSTRACT: In this paper, we prove common fixed points theorems for a variety of mappings, including various contraction, commuting mappings, weakly commuting mappings and their variants, as well as weakly compatible mappings and weakly compatible maps along with property (E.A.).

Key Words: Perturbed metric space, common fixed points, commuting mappings, weakly commuting mappings, weakly compatible mappings, expansive map, Property (E.A.).

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1. Introduction

The measurement of the distance between two points is not always exact. During measurement, some errors may occur. These errors may be slightly positive, slightly negative, or sometimes zero. If error is zero, then it corresponds to the metric. To account for these, a positive error is subtracted and a negative error is added during determining the exact value of the distance function. These errors may play a significant role during measurement.

In order to overcome the difficulty, whenever error is added in metric, Mohamed Jleli and Bessem Samet [10] gave the notion of a perturbed metric space. Perturbed metric spaces is improvement over the metric spaces. The significance of perturbed metric spaces lies across a wide range of mathematical and applied disciplines.

Even though for small positive errors, the structure of these spaces still retains the properties of metric spaces. In this way, perturbed metric spaces help to bridge the gap between the mathematical models and real-world situations, where exact distance are not measurable.

In 2025, Mohamed Jleli and Bessem Samet [10] introduced a more general form of distance function, known as perturbed metric space as follows :

Definition 1.1. Let $D, P : X \times X \to [0, \infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function

$$D - P : X \times X \to \mathbb{R},$$

defined by the relation

$$(D - P)(x, y) = D(x, y) - P(x, y),$$

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for all $x, y, z \in X$, is a exact metric on X, satisfies the following conditions

(i) $(D-P)(x,y) \geq 0$;

(ii) $(D-P)(x,y) = 0$ if and only if $x = y$;

(iii) $(D-P)(x,y) = (D-P)(y,x)$;

(iv) $(D-P)(x,y) \leq (D-P)(x,z) + (D-P)(z,y)$.

P is called a *perturbing function* and $D = d + P$ be an *perturbed metric.*
The set X endowed with D and *perturbed function* P denoted by (X, D, P) is known as *perturbed metric spaces.*

Notice that a perturbed metric on X is not necessarily a metric on X. But a metric is always perturbed metric when perturbed error is zero.

Example 1.1. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x,y) = |x-y| + x^2y^4, \ \ for \ all \ x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed function

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x,y) = x^2y^4, \quad x, y \in \mathbb{R}.$$

In this case, the exact metric is the mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(x,y) = D(x,y) - P(x,y) \ , \ where$$

$$d(x,y) = |x-y|, \quad x, y \in \mathbb{R}.$$

Here we note that D is not necessarily a metric, because $D(1,1) = 1 \neq 0$ as $x = y$, but D is perturbed metric on X with respect to perturbed function P.

We now introduce topological structure in perturbed metric space.

The topological structure of the perturbed metric space (X, D, P) is induced by the exact metric $d = D - P$. The topology $\tau_{D,P}$ on X is defined as:

$$\tau_{D,P} := \tau_d = \{U \subseteq X \mid \forall x \in U, \exists r > 0 \text{ such that } B_d(x,r) \subseteq U\},$$

where the open ball with respect to d is given by:

$$B_d(x,r) = \{y \in X \mid d(x,y) = D(x,y) - P(x,y) < r\}.$$

Definition 1.2. Let (X, D, P) be a perturbed metric space with perturbed function P. A sequence $\{x_n\}$ in X is said to be

(i) *perturbed convergent sequence*, if $\{x_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.

(ii) *perturbed Cauchy sequence*, if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

(iii) (X, D, P) is a *complete perturbed metric space* if (X, d) is a complete metric space, i.e., every perturbed Cauchy sequence converges in perturbed metric space.

A mapping T defined on (X, D, P) is a *perturbed continuous mapping*, if T is continuous with respect to the metric d.

We recall some elementary properties of perturbed metric spaces [10] .

Proposition 1.1. [10] Let $D, P, Q : X \times X \to [0, \infty)$ be three given mappings and $\alpha > 0$.

(i) If (X, D, P) and (X, D, Q) be two perturbed metric spaces, then $\left(X, D, \frac{P+Q}{2}\right)$ is a perturbed metric space.

(ii) If (X, D, P) is a perturbed metric space, then $(X, \alpha D, \alpha P)$ is a perturbed metric space.

Here for the convenience of readers, we provide the proof of the proposition 1.1.

Proof.

(i) Since $D - P$ and $D - Q$ are two metrics on X, then

$$\frac{1}{2}[(D-P)+(D-Q)] = D - \frac{P+Q}{2}$$

is a metric on X, which proves (i).

(ii) Since $D - P$ is a metric on X and $\alpha > 0$, then

$$\alpha(D-P) = \alpha D - \alpha P$$

is a metric on X, which proves (ii).

Remark 1.1. If (X, D, P_1), (X, D, P_2), (X, D, P_3) $\dots (X, D, P_k)$ be k perturbed metric spaces, then $\left(X, D, \frac{(P_1+P_2+P_3+\dots P_k)}{k}\right)$ is a perturbed metric space.

In 2025, Mohamed Jleli and Bessem Samet [10] proved the Banach contraction principle in the setting of perturbed metric spaces.

Theorem 1.1. [10] Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ be a given mapping. Assume that the following conditions hold:

(i) T is a perturbed continuous mapping;

(ii) There exists $\lambda \in (0, 1)$ such that

$$D(Tu, Tv) \leq \lambda D(u, v)$$

for all $u, v \in X$.

Then, T admits unique fixed point.

In 2025, Maria Nutu, Cristina Maria Pacurar [12] proved the Kannan mappings in the setting of perturbed metric spaces.

Definition 1.3. [12] Let (X, D, P) be a perturbed metric space and $T : X \to X$ is a mapping such that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that

$$D(Tx, Ty) \leq \lambda \left[D(x, Tx) + D(y, Ty)\right],$$

for every $x, y \in X$. Then T is called a perturbed Kannan mapping.

Theorem 1.2. [12] Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ is a perturbed Kannan mapping. Then, T admits a unique fixed point.

2. Various Contractions in Perturbed Metric Spaces

Now we discuss various contractions such as Reich, Bianchini, and Chatterjea in the setting of the perturbed metric spaces.

In 1971, Reich [21] gave a contraction known as Reich contraction in metric spaces and proved the following fixed point theorem related to Reich contraction.

Theorem 2.1. [21] Let (X, d) be a metric space. $T : X \to X$ be a map satisfying the following condition:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \tag{2.1}$$

for all $x, y \in X$, where α, β, γ are non-negative numbers satisfying $\alpha + \beta + \gamma < 1$.

Then, there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is the unique fixed point of T. □

Now we prove Theorem 2.1. in the setting of perturbed metric spaces.

Theorem 2.2. *Let (X, D, P) be a complete perturbed metric space. Let $T : X \to X$ be a mapping satisfying:*

$$D(Tx, Ty) \leq \alpha D(x, y) + \beta D(x, Tx) + \gamma D(y, Ty) \tag{2.2}$$

for all $x, y \in X$, where α, β, γ are non-negative numbers such that $\alpha + \beta + \gamma < 1$.

Then, there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is the unique fixed point of T. □

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. From (2.2) we have

$$D(x_{n+1}, x_n) \leq \alpha D(x_n, x_{n-1}) + \beta D(x_n, x_{n+1}) + \gamma D(x_{n-1}, x_n). \tag{2.3}$$

Therefore,

$$D(x_{n+1}, x_n) \leq \frac{\alpha + \gamma}{1 - \beta} D(x_n, x_{n-1}). \tag{2.4}$$

Put $\lambda = \frac{\alpha+\gamma}{1-\beta}$, $\lambda \in [0, 1)$ and $D(x_{n+1}, x_n) \leq \lambda D(x_n, x_{n-1})$.

Continuing in this way,we have

$$D(x_{n+1}, x_n) \leq \lambda^n D(x_1, x_0), \quad \text{for all } n \geq 1. \tag{2.5}$$

Let $d = D - P$ be the exact metric. Then, from (2.5), we obtain that

$$d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq \lambda^n D(x_1, x_0), \quad \forall n \geq 0.$$

Since $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1})$, it follows that

$$d(x_n, x_{n+1}) \leq \lambda^n D(x_1, x_0), \quad \forall n \geq 0.$$

Then, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \lambda^n D(x_1, x_0) + \lambda^{n+1} D(x_1, x_0) + \cdots + \lambda^{n+p-1} D(x_1, x_0) \\ &= \lambda^n D(x_1, x_0)(1 + \lambda + \cdots + \lambda^{p-1}) \\ &= \lambda^n D(x_1, x_0) \left(\frac{1 - \lambda^p}{1 - \lambda} \right) \\ &\leq \frac{\lambda^n}{1 - \lambda} D(x_1, x_0). \end{aligned}$$

Proceeding limit as $n \to \infty$, we have $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), so $\{x_n\}$ is also perturbed Cauchy sequence w.r.t (X, D, P). Thus, by completeness of the perturbed metric space (X, D, P), there exists a $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

Now we prove that x^* is a fixed point of T. We have

$$D(Tx_n, Tx^*) \leq \alpha D(x_n, x^*) + \beta D(x_n, Tx_n) + \gamma D(x^*, Tx^*).$$

Taking the limit as $n \to \infty$ in the above equation, we obtain that

$$D(x^*, Tx^*) \leq \gamma D(x^*, Tx^*),$$

which implies

$$d(x^*, Tx^*) + P(x^*, Tx^*) \leq \gamma[d(x^*, Tx^*) + P(x^*, Tx^*)].$$

If $x^* \neq Tx^*$, it follows that $d(x^*, Tx^*) + P(x^*, Tx^*) \neq 0$, which implies the contradiction $\gamma \geq 1$, so $x^* = Tx^*$, i.e., x^* is a fixed point of T.

Let us assume that T admits two distinct fixed points $x^*, y^* \in X$. Then, we have

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \alpha D(x^*, y^*) + \beta D(x^*, Tx^*) + \gamma D(y^*, Ty^*),$$

i.e,

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \alpha D(x^*, y^*)\ ,$$

this implies

$$x^* = y^*.$$

If $x^* \neq y^*$, it follows that $d(x^*, y^*) + P(x^*, y^*) \neq 0$, which is a contradiction. Therefore, x^* is the unique fixed point of T. □

One can note the following :
(i) If we put $\gamma = \beta = 0$ in equation (2.2), then it yields the Banach fixed point theorem.
(ii) If we put $\alpha = 0, \beta = \gamma$ in equation (2.2), then it yields Kannan's fixed point theorem. mentioned in [13].

One can note that the Reich contraction is stronger than Banach's and Kannan's theorems.

Example 2.1. Let $X = [0, 1]$, and define a perturbed metric $D: X \times X \to \mathbb{R}$ as

$$D(x, y) = |x - y| + |x - y|^2\ ,\ for\ all\ x, y \in \mathbb{R}$$

with the perturbation function $P: X \times X \to \mathbb{R}$ given by

$$P(x, y) = |x - y|^2,\ for\ all\ x, y \in \mathbb{R}.$$

Defining $T(x) = x/3$ for $0 \leq x < 1$ and $T(1) = \frac{1}{6}$. T does not satisfy Banach's condition because it is not continuous at 1. Kannan's condition also cannot be satisfied because

$$D(T(0), T(\tfrac{1}{3})) = \tfrac{1}{2}\left(D(0, T(0)) + D(\tfrac{1}{3}, T(\tfrac{1}{3}))\right).$$

But it satisfies condition (2.2) if we put $a = \frac{1}{6}$, $b = \frac{1}{9}$, $c = \frac{1}{3}$ (these are not the smallest possible values).

In 1972, Bianchini [14] introduced the following contraction in metric spaces as follows:

Theorem 2.3. [14] Let (X, d) be a complete metric space and let f be a self-map satisfying (2.6)

$$d(fx, fy) \leq h\ max\{d(x, fx), d(y, fy)\},\ where\ h \in [0, 1) \tag{2.6}$$

for all $x, y \in X$. Then, f has a unique fixed point.
Now, we prove Bianchini fixed point theorem in setting of perturbed metric spaces.

Theorem 2.4. Let (X, D, P) be a complete perturbed metric space and $f : X \to X$ be a given mapping. Assume that the following conditions hold:

(i) f is a perturbed continuous mapping;

(ii) There exists $h \in [0, 1)$ such that

$$D(fx, fy) \leq h\ max\{D(x, fx), D(y, fy)\} \tag{2.7}$$

for all $x, y \in X$.

Then, f admits a unique fixed point.

Proof:- Let $x_0 \in X$ be fixed. Define $x_{n+1} = fx_n$ for n = 0,1,2,3 . ...

Now

$$D(x_n, x_{n+1}) = D(fx_{n-1}, fx_n).$$

$$D(fx_{n-1}, fx_n) \leq h\ max\{D(x_{n-1}, fx_{n-1}), D(x_n, fx_n)\}$$

$$D(x_n, x_{n+1}) \leq h\ max\{D(x_{n-1}, x_n), D(x_n, x_{n+1})\}.$$

Case I : $max\{D(x_{n-1}, x_n), D(x_n, x_{n+1})\} = D(x_{n-1}, x_n)$

$$D(x_n, x_{n+1}) \leq hD(x_{n-1}, x_n). \tag{2.8}$$

Continuing in this way,we have

$$D(x_n, x_{n+1}) \leq h^n D(x_{n-1}, x_n), n \in \mathbb{N} \tag{2.9}$$

Put $D(x_0, x_1) = D_0$

$$D(x_n, x_{n+1}) \leq h^n D_0, n \in \mathbb{N}.$$

Let $d = D - P$ be the exact metric,we deduce that

$$d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq h^n D_0 \quad , n \in \mathbb{N}.$$

Since,

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1}).$$

Therefore,

$$d(x_n, x_{n+1}) \leq h^n D_0 \quad , n \in \mathbb{N}.$$

Now

$$\begin{aligned} d(x_n, x_{n+p}) &\leq h^n D_0 + h^{n+1} D_0 + \cdots + h^{n+p-1} D_0 \\ &= h^n D_0 (1 + h + h^2 + \cdots + h^{p-1}) \\ &= h^n D_0 \left(\frac{1 - h^p}{1 - h} \right) \\ &\leq \left(\frac{h^n}{1 - h} \right) D_0. \end{aligned}$$

Since, $h \in [0,1)$, therefore $\{x_n\}$ is a cauchy sequence in metric space (X,d), so $\{x_n\}$ is also a perturbed Cauchy sequence in the perturbed metric space (X,D,P). By the completeness of the perturbed metric space (X,D,P), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{2.10}$$

Now,we prove that x^* is a fixed point of f. Since f is a perturbed continuous mapping.
From (2.7)

$$D(fx_n, fx^*) \leq h\ max\{D(x_n, fx_n), D(x^*, fx^*)\}.$$

$$\lim_{n\to\infty} D(fx_n, fx^*) \leq \lim_{n\to\infty} h\ max\{D(x_n, fx_n), D(x^*, fx^*)\}.$$

$$max\{D(x_n, fx_n), D(x^*, fx^*)\} = D(x_n, fx_n).$$

$$\lim_{n\to\infty} D(fx_n, fx^*) \leq \lim_{n\to\infty} hD(x_n, fx_n).$$

$$D(x^*, fx^*) \leq hD(x^*, x^*).$$

$$D(x^*, fx^*) = 0.$$

$$x^* = fx^*$$

that is, x^* is a fixed point of f.

$$max\{D(x_n, fx_n), D(x^*, fx^*)\} = D(x^*, fx^*).$$

$$\lim_{n\to\infty} D(fx_n, fx^*) \leq \lim_{n\to\infty} hD(x^*, fx^*).$$

$$D(x^*, fx^*) \leq hD(x^*, fx^*),$$

i.e,

$$d(x^*, fx^*) + P(x^*, fx^*) \leq h[d(x^*, fx^*) + P(x^*, fx^*)].$$

If $x^* \neq fx^*$,it follows that

$$d(x^*, fx^*) + P(x^*, fx^*) \neq 0 \quad ,$$

So,

$$x^* = fx^*$$

that is, x^* is a fixed point of f.
case II: $max\{D(x_{n-1}, x_n), D(x_n, x_{n+1})\} = D(x_n, x_{n+1})$

$$D(x_n, x_{n+1}) \leq hD(x_n, x_{n+1}).$$

This case is not possible as $0 \leq h < 1$. □

In 1972 known as Chatterjea [16] contraction in complete metric spaces as follows :

Theorem 2.5. [16] Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that for every $x, y \in X$ the inequality

$$d(fx, fy) \leq \lambda\left[d(x, fy) + d(y, fx)\right], \tag{2.11}$$

holds, where $0 \leq \lambda < \frac{1}{2}$. Then, T has a unique fixed point.

Now we prove analogue of Theorem 2.5. in perturbed metric spaces.

Theorem 2.6. Let (X,D,P) be a perturbed metric space and $T : X \to X$ a mapping such that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that

$$D(Tx, Ty) \leq \lambda\left[D(x, Ty) + D(y, Tx)\right], \tag{2.12}$$

for every $x, y \in X$. Then, T admits a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as $x_n = Tx_{n-1} = T^n x_0$, $n = 1, 2, \ldots$. Now,

$$\begin{aligned} D(x_n, x_{n+1}) &= D(Tx_{n-1}, Tx_n) \\ &\leq \lambda \left[D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})\right] \\ &= \lambda \left[D(x_{n-1}, x_{n+1}) + D(x_n, x_n)\right] \\ &\leq \lambda \left[D(x_{n-1}, x_n) + D(x_n, x_{n+1})\right]. \end{aligned}$$

Thus, we have

$$D(x_n, x_{n+1}) \leq \frac{\lambda}{1-\lambda} D(x_{n-1}, x_n). \tag{2.13}$$

Continuing in this way,

$$D(x_n, x_{n+1}) \leq \gamma^n D_0, \tag{2.14}$$

where $\gamma = \frac{\lambda}{1-\lambda} \in [0, 1)$ and $D_0 = D(x_0, x_1)$.

Let $d = D - P$ be the exact metric. Then, from (2.14), we obtain that

$$d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq \gamma^n D_0,\ n \in \mathbb{N}, \quad \forall n \geq 0.$$

Since $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1})$, it follows that

$$d(x_n, x_{n+1}) \leq \gamma^n D_0, \quad \forall n \geq 0.$$

Then, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \gamma^n D_0 + \gamma^{n+1} D_0 + \cdots + \gamma^{n+p-1} D_0 \\ &= \gamma^n D_0 (1 + \gamma + \cdots + \gamma^{p-1}) \\ &= \gamma^n D_0 \left(\frac{1-\gamma^p}{1-\gamma}\right) \\ &\leq \frac{\gamma^n}{1-\gamma} D_0. \end{aligned}$$

Since $\gamma \in [0, 1)$, Therefore, $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), so $\{x_n\}$ is a perturbed Cauchy sequence in (X, D, P). Thus, by completeness of the perturbed metric space (X, D, P), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

Now we prove that x^* is a fixed point of T. We have

$$D(Tx_{n-1}, Tx^*) \leq \lambda[D(x_{n-1}, Tx^*) + D(x^*, Tx_{n-1})].$$

Taking the limit as $n \to \infty$, we obtain that

$$D(x^*, Tx^*) \leq \lambda D(x^*, Tx^*),$$

which implies

$$d(x^*, Tx^*) + P(x^*, Tx^*) \leq \lambda[d(x^*, Tx^*) + P(x^*, Tx^*)].$$

If $x^* \neq Tx^*$, it follows that $d(x^*, Tx^*) + P(x^*, Tx^*) \neq 0$, which implies $\lambda \geq 1$ a contradiction, so $x^* = Tx^*$, i.e., x^* is a fixed point of T. Let us assume that T admits two distinct fixed points $x^*, y^* \in X$. Then, by (2.12), we have

$$D(x^*, y^*) = D(Tx^*, Ty^*) \leq \lambda[D(x^*, Tx^*) + D(y^*, Ty^*)] = 0,$$

which implies

$$d(x^*, y^*) + P(x^*, y^*) \leq 0.$$

If $x^* \neq y^*$, it follows that $d(x^*, y^*) + P(x^*, y^*) \neq 0$, which is a contradiction. Therefore, x^* is the unique fixed point of T. □

Example 2.2. Let $X = [0, 1]$ be endowed with the perturbed metric $D\colon X \times X \to \mathbb{R}$ defined by

$$D(x, y) = |x - y| + x^2y^2, \quad \text{for all } x, y \in \mathbb{R},$$

with the perturbation function $P\colon X \times X \to \mathbb{R}$ given by

$$P(x, y) = x^2y^2, \quad \text{for all } x, y \in \mathbb{R}.$$

Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, 1), \\ \frac{1}{2}, & \text{if } x = 1. \end{cases}$$

Here, T is not continuous. Indeed, for $x \in [0, 1)$ and $y = 1$,

$$\left|0 - \frac{1}{2}\right| \leq h \max\left\{\left|x - \frac{1}{2}\right|, |1 - 0|\right\} = h,$$

which is true for any h satisfying $\frac{1}{2} \leq h < 1$. However, in Chatterjea's contraction condition, we require $h < \frac{1}{2}$, so (2.12) does not hold.

On the other hand, suppose T satisfies Chatterjea's contraction condition (2.12). Then, on taking $x = \frac{1}{2}$ and $y = 1$ in (2.12), we get

$$\left|0 - \frac{1}{2}\right| \leq h\left[\left|\frac{1}{2} - \frac{1}{2}\right| + |1 - 0|\right] \Leftrightarrow \frac{1}{2} \leq h < \frac{1}{2},$$

a contradiction.

3. Jungck's Fixed Point Theorem in Perturbed Metric Spaces

G. Jungck [4] proved an interesting result for commutative mapping in metric spaces as follows:

Theorem 3.1. Let (X, d) be a complete metric space and f be a continuous self-mapping of (X, d). If there exists a mapping $g : X \to X$ and a constant $0 \leq \alpha < 1$ such that

(3.1) $fgx = gfx$ for every $x \in X$,

(3.2) $g(X) \subset f(X)$,

(3.3) $d(gx, gy) \leq \alpha d(fx, fy)$ for every $x, y \in X$,

then f and g have a unique common fixed point.

Now we prove the Jungck fixed point theorem in the setting of Perturbed metric spaces as follows:

Theorem 3.2. Let f be a continuous self-mapping in a complete perturbed metric space (X, D, P). If there exists a mapping $g\colon X \to X$ and a constant $0 \leq \alpha < 1$ such that

(3.4) $f(g(x)) = g(f(x))$ for every $x \in X$,

(3.5) $g(X) \subseteq f(X)$,

(3.6) $D(gx, gy) \leq \alpha\, D(fx, fy)$ for every $x, y \in X$.

Then f and g have a unique common fixed point.

Proof. (Necessary condition) suppose that $f(a) = a$ for some $a \in X$.
Define $g\colon X \to X$ by $g(x) = a$ for all $x \in X$. Then $g(f(x)) = a$ and $f(g(x)) = f(a) = a$ $(x \in X)$, so $f(g(x)) = f(g(x))$ for all $x \in X$ and g commutes with f.
Moreover, $g(x) = a = f(a)$ for all $x \in X$ so that $g(X) \subseteq f(X)$. Finally, for any $\alpha \in (0, 1)$ we have for all $x, y \in X$:

$$D(g(x), g(y)) = D(a, a) = 0 \leq \alpha\, D(f(x), f(y)).$$

Thus (3.6) holds.

On the other hand, suppose there is a mapping g of X into itself which commutes with f and for which (3.6) holds. We show that this condition is sufficient to ensure that f and g have a unique common fixed point.
Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \dots, \text{ where } x_0 \in X. \tag{3.7}$$

Therefore, from (3.6), we have

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) = D(gx_{n-1}, gx_n) \leq \alpha D(fx_{n-1}, fx_n) \\ &= \alpha D(y_{n-1}, y_n) \\ &\vdots \\ &\leq \alpha^n D(y_0, y_1). \end{aligned} \tag{3.8}$$

i.e.,

$$D(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric. Then, from (3.8), we obtain that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1) \quad \forall n \geq 0.$$

Thus,we have

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \alpha^n D(y_0, y_1) + \alpha^{n+1} D(y_0, y_1) + \cdots + \alpha^{n+p-1} D(y_0, y_1) \\ &= \alpha^n D(y_0, y_1)(1 + \alpha + \cdots + \alpha^{p-1}) \\ &= \alpha^n D(y_0, y_1)\left(\frac{1-\alpha^p}{1-\alpha}\right) \\ &\leq \frac{\alpha^n}{1-\alpha} D(y_0, y_1). \end{aligned}$$

Since $\alpha \in [0, 1)$ we obtain that $< y_n >$ is a cauchy sequence in the metric space (X, d), so from Definition (1.2) $< y_n >$ is a perturbed cauchy sequence in (X, D, P) so complete in perturbed metric spaces (X, D, P) , yields the existence of $t \in X$ such that

$$f(x_n) \to t. \tag{3.9}$$

But then (3.7) implies that

$$g(x_n) \to t. \tag{3.10}$$

Now, since f is continuous, by the statement, both f and g are continuous. From (3.9) and (3.10) we have $g(f(x_n)) \to g(t)$ and $f(g(x_n)) \to f(t)$. But f and g commute so that $g(f(x_n)) = f(g(x_n))$ for all n. Thus, $f(t) = g(t)$, and consequently $f(f(t)) = f(g(t)) = g(g(t))$ by commutativity. Therefore, we have

$$D(g(t), g(g(t))) \leq \alpha\ D(f(t), f(g(t))) = \alpha D(g(t), g(g(t))). \tag{3.11}$$

Hence, $(1-\alpha)(D(g(t), g(g(t)))) \leq 0$. Since $\alpha \in (0, 1)$,$g(t) = g(g(t))$.We now have $g(t) = g(g(t)) = f(g(t))$;i.e., $g(t)$ is a common fixed point of f and g .

To see that f and g can have only one common fixed point, suppose that $x = f(x) = g(x)$ and $y = f(y) = g(y)$. Then (3.6) implies

$$D(x,y) = D(g(x), g(y)) \leq \alpha\, D(f(x), f(y)) = \alpha D(x,y),$$

or $D(x,y)(1-\alpha) \leq 0$. Since $\alpha < 1, x = y$.

Example 3.1. Let $X = [0,1]$ and define a perturbed metric $D \colon X \times X \to \mathbb{R}$ as

$$D(x,y) = |x-y| + |x-y|^2 \ , \ for\ all\ x, y \in \mathbb{R}$$

with the perturbation function $P \colon X \times X \to \mathbb{R}$ given by

$$P(x,y) = |x-y|^2,\ for\ all\ x, y \in \mathbb{R}.$$

Now define the mappings $f, g \colon X \to X$ as

$$f(x) = \frac{1}{2}x, \quad g(x) = \frac{1}{4}x$$

Then

$$f(g(x)) = f\left(\frac{1}{4}x\right) = \frac{1}{8}x = g\left(\frac{1}{2}x\right) = g(f(x)) \Rightarrow f(g(x)) = g(f(x))\ For\ all\ x \in X,\ so\ that\ f\ and\ g\ commute.$$

Also

$$g(X) = \left[0, \frac{1}{4}\right] \subseteq \left[0, \frac{1}{2}\right] = f(X) \Rightarrow g(X) \subseteq f(X).$$

Also it is easy to show that

$$D(gx, gy) \leq \alpha D(fx, fy) \quad \text{for all } x, y \in X.$$

All the conditions of Theorem 3.2 are satisfied for the functions $f(x) = \frac{1}{2}x$ and $g(x) = \frac{1}{4}x$ in the perturbed metric space (X, D, P). Therefore, 0 is unique common fixed point.

Corollary 3.1. Let f and g be commuting mappings of a complete perturbed metric space (X, D, P) into itself. Suppose that f is continuous and $g(X) \subseteq f(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that

$$D(g^k(x), g^k(y)) \leq \alpha D(f(x), f(y)) \quad \text{for all } x, y \in X,$$

then f and g have a unique common fixed point.

Proof. Clearly, g^k commutes with f, and since $g^k(X) \subseteq g(X) \subseteq f(X)$, the theorem (from the main result) applies to g^k and f. So there is a unique $a \in X$ such that:

$$a = f(a) = g^k(a).$$

But then, since f and g commute, we can write

$$g(a) = f(g(a)) = g(f(a)) = g(a).$$

That is,

$$g(a) = g(g^k(a)) = g^{k+1}(a),$$

which shows that $g(a)$ is a common fixed point of both f and g^k. By the uniqueness of a, we get $g(a) = a$, so

$$f(a) = a = g(a).$$

Thus, a is a common fixed point of f and g and uniqueness follows from the main theorem. ■

We obtain the Banach contraction principle as a consequence of Corollary 3.1 if we set $k = 1$ and let f be the identity map $I(x) = x$. In fact, if we let f be the identity map and keep the general k, we obtain the generalization of Banach's theorem [10]. Note that Corollary 3.1 does not require g to be continuous.

4. Weakly Commuting Mappings and its Variants

In this section, we investigate the existence of common fixed points for weakly commuting mappings in the setting of perturbed metric spaces.

Definition 4.1. Two self-mappings f and g be of a metric space (X, d) are said to be commuting mapping if and only if

$$f(g(x)) = g(f(x)) \quad \text{for all } x \in X.$$

In 1982, Sessa [17] introduced the concept of weak commutativity in metric spaces as follows:
Definition 4.2. Two self-mappings f and g be of a metric space (X, d) are said to be weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx) \quad \text{for all } x \in X.$$

Pant [15] gave the notion of R-weakly commuting mappings as follows:
Definition 4.3. *A pair of self-mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting if there exists some $R > 0$ such that*

$$d(fgx, gfx) \leq Rd(fx, gx), \quad \textit{for all } x \in X.$$

This leads to a natural question: *Can fixed point theorems be extended to cases where the metric spaces is not complete and the mappings f and g are not continuous on the entire space X?* A partial response to this question was provided by Pant [15].

In 1997, Pathak, Cho and Kang [7] improved the notion of R-weakly commuting mappings to the notion of R-weakly commuting mappings of type (A_g) and R-weakly commuting mappings of type (A_f).

Definition 4.4. *A pair of self-mappings (f, g) of a metric space (X, d) is said to be*

(i) *R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that*

$$d(gfx, ffx) \leq Rd(fx, gx), \quad \text{for all } x \in X.$$

(ii) *R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that*

$$d(fgx, ggx) \leq Rd(fx, gx), \quad \text{for all } x \in X.$$

In 2010, Kumar [22] introduced the notion of R-weakly commuting mappings of type (P) as follows:

Definition 4.5. A pair of self-mappings (f, g) of a metric space (X, d) is said to be R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that

$$d(ffx, ggx) \leq Rd(fx, gx) \text{ for all } x \in X.$$

Now we introduce the analogues notions of weakly commuting mappings and their variants in setting of perturbed metric spaces.

Definition 4.6. Let f and g be two self-mappings on X in a perturbed metric space (X, D, P). Then, a pair (f, g) is said to be weakly commuting mapping if and only if

$$D(fgx, gfx) \leq D(gx, fx) \quad \text{for all } x \in X.$$

Remark 4.1. *Weakly commuting mappings need not be commuting.*
Example 4.1. Let $A, B : [0, 2] \to [0, 2]$ defined by $A(x) = \frac{x}{x+2}, B(x) = \frac{x}{2}$ for all x.
Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2y^2, \textit{ for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping P,

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2 y^2, \quad x, y \in \mathbb{R}.$$

Here, (A, B) is weakly commuting mapping but not commuting. We compute

$$A(Bx) = A\left(\frac{x}{2}\right) = \frac{\frac{x}{2}}{\frac{x}{2}+2} = \frac{x}{x+4}, \quad B(Ax) = B\left(\frac{x}{x+2}\right) = \frac{x}{2(x+2)}.$$

Clearly, $A(Bx) \neq B(Ax)$, so A and B are not commuting.
But $D(ABx, BAx) \leq D(Ax, Bx)$, so A and B are weakly commuting.

Definition 4.7. *A pair of self-mappings (f, g) of a perturbed metric space (X, D, P) is said to be R-weakly commuting if there exists some $R > 0$ such that*

$$D(fgx, gfx) \leq RD(fx, gx), \quad \textit{for all } x \in X.$$

Definition 4.8. *A pair of self-mappings (f, g) of a perturbed metric space (X, D, P) is said to be*

(i) *R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that*

$$D(gfx, ffx) \leq RD(fx, gx), \quad \text{for all } x \in X.$$

(ii) *R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that*

$$D(fgx, ggx) \leq RD(fx, gx), \quad \text{for all } x \in X.$$

Definition 4.9. *Let (X, D, P) be a perturbed metric space and $f, g : X \to X$ be two self-mappings. The pair (f, g) is said to be R-weakly commuting of type (P), if there exists a constant $R > 0$ such that*

$$D(ffx, ggx) \leq RD(fx, gx) \quad \textit{for all } x \in X.$$

Remark 4.2. Now we gave an example which show that *R-weakly commuting mappings*, *R-weakly commuting of type (A_g)*, *R-weakly commuting of type (A_f)*, and *R-weakly commuting of type (P)* are independent.

Example 4.2. Let $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by

$$D(x, y) = |x - y| + x^2 y^2, \ \textit{for all } x, y \in \mathbb{R}.$$

Then D is a perturbed metric on $\mathbb{R}$ with respect to the perturbed mapping

$$P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

given by

$$P(x, y) = x^2 y^2, \quad x, y \in \mathbb{R}.$$

Let $f, g : X \to X$ defined by $fx = x$ and $gx = x^2$, for all x, where $X = [0, 1]$. Then, by a straightforward calculation, one can show that for all $x \in X$,

$$f(f(x)) = x, \quad g(f(x)) = x^2, \quad f(g(x)) = x^2, \quad g(g(x)) = x^4,$$

and

$$\begin{aligned} D(fg(x), gf(x)) &= D(x^2, x^2) \\ &= |x^2 - x^2| + x^4 \cdot x^4 = x^8, \\ D(f(x), g(x)) &= D(x, x^2) \\ &= |x - x^2| + x^2 \cdot x^4 = |x - x^2| + x^6, \end{aligned}$$

$$D(gf(x), ff(x)) = D(x^2, x) = |x^2 - x| + x^4,$$
$$D(fg(x), gg(x)) = D(x^2, x^4) = |x^2 - x^4| + x^4 \cdot x^8 = |x^2 - x^4| + x^{12}.$$

Therefore, we conclude as follows:

1. The pair (f, g) is R-weakly commuting for all positive real values of $R \geq 1$.
2. For $R = 2$, the pair (f, g) is R-weakly commuting of type (A_f), type (A_g), and of type (P).
3. For $R = 1$, the pair (f, g) is R-weakly commuting and type (A_f), but not of type (P) and type (A_g).

Now we prove theorems for these mappings.

Theorem 4.1. Let (X, D, P) be a complete perturbed metric space and f and g be R-weakly commuting self-mappings of X satisfying the following conditions:

(4.1) $g(X) \subseteq f(X)$;

(4.2) f or g is continuous;

(4.3) $D(gx, gy) \leq \alpha\, D(fx, fy)$ for every $x, y \in X$ and $0 \leq \alpha < 1$.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (4.1) one can choose a point x_1 in X such that $fx_1 = gx_0$. In general choose

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X. \tag{4.4}$$

Therefore, from (4.4), we have

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) = D(gx_{n-1}, gx_n) \leq \alpha D(fx_{n-1}, fx_n) \\ &= \alpha D(y_{n-1}, y_n) \\ &\vdots \\ &\leq \alpha^n D(y_0, y_1). \end{aligned} \tag{4.5}$$

i.e.,

$$D(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric. Then, from (4.5), we obtain that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1) \quad \forall n \geq 0.$$

Thus, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \alpha^n D(y_0, y_1) + \alpha^{n+1} D(y_0, y_1) + \cdots + \alpha^{n+p-1} D(y_0, y_1) \\ &= \alpha^n D(y_0, y_1)(1 + \alpha + \cdots + \alpha^{p-1}) \\ &= \alpha^n D(y_0, y_1)\left(\frac{1 - \alpha^p}{1 - \alpha}\right) \end{aligned}$$

$$\leq \frac{\alpha^n}{1-\alpha} D(y_0, y_1).$$

Since $\alpha \in [0,1)$, we obtain that $< y_n >$ is a cauchy sequence in the metric space (X, d), so $< y_n >$ is a perturbed cauchy sequence in (X, D, P). By the completeness of the perturbed metric space (X, D, P), there exists $t \in X$ such that

$$f(x_n) \to t. \tag{4.6}$$

But (4.4) implies that

$$g(x_n) \to t. \tag{4.7}$$

Let us suppose that the mapping g is continuous. Therefore

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

Since f and g are R-weakly commuting,

$$D(g(f(x_n)), f(g(x_n))) \leq R\ D(g(x_n), f(x_n)). \tag{4.8}$$

By letting $n \to \infty$, we get

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gt.$$

We now prove that $t = gt$. Suppose $t \neq gt$, then $D(t, gt) > 0$.
From (4.3), on letting $x = x_n, y = gx_n$

$$D(g(x_n), g(g(x_n))) \leq \alpha\ D(f(x_n), fg(x_n)). \tag{4.9}$$

Proceeding limit as $n \to \infty$, we get

$$D(t, gt) \leq \alpha\ D(t, gt) < D(t, gt),\ \textit{a contradicition.}$$

Therefore $t = gt$. Since $g(X) \subseteq f(X)$, we can find t_1 in X such that $t = gt = ft_1$.
Now from (4.3), take $x = gx_n,\ y = t_1$, we have

$$D(g(g(x_n)), g(t_1)) \leq \alpha\ D(f(g(x_n)), f(t_1)). \tag{4.10}$$

Taking limit as $n \to \infty$, we get

$$D(gt, gt_1) \leq \alpha\ D(gt, ft_1) = \alpha D(gt, gt) = 0,$$

which implies that $gt = gt_1$, *i.e.*,

$$t = gt = gt_1 = ft_1.$$

Also, by using definition of $R-$ weak commutativity,

$$D(gt, ft) = D(g(ft_1), f(gt_1)) \leq R\ D(gt_1, ft_1) = 0,$$

which again implies that

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.
Uniqueness: We assume that $t_2 (\neq t)$ be another common fixed point of f and g.
Then $D(t, t_2) > 0$ and

$$D(t, t_2) = D(gt, gt_2) \leq \alpha\ D(ft, ft_2) = \alpha D(t, t_2) < D(t, t_2),$$

a contradiction, therefore $t = t_2$. Hence uniqueness follows.

Theorem 4.2. Theorem 4.1 remains true if $R-$ weakly commuting property is replaced by any one (retaining the rest of the hypothesis) of the following:

(a) R-weakly commuting property of type (A_g),

(b) R-weakly commuting property of type (A_f),

(c) R-weakly commuting property of type (P).

Proof: Let x_0 be an arbitrary point in X. By (4.1) one can choose a point x_1 in X such that $fx_1 = gx_0$. In general choose

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X. \tag{4.11}$$

From the proof of Theorem 4.1 we conclude that $< y_n >$ is a perturbed cauchy sequence in (X, D, P).By the completeness of the perturbed metric space (X, D, P), there exists $t \in X$ such that

$$f(x_n) \to t. \tag{4.12}$$

But (4.11) implies that

$$g(x_n) \to t. \tag{4.13}$$

Let us suppose that the mapping g is continuous. Therefore

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt.$$

(a) In case (f, g) is an R-weakly commuting property of type (A_g), then

$$D(g(f(x_n)), f(f(x_n))) \leq R\ D(g(x_n), f(x_n)). \tag{4.14}$$

By letting $n \to \infty$, we get

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} gfx_n = gt.$$

We now prove that $t = gt$. Suppose $t \neq gt$, then $D(t, gt) > 0$.
From (4.3), on letting $x = x_n, y = gx_n$

$$D(g(x_n), g(g(x_n))) \leq \alpha\ D(f(x_n), fg(x_n)). \tag{4.15}$$

Proceeding limit as $n \to \infty$, we get

$$D(t, gt) \leq \alpha\ D(t, gt) < D(t, gt),\ \textit{a contradicition.}$$

Therefore $t = gt$. Since $g(X) \subseteq f(X)$, we can find t_1 in X such that $t = gt = ft_1$.
Now from (4.3), take $x = gx_n,\ y = t_1$, we have

$$D(g(g(x_n)), g(t_1)) \leq \alpha\ D(f(g(x_n)), f(t_1)). \tag{4.16}$$

Taking limit as $n \to \infty$, we get

$$D(gt, gt_1) \leq \alpha\ D(gt, ft_1) = \alpha D(gt, gt) = 0,$$

which implies that $gt = gt_1$, *i.e.*,

$$t = gt = gt_1 = ft_1.$$

Also, by using definition of $R-$ weak commutativity of type (A_g),

$$D(gt, ft) = D(g(ft_1), f(ft_1)) \leq R\ D(gt_1, ft_1) = 0,$$

which again implies that

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.
(b) In case (f, g) is an R-weakly commuting property of type (A_f), then

$$D(f(g(x_n)), g(g(x_n))) \leq R\ D(g(x_n), f(x_n)). \tag{4.17}$$

By letting $n \to \infty$, we get

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} gfx_n = gt.$$

Similarly, using above proof we can show

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.
(c) In case (f, g) is an R-weakly commuting property of type (P), then

$$D(f(f(x_n)), g(g(x_n))) \leq R\ D(g(x_n), f(x_n)). \tag{4.18}$$

By letting $n \to \infty$, we get

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} gfx_n = gt.$$

Similarly, using above proof we can show

$$ft = gt = t.$$

Thus t is a common fixed point of f and g.

5. Weakly Compatible Mappings and Property (E.A.)

In this section, we investigate the existence of common fixed points for weakly compatible mappings in the setting of perturbed metric spaces. We consider these mappings under the additional assumption of property (E.A.), which allows us to establish fixed point results without requiring completeness of the space.

We begin with the preliminaries.

In 1996, Jungck and Rhoades [6] introduced the notion of weakly compatible mappings as follows:
Definition 5.1. Two self-maps f and g defined on X (X may be a metric space, perturbed metric space) are said to be weakly compatible if the maps commute at their coincidence points.

Example 5.1. Let $X = [0, 3]$. Define self maps f and g on X as $fx = \frac{x}{2}$ and $gx = x$, then $f(0) = g(0)$ and $fg(0) = gf(0)$. Hence f and g are weakly compatible.

Aamri and El Moutawakil [1] introduced notion *property (E.A.)* and proved common fixed point theorems for the property (E.A.) along with weakly compatible maps. A major benefit of property (E.A.) is that it ensures convergence of desired sequences without completeness. Aamri and Moutawakil [1] introduced the notion of property (E.A.) as follows:

Definition 5.2. *Let S and T be two self-maps in a perturbed metric space (X, D, P). The pair (S, T) is said to satisfy property (E.A.), if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Example 5.2. Let $X = [0, +\infty)$. Define $S, T : X \to X$ by $Tx = \dfrac{x}{2}$ and $Sx = \dfrac{3x}{5}$, for all $x \in X$. Consider the sequence $x_n = \dfrac{1}{n}$. Clearly,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0.$$

Then S and T satisfy property (E.A.).

Example 5.3. Let $X = [4, +\infty)$. Define $S, T : X \to X$ by $Tx = x + 2$ and $Sx = 3x + 2$, for all $x \in X$. Suppose that the property (E.A.) holds. Then, there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = z-2 \quad \text{and} \quad \lim_{n\to\infty} x_n = \frac{z-2}{3}.$$

Thus, $z = 2$, which is a contradiction, since $2 \notin X$. Hence S and T do not satisfy property (E.A.).

Notice that weakly compatible and property (E.A.) are independent of each other in perturbed metric spaces.

Example 5.4. Let $X = \mathbb{R}^+$ and D be the perturbed metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 0, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \text{ or } x = 0 \end{cases} \quad \text{and} \quad gx = \lfloor x \rfloor,$$

the greatest integer less than or equal to x, for all $x \in X$.

Consider a sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$, $n \geq 2$ in $(1, 2)$, then we have

$$\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n.$$

Similarly, for the sequence $\{y_n\} = \left\{1 - \frac{1}{n}\right\}$, $n \geq 2$ in $(0, 1)$, we have

$$\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n.$$

Thus the pair (f, g) satisfies property (E.A.). However, f and g are not weakly compatible as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2, \ldots$ but none of these points are coincidence points of f and g.

Thus we can conclude that, property (E.A.) does not imply weak compatibility.

Now we prove fixed point theorems in the setting of perturbed metric space for a pair of weakly compatible maps and property (E.A.).

Theorem 5.1. Let (X, D, P) be a complete perturbed metric space and f and g be self-maps of X satisfying the condition

$$D(gx, gy) \leq \alpha\, D(fx, fy), \tag{5.1}$$

where $0 \leq \alpha < 1$ and $g(X) \subseteq f(X)$. If one of the subspaces $f(X)$ or $g(X)$ is a complete subspace in X, then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof : Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X. \tag{5.2}$$

Therefore, from (5.1), we have

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) = D(gx_{n-1}, gx_n) \leq \alpha D(fx_{n-1}, fx_n) \\ &= \alpha D(y_{n-1}, y_n) \\ &\vdots \\ &\leq \alpha^n D(y_0, y_1). \end{aligned} \tag{5.3}$$

By induction,we obtain that

$$D(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1).$$

Let $d = D - P$ be the exact metric.Then, from (5.3), we obtain that

$$d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq \alpha^n D(y_0, y_1) \quad \forall n \geq 0.$$

Thus,we have

$$\begin{aligned} d(y_n, y_{n+p}) &\le \alpha^n D(y_0, y_1) + \alpha^{n+1} D(y_0, y_1) + \cdots + \alpha^{n+p-1} D(y_0, y_1) \\ &= \alpha^n D(y_0, y_1)(1 + \alpha + \cdots + \alpha^{p-1}) \\ &= \alpha^n D(y_0, y_1)\left(\frac{1-\alpha^p}{1-\alpha}\right) \\ &\le \frac{\alpha^n}{1-\alpha} D(y_0, y_1). \end{aligned}$$

Since, $\alpha \in [0, 1)$, therefore $\langle y_n \rangle$ is a cauchy sequence in metric space (X, d), so $\langle y_n \rangle$ is also a perturbed Cauchy sequence in the perturbed metric space (X, D, P). By the completeness of the perturbed metric space (X, D, P), there exists $t \in X$ such that

$$f(x_n) \to t. \tag{5.4}$$

But then (5.2) implies that

$$g(x_n) \to t. \tag{5.5}$$

Without loss of generality, assume $f(X)$ is a complete subspace of X, so there exists a point p such that $f(p) = z$.
Now from (5.1), we have,

$$D(gp, gx_n) \le \alpha D(fp, fx_n) = \alpha D(z, y_n).$$

Taking limit as $n \to \infty$, we get $g(p) = z$. Since f and g are weakly compatible, therefore, $fgp = gfp$, i.e., $fz = gz$.

Now we show that z is a common fixed point of f and g. From (5.1),

$$D(gp, gx_n) \le \alpha D(fp, fx_n).$$

Proceeding with the limit $n \to \infty$, we get $gz = z$. Hence, z is a common fixed point of f and g.
Uniqueness:

Let $w(\neq z)$ be another fixed point of f and g.

$$\begin{aligned} D(z, w) &= D(gz, gw) \\ &\le \alpha D(fz, fw) \\ &= \alpha D(z, w), \end{aligned}$$

a contradiction, as $\alpha < 1$. Hence, $z = w$.

Now we prove Theorem 5.1 with minor modifications as follows:

Theorem 5.2. Let (X, D, P) be a complete perturbed metric space and f and g be self-maps of X satisfying the condition:

$$D(gx, gy) \le \alpha\, D(fx, fy), \tag{5.6}$$

where $0 \le \alpha < 1$ and $g(X) \subseteq f(X)$. If one of the subspaces $f(X)$ or $g(X)$ is a closed subset of X, then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof : Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X. \tag{5.7}$$

From the proof of Theorem 3.1, we conclude that $\langle y_n \rangle$ is a Cauchy sequence in X. Since either $f(X)$ or $g(X)$ is closed, for definiteness assume that $g(X)$ is a closed subset of X, so it has a limit point in $g(X)$, call it z. Therefore for some $p \in X$ we have $gp = z$. Now we show that $fp = z$.

From (5.6), we have,

$$D(gx_n, gp) \leq \alpha D(fx_n, fp),$$
$$D(gx_n, gp) \leq \alpha D(y_n, z).$$

Let $\lim_{n\to\infty}$, we get $fp = z$. Rest part of the proof follows from Theorem 5.1.

Now we proved a fixed point theorem for a pair of weakly compatible maps satisfying property (E.A.) in the setting of perturbed metric spaces.

Theorem 5.3. Let (X, D, P) be a perturbed metric space and f and g be self-maps of X satisfying the condition

$$D(gx, gy) \leq \alpha\, D(fx, fy), \tag{5.8}$$

where $0 \leq \alpha < 1$ and f and g satisfy the property (E.A.). Further, $f(X)$ is a closed subspace of X. then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n \quad \text{for all } n = 0, 1, 2, \ldots$$

where $x_0 \in X$.

Since f and g satisfy the property (E.A.), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X.$$

Since $f(X)$ is a closed subspace of X and $\lim_{n\to\infty} fx_n = z$, it follows that

$$z = fp = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \quad \text{for some } p \in X.$$

This implies $z = fp \in f(X)$. Now we show that $fp = gp$.

From condition (5.8), we have

$$D(gp, gx_n) \leq \alpha D(fp, fx_n).$$

Since $y_n = gx_{n+1} = fx_n$ and taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} D(gp, gx_n) \leq \lim_{n\to\infty} D(fp, fx_n).$$

Hence, $fp = gp = z$.

The rest of the proof follows from Theorem 5.2.

Example 5.5. *Consider $X = [0, 2]$ with perturbed metric D. Define the self maps f and g on X as follows:*

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$.

Then f and g satisfy property (E.A.). Also $g(X) = \{0, 1\}$ and $f(X) = \{0, 2\}$. Here we note that neither $f(X)$ is contained in $g(X)$ nor $g(X)$ is contained in $f(X)$. Theorem 5.3 holds for $\frac{1}{2} \leq \alpha < 1$.

6. Expansion Mapping Theorems

In this section, we discuss fixed point theorems for expansion mappings in perturbed metric spaces. In 1984, Wang, Li, Gao and Iséki [18] and Rhoades [2] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces.

Now we prove expansion mappings in the setting of perturbed metric spaces, which correspond to some contractive mappings in metric spaces.

Let f be a mapping of a perturbed metric space (X, D, P) into itself. Then f is said to be an expansive mapping if there exists a constant $\alpha > 1$ such that for all $x, y \in X$, we have

$$D(fx, fy) \geq \alpha D(x, y).$$

Theorem 6.1. *Let (X, D, P) be a complete perturbed metric space and let $T : X \to X$ be a surjective mapping. Suppose that $\alpha > 1$ such that*

$$D(Tx, Ty) \geq \alpha D(x, y), \tag{6.1}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define

$$x_n = Tx_{n+1}, \quad n = 0, 1, 2, \ldots.$$

If $x_0 = x_1$, then x_1 is a fixed point of X and the proof is complete. Suppose $x_0 \neq x_1$. Thus $D(x_1, x_0) \geq 0$, without loss of generality we assume $x_n \neq x_{n+1}$. So $D(x_{n+1}, x_n) > 0$ for all $n = 0, 1, 2, \ldots$. Therefore, from (6.1),

$$\begin{aligned}
D(x_{n+1}, x_n) &= D(Tx_{n+2}, Tx_{n+1}) \geq \alpha D(x_{n+2}, x_{n+1}),\\
\frac{1}{\alpha} D(Tx_{n+2}, Tx_{n+1}) &\geq D(x_{n+2}, x_{n+1}),\\
\frac{1}{\alpha} D(x_{n+1}, x_n) &\geq D(x_{n+2}, x_{n+1}),\\
\Rightarrow \quad D(x_{n+2}, x_{n+1}) &\leq \frac{1}{\alpha} D(x_{n+1}, x_n)\\
&\leq \frac{1}{\alpha^2} D(x_n, x_{n-1})\\
&\leq \frac{1}{\alpha^3} D(x_{n-1}, x_{n-2})\\
&\vdots\\
&\leq \frac{1}{\alpha^{n+1}} D(x_1, x_0).
\end{aligned} \tag{6.2}$$

Let $d = D - P$ be the exact metric. Then, from (6.2), we obtain that

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq \frac{1}{\alpha^{n+1}} D(x_0, x_1) \quad \forall n \geq 0.$$

$$d(x_n, x_{n+1}) \leq \frac{1}{\alpha^{n+1}} D(x_0, x_1) \quad \forall n \geq 0.$$

$$d(x_n, x_{n+p}) \leq \frac{1}{\alpha^{n+1}} D(x_0, x_1) + \frac{1}{\alpha^{n+2}} D(x_0, x_1) + \cdots + \frac{1}{\alpha^{n+p-1}} D(x_0, x_1).$$

Therefore,

$$d(x_n, x_{n+p}) \leq D(x_0, x_1) \cdot \frac{1 - \frac{1}{\alpha^p}}{1 - \frac{1}{\alpha}} \cdot \frac{1}{\alpha^n}.$$

As $n \to \infty$, the term $\frac{1}{\alpha^n} \to 0$, and the remaining factors are bounded. Hence

$$d(x_n, x_{n+p}) \to 0 \; as \; n \to \infty$$

the sequence $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). So $\{x_n\}$ is also a perturbed Cauchy sequence in a perturbed metric space (X, D, P). By the completeness of the perturbed metric spaces (X, D, P), therefore, the sequence $\{x_n\}$ converges to a point, say $z \in X$.

Suppose $D(z, Tz) > 0$ for all $z \in X$,

$$D(x_{n-1}, Tz) = D(Tx_n, Tz) \leq \frac{1}{\alpha} D(x_n, z).$$

Taking limit as $n \to \infty$, we have

$$D(z, Tz) = 0,$$

which implies $Tz = z$.

Hence z is the fixed point of T.

Uniqueness: Let $w (\neq z)$ be another fixed point for T.

Since

$$D(w, z) \leq \frac{1}{\alpha} D(Tw, Tz) = \frac{1}{\alpha} D(w, z),$$

a contradiction since $\alpha > 1$. This implies $z = w$.

Hence T has a unique fixed point in X. □

We further generalize Theorem 6.1. for a pair of weakly compatible mappings in perturbed metric spaces.

Theorem 6.2. : Let (X, D, P) be a complete perturbed metric space. Let f and g be weakly compatible self maps on X and there exists a number $1 < \alpha$ such that

$$D(fx, fy) \geq \alpha\, D(gx, gy) \quad \text{for every } x, y \in X, \tag{6.3}$$

and the following $g(X) \subseteq f(X)$.

If one of the subspaces $g(X)$ or $f(X)$ is complete, then f and g have a unique common fixed point.

Proof: Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n-1} = fx_n, \quad \text{for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X.$$

Therefore, from (6.3), we have

$$\begin{aligned} D(y_n, y_{n+1}) = D(fx_n, fx_{n+1}) = D(gx_{n-1}, gx_n) &\leq \frac{1}{\alpha} D(fx_{n-1}, fx_n) \\ &= \frac{1}{\alpha} D(y_{n-1}, y_n) \\ &\vdots \\ &\leq \frac{1}{\alpha^n} D(y_0, y_1). \end{aligned} \tag{6.4}$$

By induction,we obtain that

$$D(y_n, y_{n+1}) \leq \frac{1}{\alpha^n} D(y_0, y_1). \tag{6.5}$$

Let $d = D - P$ be the exact metric.Then, from (6.5), we obtain that

$$d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}) + P(y_n, y_{n+1}) \leq \frac{1}{\alpha^n} D(y_0, y_1) \quad \forall n \geq 0.$$

$$d(y_n, y_{n+1}) \leq \frac{1}{\alpha^n} D(y_0, y_1) \quad \forall n \geq 0.$$

$$d(y_n, y_{n+p}) \leq \frac{1}{\alpha^n} D(y_0, y_1) + \frac{1}{\alpha^{n+1}} D(y_0, y_1) + \cdots + \frac{1}{\alpha^{n+p-1}} D(y_0, y_1).$$

Therefore,

$$d(y_n, y_{n+p}) \leq D(y_0, y_1) \cdot \frac{1 - \frac{1}{\alpha^p}}{1 - \frac{1}{\alpha}} \cdot \frac{1}{\alpha^n}.$$

As $n \to \infty$, the term $\frac{1}{\alpha^n} \to 0$ and the remaining factors are bounded. Hence,

$$d(y_n, y_{n+p}) \to 0 \ as \ n \to \infty.$$

Thus, the sequence $\{y_n\}$ is a Cauchy sequence in the metric space (X, d), so $< y_n >$ is a perturbed cauchy sequence in (X, D, P), which by completeness of the perturbed metric space (X, D, P) yields the existence of $z \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n-1} = z.$$

Without loss of generality, assume $f(X)$ is a complete subspace of X, so there exists a point p such that $fp = z$.

Now from (6.4), we have,

$$D(gp, gx_n) \leq \frac{1}{\alpha} D(fp, fx_n) = \frac{1}{\alpha} D(z, y_n).$$

Taking limit as $n \to \infty$, we get $gp = z$. Since f and g are weakly compatible, therefore, $fgp = gfp$, i.e., $fz = gz$.

Now we show that z is a common fixed point of f and g. From (6.4),

$$D(gp, gx_n) \leq \frac{1}{\alpha} D(fp, fx_n).$$

Proceeding with the limit $n \to \infty$, we get $gz = z$. Hence, z is a common fixed point of f and g.
Uniqueness: Let $w(\neq z)$ be another fixed point of f and g.

$$\begin{aligned} D(z, w) &= D(gz, gw) \\ &\leq \frac{1}{\alpha} D(fz, fw) \\ &= \frac{1}{\alpha} D(z, w), \end{aligned}$$

a contradiction, as $q > 1$. Hence, $z = w$.
Now we prove the following results after relaxing the condition of completeness of the space:
Theorem 6.3. Let (X, d) be a metric space. Let f and g be weakly compatible self maps of X satisfying conditions (6.3) and the following $g(X) \subseteq f(X)$. If one of the subspaces $g(X)$ or $f(X)$ is complete, then f and g have a unique common fixed point.

Proof: From the proof of Theorem 6.2, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Now suppose that $f(X)$ is a complete subspace of X, then the subsequence of $\{y_n\}$ must get a limit in $f(X)$. Call it be v and $u \in f^{-1}v$. Then $fv = u$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. On setting $x = v$ and $y = x_n$ in (6.3),

$$D(gv, gx_n) \leq D(fv, fx_n)/q,$$

which implies that $fv = gv = u$, which shows that the pair (f, g) has a point of coincidence. Since f and g are weakly compatible, therefore,

$$fgv = gfv, \ i.e., fu = gu.$$

Now we show that u is a fixed point of f and g. From (6.3)

$$D(fu, fx_n) \geq q\ D(gu, gx_n).$$

Proceeding limit as $n \to \infty$, we have

$$D(fu, u) \geq q\ D(fu, u),$$

which implies that $fu = u$. Hence u is a fixed point of f and g. Uniqueness follows easily.

7. Conflict of Interests

The authors declare that there is no conflict of interests.

References

1. Aamri, M. and Moutawakil, D. El., Some new common fixed point theorems under strict contractive conditions. *Journal of Mathematical Analysis and Applications*, 270(1), (2002) , 181–188.
2. B.E. Rhoades , An expansion mapping theorem, Jnanabha, 23 (1993), 151–152.
3. Edelstein, Onfixed and periodic points under contractive mappings, J.London Math.Soc.37(1962),74-79.
4. G. Jungck, Commuting mappings and fixed points, *The American Mathematical Monthly*, 83 (4) (1976), 261–263.
5. G. Jungck and B.E. Rhoades, Fixed point theorems for occasionally weakly compatible maps, *Fixed Point Theory*, 3(2) (2002), 203–214.
6. G. Jungck and B. E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian Journal of Pure and Applied Mathematics, 29(3) (1998), 227–238.
7. H. K. Pathak, Y. J. Cho, and S. M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, *Bulletin of the Korean Mathematical Society*, (1997), 34(2), 247–257.
8. I. A. Rus, *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979.
9. J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Transactions of the American Mathematical Society, 215 (1976), 241–251.
10. M. Jleli, B. Samet. On Banach's fixed point theorem in perturbed metric spaces. Journal of Applied Analysis and Computation, 15(2) 2025,993- 1001.
11. M.A. Khamsi, Remarks on cone metric spaces and xed point theorems of contractive mappings, Fixed Point Theory Appl., (2010), Art. ID 315398, 7 pages.
12. Nutu, Maria, and Cristina Maria Pacurar. More general contractions in perturbed metric spaces. arXiv preprint arXiv:2502.12936 (2025).
13. R. Kannan, Some results on fixed points, Amer. Math. Monthly 76 (1969), 405-408.
14. R.M.T.Bianchini, Su un problema di S.Reich riguardante la teoria dei punti fissi,Boll. Un. Mat. Ital.5(1972), 103-108.
15. R. P. Pant, *Common fixed points of non-commuting mappings*, Journal of Mathematical Analysis and Applications, 188, (1994), 436–440.
16. S.K. Chatterjea, Fixed point theorems. C.R. Acad. Bulgare Sci. 25 (1972), 727–730 .
17. S. Sessa, 'On a weak commutativity condition of mappings in fixed point considerations, *Publications de l'Institut Mathématique (Beograd)*, 32(46) (1982), 149–153.
18. S.Z. Wang, B.Y. Li, Z.M. Gao, and K. Iséki, Some fixed point theorems on expansion mappings, *Math. Japon.*, 29 (1984), 631–636.
19. S. Reich, Fixed points of contractive functions, *Bollettino dell'Unione Matematica Italiana*, 4(5) (1971), 26–42 .
20. S. Gahler, Linear 2-nonmeirte Raume, Math. Narchr. 27 (1964), 1–43.
21. S.Reich, Some remarks concering contration mappings,Canad.Math.Bull. 14(1971),121-124.
22. Sanjay Kumar, *A note on Jungck's fixed point theorem*, Fasciculi Mathematici, 45, (2010), 87-91.

Kajal Kharb
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonipat
India.
E-mail address: kajalkharb18799@gmail.com

and

Sanjay Kumar
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonipat
India.
E-mail address: drsanjaykumar.math@dcrustm.org